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Yau's problem on Einstein field equation

By SHARIEF DESHMUKH (Riyadh)

Abstract. In this short note it is shown that on an n-dimensional compact connected positively curved Riemannian manifold (M,g) without boundary, a symmetric tensor field $T(X,Y)=g(A(X),Y)$ satisfies the Einstein field equation $R_{ij}-\frac{S}{2}g_{ij}=T_{ij}$ if and only if the following conditions are satisfied:

(i) $tr.A=-\frac{(n-2)}{2}S$,

(ii) $(\nabla A)(X,Y)-(\nabla A)(Y,X)=\frac{1}{2}R_0(X,Y)\,\mathrm{grad}\,S-F(X,Y)$, where R_{ij} is the Ricci tensor, S the scalar curvature, F the divergence of the curvature tensor field and $R_0(X,Y)Z=g(Y,Z)X-g(X,Z)Y$.

1. Introduction

In a problem suggested by Yau, it is required to find necessary and sufficient conditions on a symmetric tensor T_{ij} on a compact manifold so that one can find a metric g_{ij} to satisfy the Einstein field equation

$$R_{ij}-\frac{S}{2}g_{ij}=T_{ij},$$

where R_{ij} is the Ricci tensor and S is the scalar curvature (cf. [3], Problem 20, p. 675). Let (M,g) be an n-dimensional compact Riemannian manifold with covariant derivative operator ∇ with respect to the Riemannian connection. Then the divergence of the curvature tensor field R is a tensor field F of type $(1,2)$ defined by

$$F(X,Y)=\sum_i(\nabla_{e_i}R)(X,Y)e_i,\quad X,Y\in\mathfrak{X}(M),$$

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where $\{e_1, \dots, e_n\}$ is a local orthonormal frame and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. For a symmetric tensor field T of type $(0, 2)$, there is an associated tensor field A of type $(1, 1)$ given by $T(X, Y) = g(AX, Y)$, $X, Y \in \mathfrak{X}(M)$, and its covariant derivative is given by

$$(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y), \quad X, Y \in \mathfrak{X}(M).$$

We also consider a tensor field R_0 of type $(1, 3)$ defined by

$$R_0(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad X, Y, Z \in \mathfrak{X}(M).$$

In this paper we prove the following

Theorem. *Let (M, g) be an n-dimensional compact and connected positively curved Riemannian manifold without boundary. Then a symmetric tensor field $T(X, Y) = g(AX, Y)$ on M satisfies the Einstein field equation*

$$R_{ij} - \frac{S}{2} g_{ij} = T_{ij}$$

if and only if the following conditions are satisfied:

(i) $tr.A = -\frac{(n-2)}{2} S$

(ii) $(\nabla A)(X, Y) - (\nabla A)(Y, X) = \frac{1}{2} R_0(X, Y) \operatorname{grad} S - F(X, Y)$, *where R_{ij} is the Ricci tensor, S the scalar curvature, F the divergence of the curvature tensor field and $\{e_1, \dots, e_n\}$ is a local orthonormal frame on M.*

This theorem can be considered as a result in the direction of the above mentioned problem of Yau [3].

2. Preliminaries

Let (M, g) be an n-dimensional Riemannian manifold and Ric be the Ricci tensor field of M. The Ricci operator $Q : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$\operatorname{Ric}(X, Y) = g(Q(X), Y), \quad X, Y \in \mathfrak{X}(M),$$

and the scalar curvature S of M is given by

$$S = \sum_i \operatorname{Ric}(e_i, e_i),$$

where $\{e_i, \dots, e_n\}$ is a local orthonormal frame on M.

We have the following well known (cf. [2])

Lemma 2.1. $\frac{1}{2}\operatorname{grad} S = \sum_i (\nabla Q)(e_i, e_i)$.

