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Finiteness conditions and sums of rings

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Abstract. Let R be a ring, which is a sum of its additive subgroups R_s, $s \in S$. Suppose that all rings among the R_s satisfy one of the following classical finiteness properties: right or left Artinian or Noetherian, right or left perfect, semilocal or semiprimary. We provide new conditions on the interaction of the R_s sufficient for the whole ring R to enjoy the same property.

This paper is devoted to several well-known finiteness conditions which play important roles in ring theory. We start with the following natural problem. Let $\mathcal{K}$ be a class of associative rings, S a set, R a ring, and let $R = \sum_{s\in S} R_s$ be a sum of additive subgroups R_s of R. Suppose that all rings among the R_s belong to $\mathcal{K}$. Find conditions sufficient for R to belong to $\mathcal{K}$.

In full generality this problem was first recorded in [7]. However, its many interesting special cases have been actively investigated by a number of authors including Bahturin, Beidar, Bokut', Chick, Ferrero, Fukshansky, Gardner, Kegel, Kelarev, Kepczyk, McConnell, Mikhalev, Petravchuk, Puczyłowski and Salwa (see [3]–[8] for references).

We use two restrictions on the interaction of the components R_s introduced in [7]. Let S be a semigroup, $R = \sum_{s\in S} R_s$ a sum of additive subgroups R_s of R. If $T \subseteq S$, then we put $R_T = \sum_{s\in T} R_s$. We say that R is a *structural S-sum* if and only if, for each subsemigroup (left ideal, right ideal) T of S, the sum R_T is a subring (respectively, left ideal, right

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ideal) of R. For any $s \in S$, denote by $\langle s \rangle$ the subsemigroup generated in S by s, and put $R^s = R_{\langle s \rangle}$. We say that R is an S*-sum* if $R_s R_t \subseteq R^{st}$ for all $s, t \in S$. Many ring constructions are examples of structural S-sums and S-sums.

This paper investigates the question of when several classical finiteness conditions are preserved by S-sums and structural S-sums. First, we deal with left or right Artinian or Noetherian rings.

A semigroup entirely consisting of idempotents is called a *band*. A band is said to be a *semilattice* (*rectangular band; left zero band; right zero band; left regular band; right regular band*) if it satisfies the identity $xy = yx$ ($xyx = x$; $xy = x$; $xy = y$; $xyx = xy$; $xyx = yx$).

Theorem 1. *For any semigroup S, the following are equivalent:*

(i) *for every structural S-sum $R = \sum_{s \in S} R_s$, if all rings among the R_s are right Artinian (Noetherian), then R is right Artinian (Noetherian), too;*

(ii) *for every S-sum $R = \sum_{s \in S} R_s$, if all rings among the R_s are right Artinian (Noetherian), then R is right Artinian (Noetherian), too;*

(iii) *S is a finite left regular band.*

Next, we look at semilocal, semiprimary, and left or right perfect rings. Let $\mathcal{J}(R)$ be the Jacobson radical of R. A ring R is *semilocal* if $R/\mathcal{J}(R)$ is Artinian. Recall that R is said to be *right (left) T-nilpotent* if, for every sequence of elements $r_1, r_2, \ldots$ of R, there exists n such that $r_n \ldots r_1 = 0$ (respectively, $r_1 \ldots r_n = 0$). A semilocal ring is *semiprimary* (*right perfect, left perfect*) if $\mathcal{J}(R)$ is nilpotent (right T-nilpotent; left T-nilpotent). A semigroup is said to be *combinatorial* if all its subgroups are singletons. We obtain the following conditions sufficient for preservation of these properties by structural sums of rings.

Theorem 2. *Let S be a periodic combinatorial semigroup with a finite number of idempotents and locally nilpotent (nilpotent, right T-nilpotent, left T-nilpotent) nil factors, and let $R = \sum_{s \in S} R_s$ be an S-sum. If all rings among the R_s are semilocal (respectively, semiprimary, right perfect, left perfect), then R is semilocal (respectively, semiprimary, right perfect, left perfect), too.*

A semigroup is said to be *semisimple* if all its principal factors are simple or 0-simple.

Theorem 3. *Let S be a semisimple periodic combinatorial semigroup with a finite number of idempotents, and let $R = \sum_{s\in S} R_s$ be a structural S-sum. If all rings among the R_s are semilocal (or semiprimary, or right perfect, or left perfect), then R possesses the same property, too.*

Proof of Theorem 1. The case where $|S| = 1$ is trivial, and so throughout we assume that S is not a singleton. The implication (i)$\Rightarrow$(ii) is obvious, because every S-sum is a structural S-sum.

