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Scalar concomitants of a system of vectors in pseudo-Euclidean geometry of index 1

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Abstract. In this paper we solve the functional equation

$$F\left(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{s}{u}\right) = F\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}\right)$$

for an arbitrary pseudo-orthogonal matrix $A \in O(n,1,R)$ and an arbitrary system of vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}$, where $1 \le s \le n$, and we determine all scalar concomitants of this system in the pseudo-Euclidean geometry of index one $\mathbb{E}^n_1$.

1. Introduction

Referring to Klein's famous Erlangen program, a Klein space was defined by M. Kucharzewski in [5] as a triple (M, G, f), where M is a non-empty set, G denotes a group and f is an effective action of G on the set M, i.e. f is the mapping $f : M \times G \to M$ which satisfies the following conditions:

(1) $$\bigwedge_{x\in M} \bigwedge_{g_1,g_2\in G} f(f(x,g_1),g_2) = f(x, g_2 \circ g_1)$$

(2) $$\bigwedge_{x\in M} f(x,e) = x$$

(3) $$\bigwedge_{x\in M} f(x,g) = x \Rightarrow g = e.$$

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where $\circ$ denotes the group operation and e the unit element of G.

Every triple (M_i, G, f_i), where $f_i : M_i \times G \to M_i$ is an action of G on the set M_i (it satisfies conditions (1), (2)) not necessarily effective, is said to be a geometrical object associated with the Klein space (M, G, f). The class of geometrical objects $\{(M_i, G, f_i) \text{ where } i \in I\}$ which are associated with the Klein space (M, G, f) constitutes a category if we take as morphisms equivariant mappings $F_{ij} : M_i \to M_j$, i.e. the mappings which satisfy the condition

$$\bigwedge_{i,j\in I} \bigwedge_{x\in M_i} \bigwedge_{g\in G} F_{ij}\left(f_i\left(x,g\right)\right) = f_j\left(F_{ij}\left(x\right), g\right). \tag{4}$$

Following M. KUCHARZEWSKI we call this category the Klein geometry of the group G ([5]). If the equivariant mapping F_{ij} is surjective then the object (M_j, G, f_j) is said to be a concomitant of the object (M_i, G, f_i) and we say that the mapping F_{ij} determines this concomitant. If the F_{ij} are injective, then the respective objects are said to be equivalent. In the study of Klein geometry the essential problem is to determine the objects of this geometry and their classification with respect to equivalence, as well as to determine those concomitants of a given object which are objects of a given type.

2. Pseudo-Euclidean geometry of index one

Omitting details about n-dimensional ($n \geq 2$) pseudo-Euclidean geometry of index one $\mathbb{E}_1^n$, which particular in the case $n = 4$ in connection with the theory of relativity are included in a number of journal papers, we give here indispensable notations only. For $n \geq 2$ let be given a matrix $E_1 = [e_{ij}] \in GL\,(n, R)$, where

$$e_{ij} = \begin{cases} 0 & \text{for } i \neq j, \\ +1 & \text{for } i = j \neq n, \\ -1 & \text{for } i = j = n. \end{cases}$$

Definition 1. A pseudo-orthogonal group of index one we call a subgroup of the group $GL\,(n, R)$ if it satisfies

$$G_1 = O(n,1,R) = \left\{A : A \in GL\,(n,R) \wedge A^T \cdot E_1 \cdot A = E_1\right\}. \tag{5}$$

The group G_1 determines a subgroup of the affine group

$$G = \{(A, a) : A \in G_1 \wedge a \in R^n\}\,. \tag{6}$$

Definition 2. A pseudo-Euclidean geometry of index one $\mathbb{E}_1^n$ we call a category of geometrical objects associated with a pseudo-Euclidean space of index one (R^n, G, f).

In particular, to a geometrical object in the geometry $\mathbb{E}_1^n$ there belongs a contravariant vector (or a vector simply)

$$(R^n, G, f_1)\,, \text{ where } \bigwedge_{u\in R^n} \bigwedge_{(A,a)\in G} f_1\,(u, (A, a)) = A \cdot u \tag{7}$$

a covariant vector (or a covector simply)

$$(R^n, G, f_2)\,, \text{ where } \bigwedge_{\overset{*}{v}\in R^n} \bigwedge_{(A,a)\in G} f_2\left(\overset{*}{v}, (A, a)\right) = \overset{*}{v} \cdot A^{-1} \tag{8}$$

and a scalar

$$(R, G, f_3)\,, \text{ where } \bigwedge_{x\in R} \bigwedge_{(A,a)\in G} f_3\,(x, (A, a)) = x. \tag{9}$$

