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On the functional equation of the bisectrix transform

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Abstract. We solve a functional equation characterizing certain maps on real normed spaces transforming bisectrices.

Let $(E_1, \| \, \|_1)$ and $(E_2, \| \, \|_2)$ be real normed spaces such that $\dim E_1 \geq 2$ and $\dim E_2 \geq 2$. For any couple of non-zero vectors x and y in E_1 we consider *the bisectrix segment* $\left\{\lambda\left(\frac{\|x\|_1}{\|x\|_1+\|y\|_1}y + \frac{\|y\|_1}{\|x\|_1+\|y\|_1}x\right), \, 0 \leq \lambda \leq 1\right\}$. Our aim in this paper is to study mappings $f : E_1 \to E_2$ which transform some points in the bisectrix segment generated by any couple x, y in E_1 into the corresponding points of the bisectrix segment in E_2 generated by $f(x)$ and $f(y)$, i.e., we will deal with the equations:

$$
\begin{aligned}
(1)\quad & f\left(\lambda\left(\frac{\|x\|_1}{\|x\|_1+\|y\|_1}y + \frac{\|y\|_1}{\|x\|_1+\|y\|_1}x\right)\right) \\
&\qquad = g(\lambda)\left[\frac{\|f(x)\|_2}{\|f(x)\|_2+\|f(y)\|_2}f(y) + \frac{\|f(y)\|_2}{\|f(x)\|_2+\|f(y)\|_2}f(x)\right],
\end{aligned}
$$

for some $g : \{0, \lambda_0, 1\} \to \{0, k_0, 1\}$. This study of (1) will be made in normed spaces and at the end we will see that in the case of inner product spaces (1) characterizes linear similitudes. We first prove how to derive from (1) the positive linear homogeneity of f.

Lemma. *Given λ_0, $k_0 \in (0,1)$, let g be such that $g(0) = 0$, $g(\lambda_0) = k_0$ and $g(1) = 1$, and let $f : E_1 \to E_2$ be a mapping, continuous at 0, such*

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that $f(x) = 0$ if and only if $x = 0$. If (1) is satisfied then necessarily $f(\alpha x) = \alpha f(x)$, for all x in E_1, and all positive reals $\alpha > 0$, and $\lambda_0 = k_0$.

Proof. For $\lambda = 1$ equation (1) gives the following equality:

$$(2)\qquad f\left(\frac{\|x\|_1}{\|x\|_1+\|y\|_1}y+\frac{\|y\|_1}{\|x\|_1+\|y\|_1}x\right) = \frac{\|f(x)\|_2}{\|f(x)\|_2+\|f(y)\|_2}f(y)+\frac{\|f(y)\|_2}{\|f(x)\|_2+\|f(y)\|_2}f(x).$$

The substitution $y = \alpha x$ into (2), with $x \neq 0$ and α real, yields:

$$(3)\qquad f\left(\frac{|\alpha|+\alpha}{|\alpha|+1}x\right) = \frac{\|f(\alpha x)\|_2 f(x)+\|f(x)\|_2 f(\alpha x)}{\|f(x)\|_2+\|f(\alpha x)\|_2}.$$

Since for any $\alpha < 0$, $|\alpha| + \alpha = 0$ and $f(0) = 0$, by (3) we have for $\alpha < 0$, $x \neq 0$,

$$(4)\qquad f(\alpha x) = -\frac{\|f(\alpha x)\|_2}{\|f(x)\|_2}f(x).$$

Thus for any $\alpha > 0$, writing $f(\alpha x) = f((-\alpha)(-x))$ and by repeated use of (4) we immediately conclude

$$(5)\qquad f(\alpha x) = \frac{\|f(\alpha x)\|_2}{\|f(x)\|_2}f(x).$$

Fixing $u \neq 0$, for any reals $\alpha, \beta > 0$ the substitutions $x = \alpha u$ and $y = \beta u$ into (2) give, in view of (5):

$$f\left(\frac{2\alpha\beta}{\alpha+\beta}u\right) = \frac{2\|f(\alpha u)\|_2\|f(\beta u)\|_2}{\|f(\alpha u)\|_2+\|f(\beta u)\|_2}\cdot\frac{1}{\|f(u)\|_2}f(u),$$

and taking norms

$$(6)\qquad \left\|f\left(\frac{2\alpha\beta}{\alpha+\beta}u\right)\right\|_2 = \frac{2\|f(\alpha u)\|_2\|f(\beta u)\|_2}{\|f(\alpha u)\|_2+\|f(\beta u)\|_2}.$$

Since u is fixed, let us introduce the function $h : (0,\infty) \to (0,\infty)$, $h(t) = \|f(tu)\|_2$. Then by (6) h is a morphism with respect to the harmonic mean, i.e.,

$$h\left(\frac{2\alpha\beta}{\alpha+\beta}\right) = \frac{2h(\alpha)h(\beta)}{h(\alpha)+h(\beta)}.$$

