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Geometry of multiparametrized Lagrangians

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Dedicated to Professor Lajos Tamássy on his 70th birthday

1. Introduction

There are many problems in theoretical physics and variational calculus in which the Euler–Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial y^i}\right)-\frac{\partial \mathcal{L}}{\partial x^i}=0\,, \tag{1.1}$$

where $y^i=\frac{dx^i}{dt}$, are fundamental.

The function $\mathcal{L}$ depending on (x^i) and (y^i), $i=1,2,\dots,n$, is called a Lagrangian function or simply a Lagrangian. From a geometrical point of view a Lagrangian (of M) is a function $\mathcal{L}:TM\to R$, where TM is the total space of the tangent bundle (TM,τ,M) to a smooth (C^∞) manifold M. A point $v\in TM$ has the local coordinates (x^i,y^i), where (x^i) are the local coordinates of $x=\tau(v)$, and $v_x=y^i\left(\frac{\partial}{\partial x^i}\right)_x$. Thus

$$\mathcal{L}:(x^i,y^i)\to\mathcal{L}(x^i,y^i),\quad i=1,2,\dots,n=\dim M\,.$$

Expanding the time derivative, eq. (1.1) becomes

$$\frac{\partial^2\mathcal{L}}{\partial y^i\partial y^j}\frac{d^2x^j}{dt^2}=\frac{\partial \mathcal{L}}{\partial x^i}-\frac{\partial^2\mathcal{L}}{\partial y^i\partial x^j}\frac{dx^j}{dt} \tag{1.2}$$

and in order to put it in a normal form we must assume

$$\det\left(\frac{\partial^2\mathcal{L}}{\partial y^i\partial y^j}\right)\neq 0\,. \tag{1.3}$$

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If (1.3) holds, $\mathcal{L}$ is called a regular Lagrangian.

If we put $\theta_{\mathcal{L}} = \frac{\partial \mathcal{L}}{\partial x^i} dx^i$, then $\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}}$ gives a symplectic two forms on TM which is nondegenerate if $\mathcal{L}$ is a regular Lagrangian. Thus the results of symplectic geometry may be applied.

A different point of view was proposed by R. Miron. Namely, using $g_{ij} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j}$ and a nonlinear connection on TM, a metrical structure on TM is introduced so that the results of the so-called Lagrange geometry may be applied.

The origin of Lagrange geometry is as follows. It is usual to call the pair $(M, \mathcal{L})$, where $\mathcal{L}$ is a regular Lagrangian, a Lagrange space. If (M, F) is a Finsler space (cf. M. Matsumoto [5]) then taking $\mathcal{L} = F^2$ we see that any Finsler space is a Lagrange space. As is well-known Finsler geometry has a long tradition and its body of results is very large.

On the other hand, as Professor Radu Miron showed in [6] and [7], the main results of Finsler geometry regarding the nonlinear connection, the Cartan connection and so on can be extended to Lagrange spaces. Thus the geometry of the pair $(M, \mathcal{L})$, called Lagrange geometry, was developed in the last ten years.

In dynamics as well as in variational calculus (see R. Hermann [4] p.117) time dependent Lagrangians are also considered.

A regular time dependent Lagrangian is a function $\mathcal{L} : TM \times R \to R$, $(x^i, y^i, t) \to L(x^i, y^i, t)$ such that $\det\left(\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j}\right) \neq 0$. We developed a geometrical theory of such Lagrangians in some recent papers ([1]–[3]). Now we consider multiparametrized Lagrangians i.e. functions $\mathcal{L} : TM \times R^m \to R$, $(x^i, y^i, t^1, \dots, t^m) \to \mathcal{L}(x^i, y^i, t^1, \dots, t^m)$. Such Lagrangians appear in variational problems for wich the constraints are considered (cf. R. Hermann [4]).

We shall derive here the main facts from the geometry of the manifold $E = TM \times R^m$ fibered over M, endowed with a regular multiparametrized Lagrangian $\mathcal{L}$, i.e. $\det\left(\frac{\partial^2 \mathcal{L}}{\partial y^i \partial y^j}\right) \neq 0$ be assumed.