A vector and a covector are equivalent. The mapping H which is given by the formula $\overset{*}{u} = H(u) = (E_1 \cdot u)^T$ is equivariant and bijective. To determine different types of concomitants of the system of s contravariant vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}$, which will be studied in a forthcoming paper, it is necessary to know the scalar concomitants of this system. To describe these concomitants one must solve the functional equation (4), which, applying the transformation rules (7) and (9), may be rewritten in the form

$$\bigwedge_{A\in G_1} F\left(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{s}{u}\right) = F\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}\right). \tag{10}$$

In the special case $s = 2$ we have

Lemma 3. *For two covariant vectors u and v the mapping*

$$p\,(u, v) = u^T \cdot E_1 \cdot v = u^1v^1 + u^2v^2 + \dots + u^{n-1}v^{n-1} - u^nv^n \tag{11}$$

describes a scalar concomitant.

Proof. For any matrix $A \in G_1$ we have

$$p(Au, Av) = (Au)^T \cdot E_1 \cdot (Av) = u^T \left(A^T E_1 A\right) v = u^T E_1 v = p(u, v).$$

$\square$

Let us observe that for arbitrary vectors u, v, w and arbitrary reals α, β we have

$$\begin{aligned} *) \qquad & p(u, v) = p(v, u) \\ **) \qquad & p(\alpha u + \beta v, w) = \alpha \cdot p(u, w) + \beta \cdot p(v, w). \end{aligned}$$

We want to give a general solution F of the functional equation (10). For this we will construct a special pseudo-orthogonal matrix $A = A\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}\right)$.

3. Type of a subspace and signature of a sequence of spanned subspaces

Let in $\mathbb{E}_1^n$ be given a sequence of linearly independent contravariant vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}, \dots, \underset{n}{u}$. Let us denote the scalars by

$$p_{ij} = p\left(\underset{i}{u}, \underset{j}{u}\right) \qquad \text{for } i, j = 1, 2, \dots, n \tag{12}$$

and let

$$\varepsilon_s = \operatorname{sign} \begin{vmatrix} p_{11} & p_{12} & \dots & p_{1s} \\ p_{21} & p_{22} & \dots & p_{2s} \\ \dots & \dots & \dots & \dots \\ p_{s1} & p_{s2} & \dots & p_{ss} \end{vmatrix} = \operatorname{sign} \det \left[p_{ij}\right]_1^s \qquad \text{for } s = 1, 2, \dots, n. \tag{13}$$

Definition 4. We say that the linear subspace $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}\right)$ generated by the vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}$, where $s = 1, 2, \dots, n$ is:

$*)$ Euclidean or of type $+1$ if $\varepsilon_s = 1$,

$**)$ pseudo-Euclidean or of type -1 if $\varepsilon_s = -1$,

$***)$ isotropic or of type 0 if $\varepsilon_s = 0$.

Corollary 5. *A type of a subspace* $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}\right)$ *is invariant by an arbitrary permutation of the vectors* $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}$.

Taking in mind that the vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}$ are linearly independent, from Cauchy's Theorem it follows that

$$\varepsilon_n = \operatorname{sign}\det\,[p_{ij}]_1^n = \operatorname{sign}\Big(\det\Big(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}\Big) \cdot \Big(-\det\Big(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}\Big)\Big)\Big) = -1.$$

Corollary 6. *A type of the space which is spanned by linearly independent vectors* $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}$ *is equal* -1.

Let us take $\varepsilon_0 = +1$ and let formulate the

Definition 7. Let a sequence of subspaces $L\left(\underset{1}{u}\right), L\left(\underset{1}{u}, \underset{2}{u}\right), \dots$ $\dots, L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}\right)$ which are spanned by a sequence of linearly independent vectors $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}$ be given. The sequence $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}, \varepsilon_n) = (+1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}, -1)$ will be called the signature of the sequence $L\left(\underset{1}{u}\right), L\left(\underset{1}{u}, \underset{2}{u}\right), \dots, L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}\right)$, or the signature of the sequence $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}$.

Corollary 8. *For an arbitrary permutation* σ *of* $(1, 2, \dots, n)$*, the signature of a sequence of vectors* $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}$ *is in general different from the signature of the sequence* $\underset{\sigma(1)}{u}, \underset{\sigma(2)}{u}, \dots, \underset{\sigma(n)}{u}$.