Then (see (ACZÉL, 1966)) the function $J(t) = 1/h(1/t)$ satisfies the classical Jensen equation and it is bounded below by 0, so and there must be two positive constants a, b such that $J(t) = at+b$, i.e., $h(t) = t/(a+bt)$. Now we let u vary again, so a and b will be functions of u and $\|f(tu)\|_2 = t/(a(u)+b(u)t)$. Since for $t = 1$ we have $a(u)+b(u) = 1/\|f(u)\|_2$ we obtain

$$\|f(tu)\|_2 = \frac{t\|f(u)\|_2}{1 + b(u)(t-1)\|f(u)\|_2}, \tag{7}$$

for all $u \neq 0$ and $t > 0$. Bearing in mind (5) we also have

$$f(tu) = \frac{t}{1 + b(u)(t-1)\|f(u)\|_2} f(u). \tag{8}$$

By (7) we have for all $u \neq 0$ and $t > 0$, $t \neq 1$,

$$b(u) = \frac{1}{t-1}\left[\frac{t}{\|f(tu)\|_2} - \frac{1}{\|f(u)\|_2}\right],$$

using this expression for computing $b(tu)$ and with the help of (7) it is easy to verify that b must be homogeneous of degree zero:

$$b(tu) = b(u), \tag{9}$$

for all $t > 0$ and $u \neq 0$. If in (1) we substitute $x = y = u$ we obtain for $\lambda = \lambda_0$

$$f(\lambda_0 u) = k_0 f(u), \tag{10}$$

and combining (8) and (10):

$$k_0 = \frac{\lambda_0}{1 + b(u)(\lambda_0 - 1)\|f(u)\|_2}, \tag{11}$$

and therefore by any $u, v \neq 0$ we must have

$$b(u)\|f(u)\|_2 = b(v)\|f(v)\|_2,$$

i.e., $b(u)\|f(u)\|_2 = C$ for some constant C. We claim that C must be zero. In fact, in view of (9) we have for all $t > 0$

$$b(u)\|f(tu)\|_2 = b(tu)\|f(tu)\|_2 = C,$$

and using (7)

$$\frac{Ct}{1+C(t-1)} = \frac{tb(u)\|f(u)\|_2}{1+b(u)(t-1)\|f(u)\|_2} = C,$$

whence either $C=0$ or $C=1$. The possibility $C=1$, i.e., $b(u)\|f(tu)\|_2=1$ is inconsistent with the continuity of f at 0. Thus $C=0$, and by (11) $k_0=\lambda_0$ and by (8), $f(tu)=tf(u)$ for all $t>0$ and any u in E_1. $\square$

Now we can solve (1).

Theorem. *Let $g:\{0,\lambda_0,1\}\to\{0,k_0,1\}$, with $\lambda_0, k_0\in(0,1)$ be such that $g(0)=0$, $g(\lambda_0)=k_0$ and $g(1)=1$ and let $f:E_1\to E_2$ be a mapping continuous at 0 such that $f(x)=0$ if and only if $x=0$. Then f satisfies (1) if and only if $\lambda_0=k_0$ and f satisfies the following conditions:*

(i) *$f(tu)=tf(u)$, for all $t>0$ and any u in E_1;*

(ii) *$f(u+v)=f(u)+f(v)$ whenever $\|u\|_1=\|v\|_1$;*

(iii) *$\|f(u)\|_2=C\|u\|_1$, for some constant $C>0$ and for any u.*

Proof. Sufficiency is a straightforward verification. So let us assume that f satisfies (1). By the previous Lemma we already know that necessarily $\lambda_0=k_0$ and we have that (i) holds. Thus by (1) we have for all u, v such that $\|u\|_1=\|v\|_1$

$$f(u+v)=2f\left(\frac{u+v}{2}\right)=\frac{2\|f(v)\|_2 f(u)+2\|f(u)\|_2 f(v)}{\|f(u)\|_2+\|f(v)\|_2}. \tag{12}$$

On the other hand by (1) and (i) we also have

$$\begin{aligned} f(\|y\|_1 x+\|x\|_1 y) = {} & \frac{\|x\|_1+\|y\|_1}{\|f(x)\|_2+\|f(y)\|_2} \\ & \times\left(\|f(y)\|_2 f(x)+\|f(x)\|_2 f(y)\right), \end{aligned} \tag{13}$$

