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On weakly conformally symmetric spaces

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Abstract. The object of the present paper is to establish the existence of *weakly conformally symmetric spaces* by an example and to investigate the decomposability of such spaces.

1. Introduction

The notions of weakly symmetric and weakly projective symmetric spaces were introduced by Tamássy and Binh [5]. In a subsequent paper Binh [1] studied decomposable weakly symmetric spaces. A non-flat Riemannian space $V^n (n > 2)$ is called weakly symmetric if the curvature tensor R_{hijk} satisfies the condition

(1.1) $$R_{hijk,m} = A_m R_{hijk} + B_h R_{mijk} + D_i R_{hmjk} + E_j R_{himk} + F_k R_{hijm},$$

where A, B, D, E, F are 1-forms (non-zero simultaneously) and the comma ' , ' denotes covariant differentiation with respect to the metric tensor of the space. The 1-forms are called the associated 1-forms of the space and an n-dimensional space of this kind is denoted by $(WS)_n$.

On the analogy of $(WS)_n$ Tamássy and Binh [6] introduced the notion of weakly Ricci symmetric spaces $(WRS)_n$. A Riemannian space V^n is called weakly Ricci symmetric if there exist 1-forms A, B, D such that $R_{ij,k} = A_k R_{ij} + B_i R_{kj} + D_j R_{ik}$, where the Ricci tensor $R_{ij} \neq 0$.

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The present paper deals with non conformally-flat Riemannian spaces $V^n (n > 3)$ whose conformal curvature tensor C_{hijk} satisfies the condition:

$$C_{hijk,m} = A_m C_{hijk} + B_h C_{mijk} + D_i C_{hmjk} + E_j C_{himk} + F_k C_{hijm}, \tag{1.2}$$

where

$$C_{hijk} = R_{hijk} - \frac{1}{n-2}(g_{hk}R_{ij} - g_{hj}R_{ik} + g_{ij}R_{hk} - g_{ik}R_{hj}) + \frac{R}{(n-1)(n-2)}(g_{hk}g_{ij} - g_{hj}g_{ik}), \tag{1.3}$$

R is the scalar curvature and A, B, D, E, F are 1-forms (non-zero simultaneously). Such a space will be called a *weakly conformally symmetric space* and denoted by $(WCS)_n$. The dimension has been taken greater than 3 because the conformal curvature tensor vanishes for $n = 3$.

In our previous paper [3], by using the skew-symmetric property of the curvature tensor, the defining relation (1.1) has been reduced to the modified form

$$R_{hijk,m} = A_m R_{hijk} + B_h R_{mijk} + B_i R_{hmjk} + D_j R_{himk} + D_k R_{hijm}. \tag{1.4}$$

Since also the conformal curvature tensor satisfies the same skew-symmetric property as the curvature tensor, the defining relation (1.2) of a $(WCS)_n$ can be expressed in the following form:

$$C_{hijk,m} = A_m C_{hijk} + B_h C_{mijk} + B_i C_{hmjk} + D_j C_{himk} + D_k C_{hijm}. \tag{1.5}$$

In Section 2 an example of a $(WCS)_n$ is given. In Section 3 it is shown that a decomposable $(WCS)_n$ is of zero scalar curvature.

2. Example of a $(WCS)_n$

In this section we want to construct concrete $(WCS)_n$ spaces. On the coordinate space R^n (with coordinates $x^1, \dots, x^n$) we define a Riemannian space V^n. We calculate the components of the curvature tensor, the Ricci tensor, the conformal curvature tensor and its covariant derivative, and then we verify the defining relation (1.5).

Let each Latin index run over $1, 2, \ldots, n$ and each Greek index over $2, 3, \ldots, n-1$. We define a Riemannian metric on the R^n $(n \geq 4)$ by the formula

$$ds^2 = \phi (dx^1)^2 + K_{\alpha\beta} dx^\alpha dx^\beta + 2dx^1 dx^n, \tag{2.1}$$

where $[K_{\alpha\beta}]$ is a symmetric and non-singular matrix consisting of constants and ϕ is a function of $x^1, x^2, \ldots, x^{n-1}$ and independent of x^n. In the metric considered, the only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are (see [4])