2. On the manifold $TM \times R^m$ fibered over M

A transformation $(x^i, y^i, t^\alpha) \to (\bar{x}^i, \bar{y}^i, \bar{t}^\alpha)$ of local coordinates on $TM \times R^m$ is of the form

$$
\begin{aligned}
\bar{x}^i &= \bar{x}^i(x^1, \dots, x^n), \quad \operatorname{rank}\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) = n \\
\bar{y}^i &= \frac{\partial \bar{x}^i}{\partial x^j} y^j, \qquad \bar{t}^\alpha = t^\alpha .
\end{aligned}
\tag{2.1}
$$

Here and in the sequel the indices $i, j, \dots,$ run from 1 to $n = \dim M$ and $\alpha, \beta, \gamma, \dots,$ run from 1 to m.

Sometimes we shall set $(y^i, t^\alpha) = z^a,\ a, b, c, \ldots = 1, \ldots, n+m$ and then (2.1) becomes

$$\begin{aligned} &\bar{x}^i = \bar{x}^i(x^1, \ldots, x^n), \text{ rank}\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) = n \\ &\bar{z}^a = M^a_b(x) z^b, \qquad \text{rank}(M^a_b) = n+m\,. \end{aligned} \tag{2.2.}$$

Computing the Jacobian matrix of the mapping (2.2) we obtain

Theorem 2.1. *The manifold $TM \times R^m$ is orientable.*

The triad $(TM \times R^m, \pi, M)$, $\pi(v, t^\alpha) = \tau(v)$ is a vector bundle of rank $n+m$ (the local fibre is $T_x M \times R^m$, at $x = \tau(v)$).

By a general result on vector bundles (R. Miron and M. Anastasiei [7]) we have

Theorem 2.2. *If M is paracompact then $TM \times R^m$ is paracompact.*

Thus if M is paracompact then $TM \times R^m$ admits smooth partitions of unity.

Let us set $E = TM \times R^m$ and let $\pi^T : TE \to TM$ be the tangent mapping of π. Then $VE = \ker \pi^T$ is a vector subbundle of the tangent bundle (TE, τ_E, E) to E. We call it the vertical bundle over E. The natural basis in $T_u E$, $u \in E$, is $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t^\alpha}\right)$ and $\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial t^\alpha}\right)$ gives a basis of $V_u E$.

These bases transform under (2.1) as follows:

$$\begin{aligned} &\frac{\partial}{\partial x^i} = \frac{\partial^2 \bar{x}^h}{\partial x^i \partial x^j} y^j \frac{\partial}{\partial \bar{y}^h} + \frac{\partial \bar{x}^h}{\partial x^i} \frac{\partial}{\partial \bar{x}^h} \\ &\frac{\partial}{\partial y^i} = \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial}{\partial \bar{y}^k}, \qquad \frac{\partial}{\partial t^\alpha} = \frac{\partial}{\partial \bar{t}^\alpha}\,. \end{aligned} \tag{2.3}$$

By (2.3) it comes out that $J : T_u E \to T_u E$ given by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad J\left(\frac{\partial}{\partial z^a}\right) = 0 \ \text{ is well-defined.} \tag{2.4}$$

By a direct calculation one gets

Theorem 2.3.

$$\begin{aligned} &\textit{(i)}\ J^2 = 0, \qquad \textit{(i)}\ \operatorname{Ker} J = VE,\ \operatorname{Im} J \subset VE, \\ &\textit{(iii)}\ N_J = 0, \qquad \textit{(iv)}\ L_C J = -J. \end{aligned}$$

Here N_J denotes the Nijenhuis tensor field associated to J and $L_C J$ denotes the Lie derivative of J with respect to the Liouville vector field

$C = z^a \frac{\partial}{\partial z^a}$:

$$(L_C J)(X) = [C, JX] - J[C, X], \quad X \in \mathfrak{X}(E)\,.$$

Thus, the manifold $TM \times R^m$ is endowed with an almost tangent structure (cf. (i)) which is integrable (cf. (iii)) and homogeneous of degree 0 (cf. (iv)).

3. Nonlinear connections on $E = TM \times R^m$

Now we shall regard the vertical bundle over E as a distribution $u \to V_u E,\ u \in E$, on E.

Definition 3.1. A nonlinear connection on E is a distribution $u \to H_u E$ which is supplementary to the vertical distribution on E, that is,

$$T_u E = H_u E \oplus V_u E,\ u \in E \tag{3.1}$$

holds good.

