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Oscillatory properties of equations of mathematical physics with time-dependent coefficients

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Abstract. A condition ensuring that all solutions of certain partial differential equations with time-dependent coefficients are globally oscillatory is given.

The behaviour of various systems with distributed parameters (strings, beams, membranes, plates, etc.) is described by an equation of the type

$$\begin{aligned} u_{tt} + 2\alpha_0(t)u_t - 2\alpha_1(t)\Delta u_t + 2\alpha_2(t)\Delta^2 u_t + \beta_0(t)u \\ -\beta_1(t)\Delta u + \beta_2(t)\Delta^2 u = 0, \end{aligned} \tag{1}$$

where $u = u(t,x)$, $t \in J_0 = [t_0, \infty)$ for some $t_0 \in \mathbb{R}$, $x \in \Omega$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with sufficiently regular boundary $\partial\Omega$, Δ^2 is the second power of the Laplacian, and $\alpha_i \in W^{1,\infty}(J_0)$, $\beta_i \in L^\infty(J_0)$ for $i = 0, 1, 2$.

The equation is supposed to be complemented by the boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega \tag{2}$$

and, moreover, if $|\alpha_2(t)| + |\beta_2(t)| \not\equiv 0$, then

$$\Delta u = 0 \quad \text{on} \quad \partial\Omega. \tag{3}$$

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By a solution we mean any weak solution satisfying

$$\begin{aligned}\frac{d^2}{dt^2}(u(t),w) + 2\alpha_0(t)\frac{d}{dt}(u(t),w) + 2\alpha_1(t)\frac{d}{dt}(u(t),\Delta w)\\ + 2\alpha_2(t)\frac{d}{dt}(u(t),\Delta^2 w) + \beta_0(t)(u(t),w)\\ + \beta_1(t)(u(t),\Delta w) + \beta_2(t)(u(t),\Delta^2 w) = 0\end{aligned}$$

for any $w \in W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega)$ in case (2) and, moreover, $\Delta w \in W^{2,2}(\Omega) \cap \dot{W}^{1,2}(\Omega)$ in case (3), respectively in the sense of distributions on J_0. (Of course, $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$.)

The coefficients α_i and β_i physically related to various kinds of damping are assumed to satisfy certain conditions (the enumeration of which is not the purpose of this paper) to ensure both the existence of solutions and the uniqueness of the corresponding initial-boundary value problem or at least the following (unique continuation) property: if u is a solution on $J_0 \times \Omega$, $T \geq t_0$, $\epsilon > 0$, then

$$u = 0 \text{ on } (T, T+\epsilon) \times \Omega \implies u \equiv 0 \text{ on } J_0 \times \Omega. \tag{4}$$

Oscillatory properties of solutions of differential equations have been studied by many authors from various points of view (see e.g. [1]–[9]). In Theorem 1 we give a suitable condition ensuring that any solution of the differential equation (1) satisfying the boundary condition (2) (and (3), respectively) has, roughly speaking, a zero in any domain $J \times \Omega$ where $J \subset J_0$ is an interval the length of which is sufficiently large and this length can be chosen independently of J.

This property is more precisely expressed by the following definition (see [1], [2], [5], [9]): A measurable function $u\colon J_0 \times \Omega \to \mathbb{R}$ is said to be globally oscillatory (about zero) if there exists (the so-called oscillatory time) $\Theta > 0$ such that for any interval $J \subset J_0$ the length $|J|$ of which is greater than Θ we have either $u \equiv 0$ ($u = 0$ a.e.) on $J_0 \times \Omega$ or simultaneously

$$\text{meas}\,\{(t,x) \in J \times \Omega \mid u(t,x) > 0\} > 0$$

and

$$\text{meas}\,\{(t,x) \in J \times \Omega \mid u(t,x) < 0\} > 0.$$

In other words, $u\colon J_0 \times \Omega \to \mathbb{R}$ is globally oscillatory if and only if there exists $\Theta > 0$ such that

$$u \geq 0 \text{ (or } u \leq 0) \quad \text{on } J \times \Omega,\ J \subset J_0 \Longrightarrow \begin{cases} \text{either } u \equiv 0 \text{ on } J_0 \times \Omega, \\ \text{or } |J| \leq \Theta. \end{cases}$$

The proof of Theorem 1 relies on the well-known fact from the theory of elliptic equations that the eigenvalues of the operator $(-\Delta)$ with the homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$ form an infinite sequence $(\lambda_k)_{k=1}^{\infty}$ and can be ordered according to increasing magnitude so that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \to \infty \quad \text{as } k \to \infty.$$

The first eigenvalue λ_1 is positive and the corresponding eigenfunction e_1 can be chosen to be positive on Ω. Hence we have

$$(\Delta + \lambda_1)\, e_1 = 0, \quad \lambda_1 > 0, \quad e_1 = 0 \text{ on } \partial\Omega, \quad e_1 > 0 \text{ on } \Omega. \tag{5}$$

Moreover, λ_1 is a simple eigenvalue, i.e. e_1 spans the null space $\ker(\Delta+\lambda_1)$.