$$\Gamma^\beta_{11} = -\frac{1}{2} K^{\alpha\beta} \phi_{\cdot\alpha}, \qquad \Gamma^n_{11} = \frac{1}{2}\phi_{\cdot 1}, \qquad \Gamma^n_{1\alpha} = \frac{1}{2}\phi_{\cdot\alpha},$$
$$R_{1\alpha\beta 1} = \frac{1}{2}\phi_{\cdot\alpha\beta}, \qquad R_{11} = \frac{1}{2}\phi_{\cdot\alpha\beta} K^{\alpha\beta},$$

and the components which can be obtained from these by the symmetry properties. Here '.' denotes partial differentation with respect to the coordinates and $K^{\alpha\beta}$ are the elements of the martix inverse to $[K_{\alpha\beta}]$. We consider $K_{\alpha\beta}$ as the Kronecker symbol $\delta_{\alpha\beta}$ and $\phi = (M_{\alpha\beta} + \delta_{\alpha\beta}) x^\alpha x^\beta e^{x^1}$, where $M_{\alpha\beta}$ are constants and satisfy the relations $M_{\alpha\beta} = 0$ for $\alpha \neq \beta$; $M_{\alpha\beta} \neq 0$ for $\alpha = \beta$, and $\sum_{\alpha=2}^{n-1} M_{\alpha\alpha} = 0$. Hence we obtain the following relations:

$$\delta_{\alpha\beta}\delta^{\alpha\beta} = n-2, \quad \delta^{\alpha\beta} M_{\alpha\beta} = \sum_{2}^{n-1} M_{\alpha\alpha} = 0 \quad \text{and} \quad \phi_{\cdot\alpha\beta} = 2(M_{\alpha\beta} + \delta_{\alpha\beta}) e^{x^1}.$$

Therefore

$$\delta^{\alpha\beta}\phi_{\cdot\alpha\beta} = 2(\delta^{\alpha\beta} M_{\alpha\beta} + \delta^{\alpha\beta}\delta_{\alpha\beta}) e^{x^1} = 2(n-2) e^{x^1}.$$

Since $\phi_{\cdot\alpha\beta}$ vanishes for $\alpha \neq \beta$, the only non-zero components for R_{hijk} and R_{ij} are

$$R_{1\alpha\alpha 1} = \frac{1}{2}\phi_{\cdot\alpha\alpha} = (1 + M_{\alpha\alpha}) e^{x^1} \text{ and } R_{11} = \frac{1}{2}\phi_{\cdot\alpha\beta}\delta^{\alpha\beta} = (n-2) e^{x^1}. \tag{2.2}$$

Hence $R = g^{ij}\, R_{ij} = g^{11} R_{11}$. Again from (2.1) we obtain $g_{ni} = g_{in} = 0$ for $i \neq 1$, which implies $g^{11} = 0$. So $R = 0$. Therefore R^n with the considered metric will be a Riemannian space V^n whose scalar curvature is zero. (1.3), (2.1) and (2.2) show that in this space the only non-zero components of C_{hijk} are

$$
\begin{aligned}
C_{1\alpha\alpha 1} &= R_{1\alpha\alpha 1} - \frac{1}{n-2}(g_{\alpha\alpha}R_{11}) \\
&= (1+M_{\alpha\alpha})e^{x^1} - \frac{1}{n-2}(n-2)e^{x^1} = M_{\alpha\alpha}e^{x^1},
\end{aligned} \tag{2.3}
$$

and the components which can be obtained from these by the symmetry properties. The only non-zero components of $C_{hijk,m}$ are

$$
C_{1\alpha\alpha 1,1} = M_{\alpha\alpha}e^{x^1} = C_{1\alpha\alpha 1} \neq 0, \tag{2.4}
$$

and the components which can be obtained from these by the symmetry properties. Hence our V^n is neither conformally flat nor conformally symmetric [2].

We want to show that this V^n is a $(WCS)_n$, that is, it satisfies (1.5). Let us consider the 1-forms

$$
\begin{aligned}
A_i(x) &= \frac{1}{3} \quad \text{for } i=1, \ = 0 \text{ otherwise} \\
B_i(x) &= \frac{2}{9} \quad \text{for } i=1, \ = 0 \text{ otherwise} \\
D_i(x) &= \frac{4}{9} \quad \text{for } i=1, \ = 0 \text{ otherwise.}
\end{aligned} \tag{2.5}
$$