Locally, the distribution $u \to H_u E$, called horizontal distribution, is completely determined by n local vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial z^a} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} - N_i^\alpha \frac{\partial}{\partial t^\alpha}\,. \tag{3.2}$$

The form of these vector fields is a consequence of the fact that $\pi^T\,|_{HE}$ is an isomorphism which carries $\frac{\delta}{\delta x^i}$ to $\frac{\partial}{\partial x^i}$. Since $u \to H_u E$ is a global distribution, the adapted frame $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial z^a}\right)$ belonging to the decomposition (3.1) must transform under (2.1) as follows:

$$\frac{\delta}{\delta x^i} = \frac{\partial \bar{x}^h}{\partial x^i} \frac{\delta}{\delta \bar{x}^h}, \quad \frac{\partial}{\partial z^a} = M_a^b(x) \frac{\partial}{\partial \bar{z}^b}\,. \tag{3.3}$$

By (3.3), the local coefficients $(N_i^a(x, y, t))$ have the following transformation law under (2.1):

$$N_i^a \frac{\partial \bar{x}^i}{\partial x^j} = M_b^a N_j^b - \frac{\partial M_b^a}{\partial x^j} z^b\,. \tag{3.4}$$

Taking in (3.4) $a = h$ and then $a = \alpha$ we see that (3.4) is equivalent to:

$$\begin{aligned} \bar{N}_i^h \frac{\partial \bar{x}^i}{\partial x^j} &= \frac{\partial \bar{x}^h}{\partial x^k} N_j^k - \frac{\partial^2 \bar{x}^h}{\partial x^j \partial x^k} y^k\,, \\ \bar{N}_i^\alpha \frac{\partial \bar{x}^i}{\partial x^j} &= N_j^\alpha\,. \end{aligned} \tag{3.5}$$

Thus $(N_i^j(x,y,t))$ transform as the coefficients of a nonlinear connection on TM and (N_i^α) as the components of a covector on M. We shall say that (N_i^α) defines an M–covector on E for every α.

Conversely, a set of functions $(N_i^j(x,y,t),\ N_i^\alpha(x,y,t))$ which satisfy (3.5) defines a nonlinear connection on E.

The following theorem says us that a regular multiparametrized Lagrangian $\mathcal{L}$ determines a nonlinear connection on E.

Let us put:

$$g_{ij}(x,y,t^\alpha) = \frac{1}{2}\frac{\partial^2\mathcal{L}(x,y,t^\alpha)}{\partial y^i\partial y^j} \tag{3.6}$$

$$G^i(x,y,t^\alpha) = \frac{1}{4}g^{ik}\left(\frac{\partial^2\mathcal{L}}{\partial y^k\partial x^j}y^j - \frac{\partial\mathcal{L}}{\partial x^k}\right). \tag{3.7}$$

Theorem 3.1. *The set of functions*

$$N_j^i(x,y,t^\alpha) = \frac{\partial G^i(x,y,t^\alpha)}{\partial y^j}; \quad N_i^\alpha = \frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial t^\alpha\partial y^i}$$

defines a nonlinear connection on $TM\times R^m$.

Proof. One checks by a tedious computation that these functions transform under (2.1) as in (3.5).

Remark 3.1. The form of G^i in (3.7) was suggested by the form (1.2) of the Euler-Lagrange equations.

Remark 3.2. The nonlinear connection determined by $\mathcal{L}$ is symmetric in the sense that its torsion

$$t_{ij}^k = \frac{\partial N_i^k}{\partial y^j} - \frac{\partial N_j^k}{\partial y^i}$$

vanishes.

4. Other geometrical structures on $TM\times R^m$

Let $N_{\mathcal{L}}$ be the nonlinear connection on E defined by $\mathcal{L}$ and $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial z^a}\right)$ the corresponding adapted frame.

If we set:

$$P\left(\frac{\delta}{\delta x^i}\right) = \frac{\delta}{\delta x^i}, \quad P\left(\frac{\partial}{\partial z^a}\right) = -\frac{\partial}{\partial z^a}, \tag{4.1}$$

we obtain an almost product structure on E, that is, $P^2 = I$, where I denotes the Kronecker tensor field.