We will determine all possible signatures of sequences of linearly independent vectors. Let be given two sequences $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}$ and $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}, \underset{s+1}{u}$ $(s = 1, 2, \dots, n-1)$ of linearly independent vectors. In connection with the first sequence we consider the symmetric matrices

$$*)\ \ \mathcal{P}(s) = \mathcal{P}\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}\right) = \left[p\left(\underset{i}{u}, \underset{j}{u}\right)\right]_1^s = [p_{ij}]_1^s$$

$$**)\ \ \mathcal{M}(s) = \left[p_{ij} + \underset{i}{u}^n \cdot \underset{j}{u}^n\right]_1^s = [m_{ij}]_1^s$$

$$***)\ \ \mathcal{D}(s) = \left[p_{ij} + 2\underset{i}{u}^n \cdot \underset{j}{u}^n\right]_1^s = [d_{ij}]_1^s.$$

The determinants of the matrices introduced above we denote by $P(s)$, $M(s)$, $D(s)$, respectively, and the cofactors of these matrices we denote by $\overset{s}{P}_{ij}$, $\overset{s}{M}_{ij}$, $\overset{s}{D}_{ij}$, respectively. Analogous notations we will use for the second sequence $\underset{1}{u}, \underset{2}{u}, \dots, \underset{s}{u}, \underset{s+1}{u}$. We have to remark that in the geometry $\mathbb{E}_1^n$ only $\mathcal{P}(s)$, $P(s)$, $\overset{s}{P}_{ij}$ are invariant, however, it is well known, that the inequalities $M\,(s) \geq 0$ and $D\,(s) > 0$ hold true and are invariant in Euclidean geometry as well as in the geometry $\mathbb{E}_1^n$. In the following we will apply

Lemma 9. *For an arbitrary square matrix $A = [a_{ij}]_1^s$ and arbitrary reals $a_1, a_2, \dots, a_s$, $c, b_1, b_2, \dots, b_s$ we have*

$$\det B = \det \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} & a_1 \\ a_{21} & a_{22} & \dots & a_{2s} & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{s1} & a_{s2} & \dots & a_{ss} & a_s \\ b_1 & b_2 & \dots & b_s & c \end{bmatrix} = c \det A - \sum_{i=1}^{s} \sum_{j=1}^{s} a_i b_j \overset{s}{A}_{ij}, \tag{14}$$

where $\overset{s}{A}_{ij}$ denote cofactors of the matrix A if $s > 1$, and by definition $\overset{1}{A}_{11} = 1$ in the case $s = 1$.

Proof. Using Laplace's formula two times for the determinant $\det B$ we get (14) immediately. □

Lemma 10. *For arbitrary reals $a_1, a_2, \dots, a_s$ we have*

$$\sum_{i=1}^{s} \sum_{j=1}^{s} a_i \underset{j}{u}^n \overset{s}{P}_{ij} = \sum_{i=1}^{s} \sum_{j=1}^{s} a_i \underset{j}{u}^n \overset{s}{M}_{ij} = \sum_{i=1}^{s} \sum_{j=1}^{s} a_i \underset{j}{u}^n \overset{s}{D}_{ij}. \tag{15}$$

Proof. Applying Lemma 9 and properties of matrices $\mathcal{P}\,(s)$, $\mathcal{M}(s)$, $\mathcal{D}(s)$ we get

$$-\sum_{i=1}^{s} \sum_{j=1}^{s} a_i \underset{j}{u}^n \overset{s}{P}_{ij} = \begin{vmatrix} & a_1 \\ \mathcal{P}(s) & \vdots \\ & a_s \\ \underset{1}{u}^n, \underset{2}{u}^n, \dots, \underset{s}{u}^n & 0 \end{vmatrix} = \begin{vmatrix} & a_1 \\ \mathcal{M}(s) & \vdots \\ & a_s \\ \underset{1}{u}^n, \underset{2}{u}^n, \dots, \underset{s}{u}^n & 0 \end{vmatrix}$$

$$= \begin{vmatrix} & a_1 \\ \mathcal{D}(s) & \vdots \\ & a_s \\ \underset{1}{u}^n, \underset{2}{u}^n, \dots, \underset{s}{u}^n & 0 \end{vmatrix}. \qquad \square$$

Theorem 11. *We have*

$$P(s)+D(s)=2M(s), \tag{16}$$

$$P(s+1)+D(s+1)=2M(s+1). \tag{17}$$

Proof. Using Lemma 9 we calculate

$$P(s)=\begin{vmatrix} & 0 \\ \mathcal{P}(s) & \vdots \\ & 0 \\ \underset{1}{u}^n,\underset{2}{u}^n,\dots,\underset{s}{u}^n & 1 \end{vmatrix}=\begin{vmatrix} & \underset{1}{u}^n \\ \mathcal{M}(s) & \vdots \\ & \underset{s}{u}^n \\ \underset{1}{u}^n,\underset{2}{u}^n,\dots,\underset{s}{u}^n & 1 \end{vmatrix}$$

$$=M(s)-\sum_{i=1}^{s}\sum_{j=1}^{s}\underset{i}{u}^n\underset{j}{u}^n\overset{s}{M}_{ij}.$$