In our V^n (1.5) reduces with these 1-forms to the equations

$$
\begin{aligned}
&\text{(i)} \quad C_{1\alpha\alpha 1,1} = A_1C_{1\alpha\alpha 1} + B_1C_{1\alpha\alpha 1} + B_\alpha C_{11\alpha 1} + D_\alpha C_{1\alpha 11} + D_1C_{1\alpha\alpha 1} \\
&\text{(ii)} \quad C_{11\alpha 1,\alpha} = A_\alpha C_{11\alpha 1} + B_1C_{\alpha 1\alpha 1} + B_1C_{1\alpha\alpha 1} + D_\alpha C_{11\alpha 1} + D_1C_{11\alpha\alpha} \\
&\text{(iii)} \quad C_{1\alpha 11,\alpha} = A_\alpha C_{1\alpha 11} + B_1C_{\alpha\alpha 11} + B_\alpha C_{1\alpha 11} + D_1C_{1\alpha\alpha 1} + D_1C_{1\alpha 1\alpha},
\end{aligned}
$$

since for the case other than (i), (ii) and (iii), the components of each term of (1.5) vanish identically and the relation (1.5) holds trivially. Now from (2.3), (2.4) and (2.5) we get the following relation for the right hand side (r.h.s.) and the left hand side (l.h.s.) of (i):

$$
\text{r.h.s. of (i)} = (A_1+B_1+D_1)C_{1\alpha\alpha 1} = 1.M_{\alpha\alpha}e^{x^1} = \text{l.h.s. of (i)}.
$$

Also r.h.s. of (ii) $= \frac{2}{9}(C_{\alpha 1\alpha 1}+C_{1\alpha\alpha 1}) = 0$ (by the skew-symmetric property of C_{hijk}) = l.h.s. of (ii). By a similar argument as in (ii), it can be shown that the relation (iii) is also true. Hence we obtain the following

Theorem 1. *Let $V^n (n \geq 4)$ be a Riemannian space with a metric of the form*

$$ds^2 = \phi(dx^1)^2 + \delta_{\alpha\beta}dx^\alpha dx^\beta + 2dx^1dx^n$$

$$\phi = (M_{\alpha\beta} + \delta_{\alpha\beta})x^\alpha x^\beta e^{x^1}$$

(where $M_{\alpha\beta}$ and $\phi(x^1, \dots, x^{n-1})$ are as described above). Then V^n is a weakly conformally symmetric space *with zero scalar curvature which is neither conformally flat nor conformally symmetric.*

3. Decomposable $(WCS)_n$

An n-dimensional Riemannian space V^n is said to be decomposable if in some coordinates its metric is given by

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{a'b'}dx^{a'}dx^{b'} \tag{3.1}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, \dots, x^r (r < n)$ denoted by $\bar{x}$, and $g^*_{a'b'}$ are functions of $x^{r+1}, \dots, x^n$ denoted by x^*; $a, b, c, \dots$ run from 1 to r and $a', b', c', \dots$ run from $r+1$ to n. The two parts of (3.1) are the metrics of a $V^r (r \geq 2)$ and a $V^{n-r} (n-r \geq 2)$ which are called the decomposition spaces of $V^n = V^r \times V^{n-r}$. Throughout this paper each object denoted by a bar is assumed to be formed from $\bar{g}_{ab}$ and of V^r, and each object denoted by a star is formed from $g^*_{a'b'}$ and of V^{n-r}.

From (3.1) we have

$$\begin{gathered} g_{ab} = \bar{g}_{ab}, \quad g_{a'b'} = g^*_{a'b'}, \quad g^{ab} = \bar{g}^{ab}, \\ g^{a'b'} = g^{*a'b'}, \quad g_{aa'} = 0 = g^{aa'}, \end{gathered} \tag{3.2}$$

and the only non-zero Christoffel symbols of the 2nd kind are as follows:

$$\Gamma^c_{ab} = \bar{\Gamma}^c_{ab}, \qquad \Gamma^{c'}_{a'b'} = \Gamma^{*c'}_{a'b'}.$$

A comma, a dot and a semicolon shall denote covariant differentiation in V^n, V^r and V^{n-r} respectively. Hence we have the following relations:

$$
\begin{aligned}
&R_{a'bcd} = 0 = R_{ab'cd'} = R_{ab'c'd'} \\
&R_{abcd,a'} = 0 = R_{ab'cd',k} = R_{ab'cd',k'} \\
&R_{abcd} = \bar{R}_{abcd}, \quad R_{a'b'c'd'} = R^*_{a'b'c'd'} \qquad (3.3)\\
&R_{ab} = \bar{R}_{ab}, \quad R_{a'b'} = R^*_{a'b'}, \quad R_{ab,c} = \bar{R}_{ab\cdot c}, \quad R_{a'b'\cdot c'} = R^*_{a'b';c'} \\
&R = g^{ij}R_{ij} = \bar{g}^{ab}\bar{R}_{ab} + g^{*a'b'}R^*_{a'b'} = \bar{R} + R^*.
\end{aligned}
$$

We suppose that $V^n = V^r \times V^{n-r}$ and $B \neq 0$.