Now let $(dx^i, \delta z^a)$ be the frame dual to $\left(\frac{\delta}{\delta x^i}\,,\ \frac{\partial}{\partial z^a}\right)$. It follows that $\delta z^a = dz^a + N_i^a dx^i$, or equivalently,

$$\begin{aligned} \delta y^i &= dy^i + N_k^i dx^k \\ \delta t^\alpha &= dt^\alpha + N_i^\alpha dx^i\,. \end{aligned} \tag{4.2}$$

Let us define a linear mapping $F : T_u E \to T_u E$ by

$$F\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \quad F\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}, \quad F\left(\frac{\partial}{\partial t^\alpha}\right) = 0\,. \tag{4.3}$$

We immediately get

Theorem 4.1. *The following equalities hold good:*

(i) $\operatorname{rank} F = 2n$,
(ii) $F^3 + F = 0$,
(iii) $F^2 = -I + \frac{\partial}{\partial t^\alpha} \otimes \delta t^\alpha$

Thus $TM \times R^m$ is framed manifold. The frame structure $\left(F,\ \frac{\partial}{\partial t^\alpha},\ \delta t^\alpha\right)$ is said to be normal if the tensor field

$$S(X,Y) = N_F(X,Y) + d(\delta t^\alpha)(X,Y)\frac{\partial}{\partial t^\alpha}, \quad X, Y \in \mathfrak{X}(E) \tag{4.5}$$

vanishes identically.

A computation in local coordinates leads to

Theorem 4.2. *The frame structure* $\left(F, \frac{\partial}{\partial t^\alpha}, \delta t^\alpha\right)$ *is normal if and only if*

1) *The curvature of* $N_{\mathcal{L}}$ *vanishes i.e.*

$$\Omega_{ij}^q := \frac{\delta N_j^a}{\delta x^i} - \frac{\delta N_i^a}{\delta x^j} = 0\,,$$

2) $\frac{\partial N_i^a}{\partial t^\alpha} = 0\,.$

It is obvious that the following tensor field

$$G = g_{ij} dx^i \otimes dx^j + g_{ij} \delta y^i \otimes \delta y^j + \sum_{\alpha=1}^{m} \delta t^\alpha \otimes \delta t^\alpha \tag{4.6}$$

defines a metric structure on $TM \times R^m$.

It is Riemannian if the quadratic form $g_{ij}\xi^i\xi^j$, $(\xi^i) \in R^n$, is positive definite.

By (4.6) we have

Theorem 4.3. *The horizontal and vertical distributions are orthogonal with respect to G.*

Some computations in local coordinates give

Theorem 4.4. *The following equations hold good:*

$$\text{(i)} \qquad G(FX, FY) = G(X, Y) - \sum_{\alpha=1}^{m} \delta t^\alpha(X)\delta t^\alpha(Y)\,,$$

$$\text{(ii)} \qquad \delta t^\alpha(X) = G\left(\frac{\partial}{\partial t^\alpha}, X\right)\,,$$

$$\text{(iii)} \qquad G(PX, Y) = G(X, PY), \quad X, Y \in \mathfrak{X}(E)\,.$$

Thus, the manifold $TM \times R^m$ possesses an almost product structure, a frame structure and a metric structure related by (i)–(iii) in Theorem 4.4.

5. M–connections on $E = TM \times R^m$

We identify, as is usual, a linear connection on E with the operator of covariant derivative D associated to it.

Definition 5.1. A linear connection D on E is said to be an M–connection if the following conditions hold good:
(i) $DP = 0$,
(ii) $DF = 0$,
(iii) $D\left(\frac{\partial}{\partial t^\alpha}\right) = 0$, $\alpha = 1, 2, \ldots, m$.

Remark 5.1. An M–connection D on E satisfies also $D(\delta t^\alpha) = 0$ by virtue (iii) in Theorem 4.1.

We have

Theorem 5.1. *A linear connection D on E is an M–connection if and only if in the frame $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial t^\alpha}\right)$ we have*

$$\begin{aligned}
&D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} = L^i_{jk} \frac{\delta}{\delta x^i}\,, \quad D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} = L^i_{jk} \frac{\partial}{\partial y^i}\,,\\
&D_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} = C^i_{jk} \frac{\delta}{\delta x^i}\,, \quad D_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} = C^i_{jk} \frac{\partial}{\partial y^i}\,,\\
&D_{\frac{\delta}{\delta t^\alpha}} \frac{\delta}{\delta x^j} = C^i_{j\alpha} \frac{\delta}{\delta x^i}\,, \quad D_{\frac{\delta}{\delta t^\alpha}} \frac{\partial}{\partial y^j} = C^i_{j\alpha} \frac{\partial}{\partial y^i}
\end{aligned} \tag{5.1}$$

where, under (2.1), L^i_{jk} change like the coefficients of a linear connection on M, C^i_{jk} change like the components of a tensor field of type (1,2) on M *and* $C^i_{j\alpha}$ $(\alpha = 1, 2, \dots, m)$ *change like the components of a tensor field of type* (1,1) *on* M.