On the other hand we have

$$D(s)=\begin{vmatrix} & 0 \\ \mathcal{D}(s) & \vdots \\ & 0 \\ \underset{1}{u}^n,\underset{2}{u}^n,\dots,\underset{s}{u}^n & 1 \end{vmatrix}=\begin{vmatrix} & -\underset{1}{u}^n \\ \mathcal{M}(s) & \vdots \\ & -\underset{s}{u}^n \\ \underset{1}{u}^n,\underset{2}{u}^n,\dots,\underset{s}{u}^n & 1 \end{vmatrix}$$

$$=M(s)+\sum_{i=1}^{s}\sum_{j=1}^{s}\underset{i}{u}^n\underset{j}{u}^n\overset{s}{M}_{ij}.$$

If we combine the above equalities, we get equality (16). Analogously we can obtain (17). □

Using Lemma 9 and Lemma 10 we can adjust the determinant $P\,(s{+}1)$ with respect to the last component of the last vector $\underset{s+1}{u}{}^n$. Namely, we have

$$\begin{aligned} P(s+1)=&-\left(\underset{s+1}{u}{}^n\right)^2 M(s)+2\,\underset{s+1}{u}{}^n\sum_{i=1}^{s}\sum_{j=1}^{s}m_{s+1,i}\underset{j}{u}^n\overset{s}{P}_{ij} \\ &+m_{s+1,s+1}P(s)-\sum_{i=1}^{s}\sum_{j=1}^{s}m_{s+1,i}m_{s+1,j}\overset{s}{P}_{ij} \end{aligned} \tag{18}$$

and if $M(s) \neq 0$ (which implies $M(s) > 0$) then the discriminant of the polynomial (18) is

$$\Delta = 4P(s)M(s+1). \tag{19}$$

Theorem 12. *If $P(s) = 0$ then $P(s+1) \leq 0$.*

Proof. If $P(s) = 0$, then using (16) we get $M(s) > 0$ and taking into account (19) we obtain $\Delta = 0$. Finally $P(s+1) \leq 0$ follows immediately from (18). □

Theorem 13. *If $P(s) < 0$ then $P(s+1) < 0$.*

Proof. We consider two cases. First, let $M(s+1) = 0$. Then (17) yields $P(s+1) = -D(s+1) < 0$.

Now, let us assume $M(s+1) \neq 0$. Then, of course, $M(s+1) > 0$ and $M(s) > 0$. In consequence of (19) and the assumption $P(s) < 0$ we have $\Delta < 0$ and finally $P(s+1) \leq 0$ follows from (18). □

In consequence of the last two theorems we have

Corollary 14. *There are two kinds of signatures of an arbitrary sequence of n linearly independent vectors, namely*

$*$) $(+1, \ldots, +1, -1, \ldots, -1)$ *if in the sequence of spanned subspaces there does not appear an isotropic subspace.*

$**$) $(+1, \ldots, +1, 0, \ldots, 0, -1, \ldots, -1)$ *if in the sequence of spanned subspaces there appear isotropic subspaces.*

Theorem 15. *If the signature of a sequence of vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{n}{u}$ is of the second kind, then there exists such a permutation σ that among the subspaces generated successively by the vectors $\underset{\sigma(1)}{u}, \underset{\sigma(2)}{u}, \ldots, \underset{\sigma(n)}{u}$ there is exactly one isotropic subspace.*

Proof. Let for any $s \in \{2, 3, \ldots, n-1\}$ be $\varepsilon_0 = \varepsilon_1 = \ldots = \varepsilon_{s-2} = +1$; $\varepsilon_{s-1} = \varepsilon_s = 0$. Since $P(s) = \det \mathcal{P}(s) = 0$, for $i, j = 1, 2, \ldots, s$ the following s^2 identities are fulfilled: $\sum\limits_{k=1}^{s} p_{ik} \overset{s}{P}_{jk} = 0$. From this and the symmetry property of the matrix $\mathcal{P}(s)$ we obtain the implication

$$\left(\bigwedge_{r=1,\ldots,s} \overset{s}{P}_{rr} = 0 \right) \Rightarrow \left(\bigwedge_{i,j=1,\ldots,s} \overset{s}{P}_{ij} = 0 \right).$$