From (1.5) we get

$$C_{a'bcd\cdot a} = A_a C_{a'bcd} + B_{a'} C_{abcd} + B_b C_{a'acd} + D_e R_{a'bad} + D_d C_{a'bca}. \qquad (3.4)$$

In view of (1.3), (3.2) and (3.3) it follows that

$$
\begin{aligned}
C_{a'bcd} = R_{a'bcd} &- \frac{1}{n-2}(g_{bc}R_{a'd} - g_{bd}R_{a'c} + g_{a'd}R_{bc} - g_{a'c}R_{bd}) \\
&+ \frac{R}{(n-1)(n-2)}(g_{a'd}g_{bc} - g_{a'c}g_{bd}) = 0
\end{aligned}
$$

and also $C_{a'bcd,a} = 0$. Hence (3.4) reduces to

$$B_{a'} C_{abcd} = 0. \qquad (3.5)$$

Similarly we get

$$B_a C_{a'b'c'd'} = 0. \qquad (3.6)$$

Since $B \neq 0$, all its components cannot vanish. Hence we consider the following two cases:

Case (i) Suppose $B_{a'} \neq 0$ for a fixed a'. Then from (3.5) $C_{abcd} = 0$ for all a, b, c, d. That is,

$$
\begin{aligned}
R_{abcd} &- \frac{1}{n-2}(g_{bc}R_{ad} - g_{bd}R_{ac} + g_{ad}R_{bc} - g_{ac}R_{bd}) \\
&+ \frac{R}{(n-1)(n-2)}(g_{ad}g_{bc} - g_{ac}g_{bd}) = 0.
\end{aligned}
$$

By (3.2) and (3.3) this equation takes the form

$$\bar{R}_{abcd} - \frac{1}{n-2}(\bar{g}_{bc}\bar{R}_{ad} - \bar{g}_{bd}\bar{R}_{ac} + \bar{g}_{ad}\bar{R}_{bc} - \bar{g}_{ac}\bar{R}_{bd}) + \frac{\bar{R}+R^*}{(n-1)(n-2)}(\bar{g}_{ad}\bar{g}_{bc} - \bar{g}_{ac}\bar{g}_{bd}) = 0.$$

Transvecting the above equation by $\bar{g}^{bc}$ we get

$$\bar{R}_{ab} - \frac{1}{n-2}(r\bar{R}_{ad} - \bar{R}_{ad} + \bar{R}\bar{g}_{ad} - \bar{R}_{ad}) + \frac{\bar{R}+R^*}{(n-1)(n-2)}(r\bar{g}_{ad} - \bar{g}_{ad}) = 0.$$

Transvecting again by $\bar{g}^{ad}$ we obtain

$$\bar{R} - \frac{2(r-1)}{n-2}\bar{R} + \frac{\bar{R}+R^*}{(n-1)(n-2)}r(r-1) = 0,$$

and hence

$$R^* = -k\bar{R}, \tag{3.7}$$

where $k = \frac{(n-r)(n-r-1)}{r(r-1)} \neq 0$ (since $n-r \geq 2$ and $r \geq 2$).
Therefore, by (3.3) we get

$$R = (1-k)\bar{R} = \left(1 - \frac{1}{k}\right)R^*. \tag{3.8}$$

Case (ii) Suppose $B_a \neq 0$ for a fixed a. Then from (3.6) it follows that $C_{a'b'c'd'} = 0$ for all a', b', c', d'. Proceeding similarly as in Case (i), we obtain

$$R^* = -\frac{1}{k}\bar{R} \tag{3.9}$$

$$R = (1-k)R^* = \left(1 - \frac{1}{k}\right)\bar{R}. \tag{3.10}$$

The relations (3.7) and (3.9) hold simultaneously only if $k = 1$. Hence from (3.8) or (3.10) we get $R = 0$. For $D \neq 0$ we obtain the same results. Thus we have the following

Theorem 2. *For $B \neq 0$ or $D \neq 0$, a decomposable $(WCS)_n$ is of zero scalar curvature.*

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