The divergence of the curvature tensor field R is a tensor field F given by

$$F(X,Y) = \sum_i (\nabla_{e_i} R)(X,Y)e_i, \quad X, Y \in \mathfrak{X}(M) \tag{2.1}$$

Using (2.1), the second Bianchi identity, and the following expression for the Ricci operator Q,

$$Q(X) = \sum_i R(X, e_i)e_i, \quad X \in \mathfrak{X}(M),$$

the following lemma can be easily proved.

Lemma 2.2. $(\nabla Q)(X,Y) - (\nabla Q)(Y,X) = -F(X,Y)$, $X, Y \in \mathfrak{X}(M)$.

For a symmetric tensor field T of type $(0,2)$ on M, we define a tensor field A of type $(1,1)$ by

$$T(X,Y) = g(AX,Y), \quad X, Y \in \mathfrak{X}(M),$$

which is also symmetric. We also define a curvature-like tensor field R_0 by

$$R_0(X,Y)Z = g(Y,Z)X - g(X,Z)Y \tag{2.2}$$

and a tensor field B of type (1,1) by

$$B = A - Q. \tag{2.3}$$

The following lemma is a direct consequence of Lemma 2.2 and equation (2.3).

Lemma 2.3. *If* $(\nabla A)(X,Y) - (\nabla A)(Y,X) = \frac{1}{2}R_0(X,Y)\operatorname{grad} S -$ $F(X,Y)$, *then the following hold:*

(i) $(\nabla B)(X,Y) - (\nabla B)(Y,X) = \frac{1}{2}R_0(X,Y)\operatorname{grad} S$

(ii) $(\nabla^2 B)(X,Y,Z) - (\nabla^2 B)(X,Z,Y) = \frac{1}{2}R_0(X,Y)\nabla_X \operatorname{grad} S$.

Lemma 2.4. *Let A be a symmetric tensor field on an n-dimensional connected Riemannian manifold (M,g) satisfying*

(i) $tr.A = -\frac{(n-2)}{2}S$

(ii) $(\nabla A)(X,Y) - (\nabla A)(Y,X) = \frac{1}{2}R_0(X,Y)\operatorname{grad} S - F(X,Y)$.

Then $\sum_i (\nabla A)(e_i,e_i) = 0$ *for a local orthonormal frame* $\{e_1,\dots,e_n\}$.

Proof. For $X \in \mathfrak{X}(M)$, as $tr.A = -\frac{(n-2)}{2}S$, we get

$$\sum_i g((\nabla A)(X,e_i),e_i) = -\frac{(n-2)}{2}g(\operatorname{grad} S, X).$$

Now, using condition (ii) of the statement, we arrive at

$$\sum_i g((\nabla A)(e_i,X),e_i) + \frac{1}{2}\sum_i g(R_0(X,e_i)\operatorname{grad} S, e_i) - \sum_i g(F(X,e_i),e_i) \\ = -\frac{(n-2)}{2}g(\operatorname{grad} S, X).$$

Using (2.2) in the above equation we arrive at

$$\sum_i g((\nabla A)(e_i,X),e_i) - \sum_i g(F(X,e_i),e_i) = \frac{1}{2}g(\operatorname{grad} S, X). \tag{2.4}$$

Now equation (2.1) gives

$$\sum_i g(F(X,e_i),e_i) = \sum_{ik}(\nabla_{e_k}R)(X,e_i;e_k,e_i) = -\sum_k(\nabla_{e_k}\operatorname{Ric})(X,e_k) \\ = -\sum_k g(X,(\nabla Q)(e_k,e_k)) = -\frac{1}{2}g(X,\operatorname{grad} S).$$

Thus the equation (2.4) gives $\sum_i g((\nabla A)(e_i,X),e_i) = 0$, or $g(X, \sum_i(\nabla A)(e_i,e_i)) = 0$, which proves the lemma. □

Finally, we prove the following lemma which is the main ingredient in the proof of the main theorem.