Let us assume that $\overset{s}{P}_{rr} = 0$ for every $r = 1, 2, \ldots, s$. Then

$$D(s) = \begin{vmatrix} & \mathcal{D}(s) & & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \underset{1}{u}^n, & \underset{2}{u}^n, \ldots, & \underset{s}{u}^n & 1 \end{vmatrix} = \begin{vmatrix} & \mathcal{P}(s) & & \begin{matrix} -2\underset{1}{u}^n \\ \vdots \\ -2\underset{s}{u}^n \end{matrix} \\ \underset{1}{u}^n, & \underset{2}{u}^n, \ldots, & \underset{s}{u}^n & 1 \end{vmatrix}$$

$$= P(s) + 2\sum_{i=1}^{s}\sum_{j=1}^{s} \underset{i}{u}^n \underset{j}{u}^n \overset{s}{P}_{ij} = 0$$

which gives a contradiction, because the vectors $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}$ are linearly independent. In what follows, there exists a principal minor of order $s-1$ which differs from zero, for instance $\overset{s}{P}_{kk} \neq 0$ for any $k \in \{1, 2, \ldots, s-1\}$. For the new sequence $\underset{1}{u}, \ldots, \underset{k-1}{u}, \underset{s}{u}, \underset{k+1}{u}, \ldots, \underset{s-1}{u}, \underset{k}{u}$ numbered by successive natural numbers we have $\bar{\varepsilon}_0 = \bar{\varepsilon}_1 = \cdots = \bar{\varepsilon}_{s-1} = 1$ and $\bar{\varepsilon}_s = 0$. □

Corollary 16. *Every n linearly independent vectors can be arranged in such a sequence $\underset{1}{u}, \underset{2}{u}, \ldots, \underset{n}{u}$ that its signature is either* $(+1, \ldots, +1, -1, \ldots$ $\ldots, -1)$ *or* $(+1, \ldots, +1, 0, -1, \ldots, -1)$.

For both cases mentioned in Corollary 16 we will give a construction the so-called Schmidt process of pseudo-orthonormality.

4. The Schmidt process of pseudo-orthonormality

Definition 17. We say that the vector u is

$*)$ the unit vector, if $p(u, u) = +1$;

$**)$ the pseudo-unit vector, if $p(u, u) = -1$.

Definition 18. Two vectors u and v satisfying the condition $p(u, v) = 0$ we call orthogonal and write $u \perp v$.

Definition 19. We say that the system of vectors $\underset{1}{e}, \underset{2}{e}, \ldots, \underset{n}{e}$ constitutes a pseudo-orthogonal basis if $\mathcal{P}\left(\underset{1}{e}, \underset{2}{e}, \ldots, \underset{n}{e}\right) = \left[p\left(\underset{i}{e}, \underset{j}{e}\right)\right]_1^n = E_1$.

Let us have n pairwise orthogonal vectors, exactly one of which is the pseudo-unit vector and all others are unit vectors. These vectors we can arrange so that they form a pseudo-orthonormal basis.

In what follows let $P(0)=1$. We prove the following

Lemma 20. *If the signature of the sequence of linearly independent vectors* $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}$ *is* $\varepsilon_0=\varepsilon_1=\dots=\varepsilon_{s-1}=1$ *and* $\varepsilon_s=\varepsilon_{s+1}=\dots=\varepsilon_n=-1$ *for* $s\in\{1,2,\dots,n\}$, *then the vectors*

$$\underset{k}{e}=\frac{\sum\limits_{i=1}^{k}\overset{k}{P}_{ki}\underset{i}{u}}{\sqrt{|P(k-1)|\cdot|P(k)|}} \qquad \text{for } k=1,2,\dots,n \tag{20}$$

constitute a pseudoorthonormal basis.

Proof. It is easy to see that

$$p\left(\underset{k}{e},\underset{j}{u}\right)=\frac{\sum\limits_{i=1}^{k}\overset{k}{P}_{ki}p_{ij}}{\sqrt{|P(k-1)|\cdot|P(k)|}}=\begin{cases}0 & \text{for } j<k\\ H_1\left(p_{lt}\right) & \text{for } j\ge k.\end{cases} \tag{21}$$

