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On a problem of Erdős–Turán

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Abstract. We find a class of real functions f having the property that the inequality $f(p_{n+1}) - 2f(p_n) + f(p_{n-1}) > 0$ holds for infinitely many positive integers n, and $f(p_{n-1}) - 2f(p_n) + f(p_{n-1}) < 0$ holds for infinitely many n.

One interesting question about the properties of the sequence $(p_n)_{n\geq 1}$ of prime numbers is:

Does there exist a positive integer n_0 such that this sequence is convex or concave for all $n > n_0$?

The answer is negative and the proof of this result is given in the paper [1] by P. Erdős and P. Turán. They established that:

For infinitely many n one has

$$p_{n+1} - 2p_n + p_{n-1} > 0,$$

and for infinitely many n:

$$p_{n+1} - 2p_n + p_{n-1} < 0.$$

C. Pomerance proved in [2], that there are infinitely many n for which:

$$2p_n < p_{n-i} + p_{n+i} \quad \text{for all } i,\ 1 \leq i \leq n-1.$$

In [3] Erdős proved that if

$$k \geq 3,\ a_1 + a_2 + \dots + a_k = 0,\ a_k \neq 0 \quad \text{and}$$
$$(k-1)a_1 + (k-2)a_2 + \dots + a_{k-1} = 0,$$

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then the sequence $x_n = a_1 p_n + a_2 p_{n-1} + \cdots + a_k p_{n+k-1}$ does not keep a constant sign.

Moreover, for fixed $k \neq 0$, $p_{n+1}^k - 2p_n^k + p_{n-1}^k > 0$ for infinitely many n and, also, $p_{n+1}^k - 2p_n^k + p_{n-1}^k < 0$ for infinitely many n.

This means that for $f(x) = x^k$, $k \neq 0$ the sequence $(f(p_n))_{n\geq 1}$ is neither convex nor concave.

We shall say that a function $f : [a_f, \infty) \to \mathbb{R}$ has property (P) if: For infinitely many n one has

$$f(p_{n+1}) - 2f(p_n) + f(p_{n-1}) > 0,$$

and for infinitely many n:

$$f(p_{n+1}) - 2f(p_n) + f(p_{n-1}) < 0.$$

To find necessary and sufficient conditions for f to have property (P) can be a difficult task.

Let $f(x) = a^x$. In case $a = 2$ one can prove immediately that the sequence is convex, so f does not have the (P) property. In case $a = 1.2$, to prove that f has the (P) property is the same as to prove that there exists an infinity of primes p for which $p+2$ too is prime. Consequently, this would amount to solving one of the greatest open problems in number theory. This example shows that to find a complete answer to our question is a very difficult task indeed. We shall restrict ourselves to finding a sufficient condition and for this purpose we shall consider a class of functions useful in this direction.

For every $f : [a_f, \infty) \to (0, \infty)$, $f \in C^1$ define $\phi : [a_f, \infty) \to \mathbb{R}$ by

$$\phi(x) = \frac{xf'(x)}{f(x)}.$$

Let $F = \{f : [a_f, \infty) \to (0, \infty),\ f \in C^1,\ \lim_{x\to\infty} \phi(x) = k \in \mathbb{R} \setminus \{0\}\}$.

One can prove that in the neighborhood of ∞ the functions of F behave somehow similarly to ax^n, because for $n > k$ one can prove that $\lim_{x\to\infty} \frac{f(x)}{x^n} = 0$, and for $n < k$, $\lim_{x\to\infty} \frac{|f(x)|}{x^n} = \infty$.

One can also notice that for real α, β, $\alpha \neq 0$ one has $f \in F$ if and only if $f^\alpha(x) \log^\beta x \in F$.

The main result of our paper is

Theorem 1. *If $f \in F$, then f has the* (P) *property. Before proving the theorem we shall need some preliminaries. One knows that*

$$p_n \sim n \log n; \tag{1}$$

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} < 0.46665 \quad \textit{(see [4])}; \tag{2}$$

$$\limsup_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = \infty \quad \textit{(see [5])}. \tag{3}$$

On the basis of the relations (2) or (3) and of Lemma 3, we now prove our theorem.