Thus by (12), (13) and (i), we have the following chain of equalities

$$\begin{aligned} &\frac{\|x\|_1+\|y\|_1}{\|x\|_1\|y\|_1(\|f(x)\|_2+\|f(y)\|_2)}(\|f(y)\|_2 f(x)+\|f(x)\|_2 f(y)) \\ &\quad = \frac{1}{\|x\|_1\|y\|_1} f(\|y\|_1 x+\|x\|_1 y) = f\left(\frac{x}{\|x\|_1}+\frac{y}{\|y\|_1}\right) \end{aligned}$$

$$= \frac{2\left\|f\left(\frac{x}{\|x\|_1}\right)\right\|_2 f\left(\frac{y}{\|y\|_1}\right) + 2\left\|f\left(\frac{y}{\|y\|_1}\right)\right\|_2 f\left(\frac{x}{\|x\|_1}\right)}{\left\|f\left(\frac{x}{\|x\|_1}\right)\right\|_2 + \left\|f\left(\frac{y}{\|y\|_1}\right)\right\|_2}$$

$$= \frac{2\|f(x)\|_2 f(y) + 2\|f(y)\|_2 f(x)}{\|y\|_1\|f(x)\|_2 + \|x\|_1\|f(y)\|_2},$$

whence

$$\left[\frac{\|x\|_1 + \|y\|_1}{\|x\|_1\|y\|_1(\|f(x)\|_2 + \|f(y)\|_2)} - \frac{2}{\|y\|_1\|f(x)\|_2 + \|x\|_1\|f(y)\|_2}\right] \times \{\|f(y)\|_2 f(x) + \|f(x)\|_2 f(y)\} = 0$$

i.e. by (13)

$$\left[\frac{\|x\|_1 + \|y\|_1}{\|x\|_1\|y\|_1(\|f(x)\|_2+\|f(y)\|_2)} - \frac{2}{\|y\|_1\|f(x)\|_2+\|x\|_1\|f(y)\|_2}\right] \times f\left(\|y\|_1 x + \|x\|_1 y\right) = 0. \tag{14}$$

Thus for u and v independent, the substitution $x = u$ and $y = \frac{2\|u\|_1}{\|v\|_1} v$ into (14) yields after straightforward manipulations that

$$\frac{\|f(u)\|_2}{\|u\|_1} = \frac{\|f(v)\|_2}{\|v\|_1}.$$

From this it is immediate to see that

$$\|f(u)\|_2 = C\|u\|_1, \tag{15}$$

for all u (including the obvious case $u = 0$). Next, using (15), (1) and (i), we obtain whenever $\|u\|_1 = \|v\|_1$

$$f(u+v) = \frac{2\|f(v)\|_2 f(u) + 2\|f(u)\|_2 f(v)}{\|f(u)\|_2 + \|f(v)\|_2}$$
$$= \frac{2C\|v\|_1 f(u) + 2C\|u\|_1 f(v)}{C\|u\|_1 + C\|v\|_1} = f(u) + f(v).$$

The Theorem is proved. □

In the case that both E_1 and E_2 are inner product spaces, i.e., $\| \ \|_1$ and $\| \ \|_2$ come from inner products, we have that the above conditions (i),

(ii) and (iii) of f are equivalent to the usual full linearity of f and the fact that f is a similitude.

Corollary. *If we add to the assumptions of the above theorem the requirement that E_1 and E_2 are inner product spaces, then the solution of* (1) *is $\lambda_0 = k_0$ and f is a linear similitude.*

Proof. We just need to pay attention to the fact that the inner product structure and the condition (iii) makes possible to deduce the full linearity of f by means of (i) and (ii). To this end, for $\|x\|_1 = \|y\|_1$ we have

$$\begin{aligned}C^2\|x\|_1^2 + C^2\|y\|_1^2 + 2C^2\langle x, y\rangle_1 &= C^2\|x+y\|_1^2 = \|f(x+y)\|_2^2\\ &= \|f(x)+f(y)\|_2^2 = \|f(x)\|_2^2 + \|f(y)\|_2^2 + 2\langle f(x), f(y)\rangle_2\\ &= C^2\|x\|_1^2 + C^2\|y\|_1^2 + 2\langle f(x), f(y)\rangle_2,\end{aligned}$$

therefore $\langle f(x), f(y)\rangle_2 = C^2\langle x, y\rangle_1$. Thus even if $\|x\|_1 \neq \|y\|_1$ we will have by the positive homogeneity of f:

$$\begin{aligned}\langle f(x), f(y)\rangle_2 &= \left\langle \|x\| f\left(\frac{x}{\|x\|}\right),\ \|y\| f\left(\frac{y}{\|y\|}\right)\right\rangle_2\\ &= \|x\|\,\|y\| C^2 \left\langle \frac{x}{\|x\|}, \frac{y}{\|y\|}\right\rangle_1 = C^2\langle x, y\rangle_1 \qquad \square\end{aligned}$$

Thus $\frac{1}{C}f$ is an orthogonal transformation from E_1 into E_2, i.e., f is a linear similitude.

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