Since $\underset{k}{e}\perp\underset{j}{u}$ for $j<k$, from (20) it follows $\underset{k}{e}\perp\underset{l}{e}$ for $k\neq l$. Now we have

$$\begin{aligned} p\left(\underset{k}{e},\underset{k}{e}\right)&=\frac{\sum\limits_{i=1}^{k}\overset{k}{P}_{ki}\sum\limits_{j=1}^{k}\overset{k}{P}_{kj}p_{ij}}{|P(k-1)|\cdot|P(k)|}=\frac{\sum\limits_{i=1}^{k}\overset{k}{P}_{ki}\delta_i^kP(k)}{|P(k-1)|\cdot|P(k)|}\\ &=\frac{P(k-1)\cdot P(k)}{|P(k-1)|\cdot|Pt(k)|}=\varepsilon_{k-1}\varepsilon_k=\begin{cases}+1 & \text{for } k\neq s\\ -1 & \text{for } k=s.\end{cases} \qquad \square\end{aligned}$$

Lemma 21. *If for any* $s\in\{1,2,\dots,n-1\}$ *we have* $P(s-1)\neq 0$ *and* $P(s)=0$ *and* $P(s+1)\neq 0$ *then* $P(s-1)\cdot P(s+1)=-\left(\overset{s+1}{P}_{s,s+1}\right)^2$.

Proof. Let us consider the Cramer system ($P(s+1)<0$) of linear equalities with the unknowns $\alpha_1,\alpha_2,\dots,\alpha_s,\alpha_{s+1}$ and the parameter a:

$$\begin{cases}\begin{cases}\alpha_1p_{11}+\alpha_2p_{12}+\dots+\alpha_{s-1}p_{1,s-1}+\alpha_sp_{1s}+\alpha_{s+1}p_{1,s+1}=0\\ \alpha_1p_{21}+\alpha_2p_{22}+\dots+\alpha_{s-1}p_{2,s-1}+\alpha_sp_{2s}+\alpha_{s+1}p_{2,s+1}=0\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \alpha_1p_{s-1,1}+\alpha_2p_{s-1,2}+\dots+\alpha_{s-1}p_{s-1,s-1}+\alpha_sp_{s-1,s}+\alpha_{s+1}p_{s-1,s+1}=0\end{cases}\\ \alpha_1p_{s,1}+\alpha_2p_{s,2}+\dots+\alpha_{s-1}p_{s,s-1}+\alpha_sp_{s,s}+\alpha_{s+1}p_{s,s+1}=1\\ \alpha_1p_{s+1,1}+\alpha_2p_{s+1,2}+\dots+\alpha_{s-1}p_{s+1,s-1}+\alpha_sp_{s+1,s}+\alpha_{s+1}p_{s+1,s+1}=a\end{cases} \tag{22}$$

From this system we easily obtain $\alpha_{s+1} = \frac{\overset{s+1}{P}_{s,s+1}}{P(s+1)}$. Omitting the last two equalities the above system reduces to another Cramer system ($P(s-1)>0$) with the unknowns $\alpha_1, \alpha_2, \dots, \alpha_{s-1}$. Putting the solution of this new system into the last but one equality of the system (22) we get $P(s-1) = -\alpha_{s+1} \overset{s+1}{P}_{s,s+1}$. □

Lemma 22. *If the signature of the sequence of linearly independent vectors* $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}$ *is* $\varepsilon_0 = \dots = \varepsilon_{s-1} = 1$; $\varepsilon_s = 0$; $\varepsilon_{s+1} = \dots = \varepsilon_n = -1$ *for* $s \in \{1, 2, \dots, n-1\}$, *then the vectors*

$$(23)\qquad \begin{aligned} \underset{k}{e} &= \frac{\sum_{i=1}^{k} \overset{k}{P}_{ki} \underset{i}{u}}{\sqrt{|P(k-1)| \cdot |P(k)|}} \quad \text{for } k = 1, 2, \dots, s-1, s+2, \dots, n \\ \underset{s}{e} &= \sum_{i=1}^{s+1} \alpha_i \underset{i}{u}; \qquad \underset{s+1}{e} = \sum_{i=1}^{s+1} \beta_i \underset{i}{u} \end{aligned}$$

form a pseudo-orthonormal basis, where α_i, β_i *are the solutions of the system* (22) *in the cases*

$$a = a_\alpha = \frac{-P(s+1) - \overset{s+1}{P}_{ss}}{2 \overset{s+1}{P}_{s,s+1}} \quad \text{and} \quad a = a_\beta = \frac{P(s+1) - \overset{s+1}{P}_{ss}}{2 \overset{s+1}{P}_{s,s+1}},$$

respectively.