Lemma 2.5. *Let A be a symmetric tensor field on an n-dimensional connected Riemannian manifold (M,g) satisfying*

(i) $tr.A = -\frac{(n-2)}{2}S$,

(ii) $(\nabla A)(X,Y) - (\nabla A)(Y,X) = \frac{1}{2}R_0(X,Y)\operatorname{grad} S - F(X,Y)$.

Then $B = A - Q$ satisfies $\|\nabla B\|^2 \geq \frac{n}{4}\|\operatorname{grad} S\|^2$, and for positively curved M the equality holds if and only if $B = -\frac{S}{2}I$.

Proof. From Lemmas 2.1 and 2.4 we have

$$\sum_i (\nabla B)(e_i, e_i) = -\frac{1}{2}\operatorname{grad} S. \tag{2.5}$$

Define $C : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $C(X) = B(X) + \frac{S}{2}X$, $X \in \mathfrak{X}(M)$. Then we have

$$(\nabla C)(X, Y) = (\nabla B)(X, Y) + \frac{1}{2}g(\operatorname{grad} S, X)Y,$$

and consequently

$$\begin{aligned}\|\nabla C\|^2 = \|\nabla B\|^2 &+ \frac{n}{4}\|\operatorname{grad} S\|^2 \\ &+ \sum_{ij} g((\nabla B)(e_i, e_j), g(\operatorname{grad} S, e_i)e_j).\end{aligned} \tag{2.6}$$

Note that B is symmetric as both A and Q are symmetric, and therefore we have

$$g((\nabla B)(X, Y), Z) = g(Y, (\nabla B)(X, Z)), \quad X, Y, Z \in \mathfrak{X}(M).$$

Using this equation and Lemma 2.3, we compute

$$\begin{aligned}&\sum_{ij} g((\nabla B)(e_i, e_j), g(\operatorname{grad} S, e_i)e_j) \\ &\quad = \sum_{ij} g(\operatorname{grad} S, e_i) g\left(B(e_j, e_i) + \frac{1}{2}R_0(e_i, e_j)\operatorname{grad} S, e_j\right) \\ &\quad = \sum_j g(\operatorname{grad} S, (\nabla B)(e_j, e_j)) + \frac{1}{2}\sum_j g(R_0(\operatorname{grad} S, e_j)\operatorname{grad} S, e_j).\end{aligned}$$

Now use this equation, (2.2), and (2.5) in (2.6) to arrive at

$$\|\nabla C\|^2 = \|\nabla B\|^2 - \frac{n}{4}\|\operatorname{grad} S\|^2,$$

which proves the inequality $\|\nabla B\|^2 \geq \frac{n}{4}\|\operatorname{grad} S\|^2$.

Next suppose that M is positively curved and the equality holds. Then we shall have $\nabla C = 0$, which gives $C = \lambda I$ (as M being positively curved, it is irreducible) for a constant λ. Thus we have $n\lambda = tr.B + \frac{nS}{2} = tr.A - tr.Q + \frac{nS}{2} = 0$, where we have used condition (i); consequently $\lambda = 0$ and this proves $B = -\frac{S}{2}I$. □

3. Proof of the Theorem

If the tensor field T satisfies the Einstein field equation, then we have $A = Q - \frac{S}{2}I$, and from Lemma 2.2 we get the conditions (i), (ii). Conversely suppose that the given conditions are satisfied. Define $f : M \to R$ by $f = \frac{1}{2}\|B\|^2$. Then choosing a local orthonormal frame $\{e_1, \dots, e_n\}$ on M, we compute the Hessian H_f of f and obtain

$$H_f(X,X) = \sum_{ij} g((\nabla B)(X,e_i),e_j)^2 + \sum_i g((\nabla^2 B)(X,X,e_i),B(e_i)).$$

Thus the Laplacian $\Delta f = \sum_k H_f(e_k,e_k)$ is given by

$$\Delta f = \|\nabla B\|^2 + \sum_{ik} g((\nabla^2 B)(e_k,e_k,e_i),B(e_i)). \tag{3.1}$$