Thus we can give an M–connection as a set of local coefficients $D\Gamma = \left(L^i_{jk},\, C^i_{jk},\, C^i_{j\alpha}\right)$.

Using these coefficients, h– and v–covariant derivatives denoted by short and long horizontal bars, respectively, of any M–tensor can be considered. For instance,

$$\begin{aligned} t_{ij}|_k =& \frac{\delta t_{ij}}{\delta x^k} - L^h_{ik} t_{hj} - L^h_{jk} t_{ih}\,, \\ t_{ij|k} =& \frac{\partial t_{ij}}{\partial y^k} - C^h_{ik} t_{hj} - C^h_{jk} t_{ih}\,, \\ t_{ij|\alpha} =& \frac{\partial t_{ij}}{\partial t^\alpha} - C^h_{i\alpha} t_{hj} - C^h_{j\alpha} t_{ih}\,. \end{aligned}$$

An M–connection on E is said to be metrical if $DG = 0$. A direct computation gives

Theorem 5.2. *An M–connection is metrical if and only if*

$$g_{ij}|_k = 0, \quad g_{ij|k} = 0, \quad g_{ij|\alpha} = 0\,, \tag{5.2}$$

holds.

Let us set $T^i_{jk} = L^i_{jk} - L^i_{kj}$, $S^i_{jk} = C^i_{jk} - C^i_{kj}$. These tensor fields are the torsions of the M–connection ΓD.

On the existence of the metrical M–connection we have the following

Theorem 5.3. *There exists a set of metrical M–connections with $T^i_{jk} = S^i_{jk} = 0$. Their local coefficients are as follows:*

$$\begin{aligned} L^k_{ij} =& \frac{1}{2} g^{kh} \left(\frac{\delta g_{hj}}{\delta x^i} + \frac{\delta g_{ih}}{\delta x^j} - \frac{\delta g_{ij}}{\delta x^h} \right) \\ C^k_{ij} =& \frac{1}{2} g^{kh} \left(\frac{\partial g_{hj}}{\partial y^i} + \frac{\partial g_{ih}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^h} \right) \\ C^k_{i\alpha} =& \frac{1}{2} g^{kh} \frac{\partial g_{ih}}{\partial t^\alpha} + O^{jk}_{ih} X^h_{j\alpha}\,, \end{aligned} \tag{5.3}$$

where $X^h_{j\alpha}$ is an arbitrary M–tensor field of type (1,1) *and* $O^{jk}_{ih} = \frac{1}{2}\left(\delta^j_i \delta^k_h - g_{ih} g^{kj}\right)$ *is the Obata operator.*

Proof. The condition $g_{ij}|_k = 0$ is equivalent to $\frac{\delta g_{ij}}{\delta x^k} = L^h_{ik} g_{hj} + L^h_{jk} g_{ih}$. Subtracting this from the sum of the other two equations obtained by a cyclic permutation in it of the indices i, j, k and using $T^i_{jk} = 0$ one gets

L^i_{jk}. C^i_{jh} is derived in a similar way. Then it is easy to check that $\frac{1}{2}g^{kh}\frac{\partial g_{ih}}{\partial t^\alpha}$ verifies $g_{ij|\alpha} = 0$. If $C^k_{i\alpha}$ is another solution of the equation $g_{ij|\alpha} = 0$, then $B^k_{i\alpha} = C^k_{i\alpha} - \frac{1}{2}g^{kh}\frac{\partial g_{ih}}{\partial t^\alpha}$ satisfies the equation $g_{ki}B^k_j + g_{jk}B^k_i = 0$.

Using the Obata operator we find that the general solution of the last equation is $B^k_{i\alpha} = O^{jk}_{ih}X^h_{j\alpha}$. $\square$

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