Proof. Beside the assertions given in Lemma 20, from (22) it also follows that

$$(24)\qquad \begin{aligned} p\left(\underset{s}{e}, \underset{j}{u}\right) &= \sum_{i=1}^{s+1} \alpha_i p_{ij} = \begin{cases} 0 & \text{for } j < s \\ 1 & \text{for } j = s \\ H_2(p_{lt}) & \text{for } j > s \end{cases} \\ p\left(\underset{s+1}{e}, \underset{j}{u}\right) &= \sum_{i=1}^{s+1} \beta_i p_{ij} = \begin{cases} 0 & \text{for } j < s \\ 1 & \text{for } j = s \\ H_3(p_{lt}) & \text{for } j > s \end{cases} \end{aligned}$$

as well as $p\left(\underset{s}{e}, \underset{s+1}{e}\right) = 0$; $p\left(\underset{s}{e}, \underset{s}{e}\right) = -1$; $p\left(\underset{s+1}{e}, \underset{s+1}{e}\right) = 1$. □

The pseudo-orthonormal basis $\underset{1}{e}, \underset{2}{e}, \dots, \underset{s-1}{e}, \underset{n}{e}, \underset{s+1}{e}, \dots, \underset{n-1}{e}, \underset{s}{e}$ constructed in accordance with the signature of $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}$ in Lemma 20 or Lemma 22 will be used to construct the pseudo-orthogonal matrix $A = A\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}\right)$. The matrix A will enable us to give a general solution of the functional equation (10).

5. Scalar concomitants of a system of vectors

Theorem 23. *Every scalar concomitant of a system of m linearly independent vectors in the geometry $\mathbb{E}_1^n$ is determined by the mapping:*

$$F\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}\right) = \Theta\left(p\left(\underset{i}{u}, \underset{j}{u}\right)\right)$$

where $i \leq j = 1, 2, \dots, m \leq n$ and Θ is an arbitrary function of $\frac{m(m+1)}{2}$ variables.

Proof. First we prove the assertion of the theorem in the case $m = n$. A given linearly independent set of n vectors we arrange into a sequence $\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}$ so that its signature is

$(*)$ $\varepsilon_0 = \dots = \varepsilon_{s-1} = 1$; $\varepsilon_s = \dots = \varepsilon_n = -1$, for $s \in \{1, 2, \dots, n\}$

or

$(**)$ $\varepsilon_0 = \dots = \varepsilon_{s-1} = 1$; $\varepsilon_s = 0$; $\varepsilon_{s+1} = \dots = \varepsilon_n = -1$, for $s \in \{1, 2, \dots, n-1\}$.

Thus by Lemma 20 or Lemma 22 we get a pseudo-orthonormal basis $\underset{1}{e}, \dots, \underset{s-1}{e}, \underset{n}{e}, \underset{s+1}{e}, \dots, \underset{n-1}{e}, \underset{s}{e}$. The covectors corresponding to this basis we number continuously: $\overset{*}{\underset{1}{e}}, \overset{*}{\underset{2}{e}}, \dots, \overset{*}{\underset{n}{e}}$. The matrix $A = A\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}\right)$ whose entries in the i-th row are the successive coefficients of the covector $\overset{*}{\underset{i}{e}}$, is a pseudo-orthogonal matrix of index one. Formulae (21), or (21) and (24) enable us to find the i-th coefficient of the image of the vector $\underset{j}{u}$.

$$\left(A\underset{j}{u}\right)^i = \overset{*}{\underset{i}{e}}\left(\underset{j}{u}\right) = p\left(\underset{k}{e}, \underset{j}{u}\right) = H\left(p\left(\underset{l}{u}, \underset{t}{u}\right)\right) \text{ where } \begin{cases} k = i & \text{if } s \neq i \neq n \\ k = n & \text{if } i = s \\ k = s & \text{if } i = n. \end{cases}$$

Now, in accordance with the equality (10) we have for
$A = A\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}\right) \in G_1$

$$F\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{n}{u}\right) = F\left(A\underset{1}{u}, A\underset{2}{u}, \dots, A\underset{n}{u}\right) = \Theta\left(p\left(\underset{i}{u}, \underset{j}{u}\right)\right).$$

Now, let $m < n$ and $P(m) = P\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}\right) \neq 0$. We construct the vectors $\underset{1}{e}, \underset{2}{e}, \dots, \underset{m}{e}$ of a pseudo-orthonormal basis in accordance with (20) or (23) and the remaining vectors $\underset{m+1}{e}, \dots, \underset{n}{e}$ can be constructed in the pseudo-orthogonal complement $L^{\perp}\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}\right)$ and the assertion of the theorem is true.