Lemma 1. *Let $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ be sequences of positive integers, $\lim_{n\to\infty} b_n = \infty$. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ then $\lim_{n\to\infty} \frac{p_{a_n}}{p_{b_n}} = c$.*

Proof. Let $y_n = \frac{a_n}{b_n}$ so $\lim_{n\to\infty} y_n = c$. Taking into account (1) it follows that

$$\lim_{n\to\infty} \frac{p_{a_n}}{p_{b_n}} = \lim_{n\to\infty} \frac{a_n}{b_n} \cdot \frac{\log a_n}{\log b_n} = c \lim_{n\to\infty} \left(1 + \frac{\log y_n}{\log b_n}\right) = c.$$

Put $F(n) = f(p_n)$. In case $f \in F$ and $\lim\limits_{x\to\infty} \phi(x) = k$, we have

Lemma 2. *Under the conditions of Lemma 1, one has*

$$\lim_{n\to\infty} \frac{F(a_n)}{F(b_n)} = c.$$

Proof. One has

$$f(x) = f(x_0) \exp \int_{x_0}^{x} \frac{\phi(t)}{t} dt = f(x_0) \exp \int_{\log x_0}^{\log x} \phi(e^u) du.$$

$$\frac{F(a_n)}{F(b_n)} = \exp \int_{\log b_n}^{\log a_n} \phi(e^u) du = \exp((\log p_{a_n} - \log p_{b_n})\phi(\theta_n)),$$

where $\min(\log p_{a_n}, \log p_{b_n}) < \theta_n < \max(\log p_{a_n}, \log p_{b_n})$ hence

$\lim_{n\to\infty} \phi(\theta_n) = k$. It follows that $\lim_{n\to\infty} \frac{F(a_n)}{F(b_n)} = \exp k \log \left(\frac{p_{a_n}}{p_{b_n}}\right) = c^k$.

Under the conditions of the previous lemmas and with the same notations one has

Lemma 3. *If the sequence* $(F(n))_{n\geq 1}$ *is convex, then*

$$F(n+1)-F(n)\sim k\frac{F(n)}{n}.$$

Proof. If the sequence $(F(n))_{n\geq 1}$ is convex it follows that for $m>n>p$:

$$\frac{F(m)-F(n)}{m-n}\geq F(n+1)-F(n)\geq \frac{F(n)-F(p)}{n-p}. \tag{4}$$

For fixed $1>\delta>0$, put $m=[(1+\delta)n]$ and $p=[(1-\delta)n]$. It follows that $m-n\sim \delta n$, $n-p\sim \delta n$.

Using Lemma 2 one obtains

$$F(m)\sim F(n)(1+\delta)^k \quad \text{and} \quad F(p)\sim F(n)(1-\delta)^k,$$

hence

$$\frac{F(m)-F(n)}{m-n}\sim \frac{F(n)((1+\delta)^k-1)}{n\delta}.$$

In the same way $\frac{F(n)-F(p)}{n-p}\sim \frac{F(n)(1-(1-\delta)^k)}{n\delta}$. Taking into account that $\lim_{\delta\to 0}\frac{(1+\delta)-1}{\delta}=\lim_{\delta\to 0}\frac{1-(1-\delta)^k}{\delta}=k$ and (4), the proof is finished.

Proof of Theorem 1. Let $(F(n))_{n\geq 1}$ be convex. One has $F(n+1)-F(n)=f(p_{n+1})-f(p_n)=(p_{n+1}-p_n)f'(\theta_n)$, $p_n<\theta_n<p_{n+1}$, hence

$$\begin{aligned}F(n+1)-F(n)&=(p_{n+1}-p_n)\cdot\frac{f(\theta_n)}{\theta_n}\phi(\theta_n)\sim k\frac{(p_{n+1}-p_n)f(p_n)}{p_n}\\&=k\frac{p_{n+1}-p_n}{\log p_n}\cdot\frac{F(n)\log p_n}{p_n}\sim k\cdot\frac{(p_{n+1}-p_n)}{\log p_n}\cdot\frac{F(n)}{n}.\end{aligned}$$

Using Lemma 3, it follows that

$$\frac{p_{n+1}-p_n}{\log p_n}\sim 1. \tag{5}$$

The same conclusion (5) is implied by the hypothesis of concavity of the sequence $(F(n))_{n\geq 1}$. In both cases we obtain (5), which manifestly contradicts (2) and (3).

Remark. The function $f(x) = x^k$ with $k \neq 0$, $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomial functions of different degrees, as well as $f(x) = \frac{x}{\log x}$ are examples of functions for which the sequence $(f(p_n))_{n \geq 1}$ is neither convex nor concave.

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