The equation (2.5) gives

$$\sum_k (\nabla^2 B)(e_i,e_k,e_k) = -\frac{1}{2}\nabla_{e_i}\operatorname{grad} S, \tag{3.2}$$

and the Ricci identity implies

$$(\nabla^2 B)(e_k,e_i,e_k) = (\nabla^2 B)(e_i,e_k,e_k) + R(e_k,e_i)Be_k - BR(e_k,e_i)e_k. \tag{3.3}$$

Thus, using (2.2), (3.2), (3.3) and Lemma 2.3 in (3.1), we arrive at

$$\begin{aligned}\Delta f = {}& \|\nabla B\|^2 - \frac{1}{2}\sum_i g(\nabla_{e_i}\operatorname{grad} S, B(e_i)) + \sum_{ik}\big[R(e_k,e_i;Be_k,Be_i)\\ &- R(e_k,e_i;e_k,B^2e_i)\big] + \frac{1}{2}\sum_{ik} g(\nabla_{e_k}\operatorname{grad} S, e_i)g(e_k,Be_i)\\ &- \frac{1}{2}\sum_{ik} g(\nabla_{e_k}\operatorname{grad} S, e_k)g(e_i,Be_i)\\ = {}& \|\nabla B\|^2 + \frac{n}{4}S\Delta S + \sum_{ik}\big[R(e_k,e_i;Be_k,Be_i) - R(e_k,e_i;e_k,B^2e_i)\big],\end{aligned} \tag{3.4}$$

where we have used $tr.B = -\frac{n}{2}S$, which follows from condition (i) and $B = A - Q$. Next, we choose a local orthonormal frame $\{e_1, \dots, e_n\}$ which diagonalizes B with $B(e_i) = \mu_i e_i$, and compute

$$\begin{aligned}
&\sum_{ik}[R(e_k, e_i; Be_k, Be_i) - R(e_k, e_i; e_k, B^2 e_i)]\\
(3.5) \qquad &= \sum_{ik} \mu_i^2 K_{ik} - \mu_i\mu_k K_{ik} = \frac{1}{2}\sum_{ik} 2\mu_i^2 K_{ik} - 2\mu_i\mu_k K_{ik},\\
&= \frac{1}{2}\left[\sum_{ik}\mu_i^2 K_{ik} + \mu_k^2 K_{ik} - 2\mu_i\mu_k K_{ik}\right] = \frac{1}{2}\sum_{ik}(\mu_i - \mu_k)^2 K_{ik},
\end{aligned}$$

where $K_{ik} = R(e_k, e_i; e_i, e_k)$ is the sectional curvature of the plane section spanned by $\{e_i, e_k\}$. Using (3.5) in (3.4) and integrating the resulting equation we get

$$\int_M \left\{\|\nabla B\|^2 + \frac{n}{4}S\Delta S + \frac{1}{2}\sum_{ik}(\mu_i - \mu_k)^2 K_{ik}\right\} dv = 0.$$

Integrating by parts the second term in the above integral, we arrive at

$$\int_M \left\{\|\nabla B\|^2 - \frac{n}{4}\|\operatorname{grad} S\|^2\right\} dv + \frac{1}{2}\int_M \left\{\sum_{ik}(\mu_i - \mu_k)^2 K_{ik}\right\} dv = 0.$$

Since $K_{ik} > 0$, the above integral together with Lemma 2.5 gives

$$\|\nabla B\|^2 = \frac{n}{4}\|\operatorname{grad} S\|^2,$$

and this equality, again by Lemma 2.5, implies $B = -\frac{S}{2}I$ and consequently the Einstein equation $A = Q - \frac{S}{2}I$.

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SHARIEF DESHMUKH
DEPARTMENT OF MATHEMATICS
KING SAUD UNIVERSITY
P.O. BOX 2455 RIYADH 11451
SAUDI ARABIA

E-mail: shariefd@ksu.edu.sa

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