Finally, let $m < n$ and $P(m) = 0$. In this case we can arrange the vectors in a sequence with signature $\varepsilon_0 = \cdots = \varepsilon_{m-1} = 1$ and $\varepsilon_m = 0$. This implies that the subspace $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}\right)$ is isotropic and the subspace $L^{\perp}\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}\right)$ is not a pseudo-orthogonal complement of $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m}{u}\right)$. The subspace $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m-1}{u}\right)$ is Euclidean and we construct the vectors $\underset{1}{e}, \underset{2}{e}, \dots, \underset{m-1}{e}$ of a pseudo-orthonormal basis in accordance with Lemma 20. In the isotropic subspace $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m-1}{u}, \underset{m}{u}\right)$ is contained exactly one one-dimensional isotropic subspace which is determined by the isotropic vector $\underset{1}{v} = \frac{1}{2P(m-1)} \sum_{i=1}^{m} \overset{m}{P}_{mi} \underset{i}{u}$. Of course, $\underset{1}{v} \perp \underset{1}{u}, \underset{2}{u}, \dots, \underset{m-1}{u}, \underset{m}{u}$. By virtue of $\overset{m}{P}_{mm} = P(m-1) > 0$ we get the equality $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m-1}{u}, \underset{m}{u}\right) = L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m-1}{u}, \underset{1}{v}\right)$. It should be remarked that there exist other isotropic subspaces of dimension m, which contain the Euclidean subspace $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m-1}{u}\right)$. There is a one-to-one relation between the set of all isotropic subspaces of dimension m inclusive the Euclidean subspace $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m-1}{u}\right)$ and the set of all points of a sphere of dimension $n - m - 1$ (see [8]). Moreover, in the extreme case $m = n-1$ both sets contain two elements. To isotropic subspaces of dimension m inclusive $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m-1}{u}\right)$ there belongs $L\left(\underset{1}{u}, \underset{2}{u}, \dots, \underset{m-1}{u}, \underset{2}{v}\right) \neq$

$L\left(\underset{1}{u},\underset{2}{u},\dots,\underset{m-1}{u},\underset{1}{v}\right)$ where $\underset{2}{v}$ is an isotropic vector (this is a consequence of the condition $p\left(\underset{m}{u},\underset{2}{v}\right)=1$ unlike $p\left(\underset{m}{u},\underset{1}{v}\right)=0$). Of course, we have $\underset{2}{v}\perp\underset{1}{u},\underset{2}{u},\dots,\underset{m-1}{u}$. The vectors $\underset{m}{e}=\underset{1}{v}-\underset{2}{v}$ and $\underset{m+1}{e}=\underset{1}{v}+\underset{2}{v}$ satisfy the conditions:

$$p\left(\underset{m}{e},\underset{m+1}{e}\right)=0,\quad p\left(\underset{m}{e},\underset{m}{e}\right)=-1,\quad p\left(\underset{m+1}{e},\underset{m+1}{e}\right)=1,$$

$$p\left(\underset{m}{e},\underset{i}{u}\right)=\begin{cases}0 & \text{for } i<m\\ -1 & \text{for } i=m\end{cases}\quad\text{and}\quad p\left(\underset{m+1}{e},\underset{i}{u}\right)=\begin{cases}0 & \text{for } i<m\\ 1 & \text{for } i=m.\end{cases}$$

The vectors $\underset{1}{e},\underset{2}{e},\dots,\underset{m-1}{e},\underset{m}{e},\underset{m+1}{e}$ constitute a pseudo-orthonormal basis of the subspace $L\left(\underset{1}{u},\underset{2}{u},\dots,\underset{m-1}{u},\underset{m}{u},\underset{2}{v}\right)$ which is pseudo-Euclidean by virtue of $P(m+1)=P\left(\underset{1}{u},\underset{2}{u},\dots,\underset{m-1}{u},\underset{m}{u},\underset{2}{v}\right)=-P(m-1)<0$. This completes the proof. $\square$

Theorem 23 may be rewritten as follows:

Theorem 24. *The sequence of linearly independent vectors* $\underset{1}{u},\underset{2}{u},\dots,\underset{m}{u}$ *and the sequence of linearly independent vectors* $\underset{1}{v},\underset{2}{v},\dots,\underset{m}{v}$ *belong to the same transitive fiber, i.e. they satisfy the condition*

$$\bigvee_{A\in G_1}\ \bigwedge_{i=1,\dots,m}\ \underset{i}{v}=A\underset{i}{u}$$

if and only if the equality of Gram's matrices

$$\mathcal{P}\left(\underset{1}{u},\underset{2}{u},\dots,\underset{m}{u}\right)=\mathcal{P}\left(\underset{1}{v},\underset{2}{v},\dots,\underset{m}{v}\right)$$

holds.

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