Theorem 1. *Let*

$$\operatorname{infess}_{t \in J_0} \left(\beta_0(t) + \lambda_1 \beta_1(t) + \lambda_1^2 \beta_2(t) - \frac{d\gamma}{dt}(t) - \gamma^2(t) \right) \geq \omega^2 > 0, \tag{6}$$

where

$$\gamma(t) = \alpha_0(t) + \lambda_1 \alpha_1(t) + \lambda_1^2 \alpha_2(t).$$

Then any solution of the problem given by (1) *and* (2) *(and* (3)*, respectively) is globally oscillatory and the oscillatory time is*

$$\Theta = \frac{\pi}{\omega}. \tag{7}$$

Proof. Let us make the projection of the equation on $\ker(\Delta + \lambda_1)$ and define

$$u_1(t) = \int_\Omega u(t,x) e_1(x)\, dx.$$

We obtain the ordinary differential equation (where $\cdot = d/dt$)

$$\ddot{u}_1(t) + 2\,\gamma(t)\,\dot{u}_1(t) + \big(\beta_0(t) + \lambda_1\,\beta_1(t) + \lambda_1^2\,\beta_2(t)\big)\ u_1(t) = 0, \quad t \in J_0.$$

Let us assume $u \geq 0$ (or $u \leq 0$) on $J \times \Omega$. Owing to (5), and the positivity of e_1 on Ω, we get $u_1 \geq 0$ (or $u_1 \leq 0$) on J. The results of [8] (Section 8) together with the assumption (6) give

$$|J| > \frac{\pi}{\omega} \Longrightarrow u_1 \equiv 0.$$

Using again (5), and the positivity of e_1, we obtain $u \equiv 0$ on $J \times \Omega$. Finally, the uniqueness property (4) yields

$$u \equiv 0 \quad \text{on } J_0 \times \Omega,$$

and this completes the proof. □

Special cases

1. Dissipative wave equations

Let $\alpha_0 = a$, $\alpha_1 = b$, $\beta_0 = c$, $\beta_1 = 1$, $\alpha_2 = \beta_2 = 0$. Then equation (1) assumes the form

$$u_{tt} - \Delta u + 2\,a(t)\,u_t - 2\,b(t)\,\Delta u_t + c(t)\,u = 0. \tag{1'}$$

The condition (6) reads now

$$\operatorname{supess}_{t \in J_0} \left(\frac{d}{dt}(a(t) + \lambda_1\, b(t)) + (a(t) + \lambda_1\, b(t))^2 - c(t) \right) < \lambda_1. \tag{6'}$$

In particular, if $a(t) = a =$ nonnegative const., $b(t) = b =$ nonnegative const., $c(t) \equiv 0$, then

$$a + \lambda_1 b < \sqrt{\lambda_1}$$

is a condition ensuring that any solution of the equation

$$u_{tt} - \Delta u + 2\,a\,u_t - 2\,b\,\Delta u_t = 0$$

with the homogeneous Dirichlet boundary condition (2) is globally oscillatory. We have

$$\omega = \sqrt{\lambda_1 - (a + \lambda_1 b)^2}.$$

For $b = 0$ we get

$$a < \sqrt{\lambda_1},$$

the well-known criterion for the telegraph equation

$$u_{tt} - \Delta\, u + 2au_t = 0$$

to have all solutions satisfying (2) globally oscillatory.

2. Dissipative beam equations

Let $n = 1$, $\Omega = (0, \ell)$, $\alpha_0 = a$, $\alpha_1 = d$, $\alpha_2 = b$, $\beta_0 = c$, $\beta_1 = p$, $\beta_2 = 1$, $\operatorname{infess}_{t\in J_0} p(t) \geq p_0 \geq 0$. The corresponding equation is

$$u_{tt} + u_{xxxx} - p(t)\, u_{xx} + 2\, a(t)\, u_t + 2\, b(t)\, u_{txxxx} - 2\, d(t)\, u_{txx} + c(t)\, u = 0,$$

and the boundary conditions are

$$u = u_{xx} = 0 \quad \text{for } x = 0, \ell.$$

Condition (6) may be written in the form

$$\text{(6”)} \qquad \operatorname{supess}_{t\in J_0} \left(\frac{d\gamma}{dt}(t) + \gamma^2(t) - c(t) \right) < \lambda_1^2 + \lambda_1 p_0$$

where

$$\gamma(t) = a(t) + \lambda_1\, d(t) + \lambda_1^2\, b(t).$$

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