(ii)$\Rightarrow$(iii): Suppose that S contains an element t such that $t \neq t^2$. Take any finite field F. Denote by P the ring of all polynomials over F in commuting variables $x_1, x_2, \dots$ without constant terms. Let I be the ideal generated in P by all polynomials $x_i x_j - x_k x_\ell$, for any positive integers i, j, k, ℓ. Consider the ring $R = P/(I+P^3)$ and its subrings $R_t = \sum_{i=1}^{\infty} F x_i$, $R_{t^2} = F x_1^2$. Put $R_s = 0$ for all $s \in S\backslash\{t, t^2\}$. Clearly, $R = \sum_{s\in S} R_s$ is an S-sum. Moreover, it is an S-graded ring. The component R_t is not a subring. The component R_{t^2} is finite, and so it is right Artinian and Noetherian. All the other components are zero. However, R is neither Artinian, nor Noetherian. This contradiction shows that S must be a band.

If S is a band, then every S-sum is an S-graded ring, and therefore [3], Corollary 6.4, shows that (iii) holds.

(iii)$\Rightarrow$(i): Suppose that S is a finite left regular band. Take any structural S-sum $R = \sum_{s\in S} R_s$ such that all rings among the R_s are right Artinian (Noetherian). Clearly, R is an S-graded ring. Therefore [3], Corollary 6.4, tells us that R is right Artinian (Noetherian), as required. □

For any semigroup S, denote by S^0 the semigroup $S \cup \{0\}$ with zero adjoined. Let G be a group, I and Λ nonempty sets, and let $P = (p_{\lambda i})$ be a $\Lambda \times I$-matrix with entries $p_{\lambda i} \in G^0$ for all $\lambda \in \Lambda$, $i \in I$. The *Rees matrix semigroup* $M^0(G; I, \Lambda; P)$ over the group G with *sandwich-matrix* P consists of zero and all triples $(g; i, \lambda)$, $i \in I$, $\lambda \in \Lambda$, and $g \in G$, where all triples of the form $(0; i, \lambda)$ are identified with zero, and multiplication is defined by the rule $(g_1; i_1, \lambda_1)(g_2; i_2, \lambda_2) = (g_1 p_{\lambda_1 i_2} g_2; i_1, \lambda_2)$. Lemma 3.1 of [3] immediately gives us the following

Lemma 4. *If S is a periodic combinatorial semigroup with a finite number of idempotents, then S^0 has a finite ideal chain*

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S^0$$

such that each factor S_{i+1}/S_i, where $0 \leq i \leq n-1$, is nil or is isomorphic to a finite Rees matrix semigroup $M^0(\{e\}; I, \Lambda; P)$.

We also need the following well-known properties of semilocal, semiprimary, right and left perfect rings (see, for example, [3], Lemma 4.3).

Lemma 5. *The classes of semilocal, semiprimary, right perfect and left perfect rings are closed for ideal extensions, right and left ideals, homomorphic images and finite sums of one-sided ideals.*

Proof of Theorem 2. Put $R_0 = 0$. Then $R = \sum_{s \in S^0} R_s$ is an S^0-sum. Lemma 4 tells us that S^0 has a finite ideal chain

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S^0$$

such that each factor S_{i+1}/S_i, where $0 \leq i \leq n-1$, is nil or is isomorphic to a finite Rees matrix semigroup $M^0(\{e\}; I, \Lambda; P)$. We show by induction on $k = 0, 1, \ldots, n$ that all rings R_{S_k} are semilocal (semiprimary, right perfect, left perfect).

The induction basis is trivial, as $R_{S_0} = 0$. Suppose that $k > 0$ and $R_{S_{k-1}}$ has the desired property. Since R_{S_k} is an extension of $R_{S_{k-1}}$ by $R_{S_k}/R_{S_{k-1}}$, in view of Lemma 5 it suffices to show that $R_{S_k}/R_{S_{k-1}}$ belongs to our class.

Clearly, $Q = R_{S_k}/R_{S_{k-1}} = \sum_{s \in S_k \setminus S_{k-1}} R_s$ is a structural S_k/S_{k-1}-sum. By the hypothesis S_k/S_{k-1} is isomorphic to a finite Rees matrix semigroup $M^0(\{e\}; I, \Lambda; P)$. The definition of multiplication in $M^0(\{e\}; I, \Lambda; P)$ implies that $M^0(\{e\}; I, \Lambda; P)$ is a union of its left ideals $L_\lambda = \{0\} \cup \{(e; i, \lambda) \mid i \in I\}$, where $\lambda \in \Lambda$. Besides, each L_λ is a union of its right ideals $I_i = \{0, (e; i, \lambda)\}$, where $i \in I$. Given that Q is an S_k/S_{k-1}-sum, it follows that Q is a sum of its left ideals Q_{L_λ}, $\lambda \in \Lambda$, and every Q_{L_λ} is a sum of its right ideals $Q_{I_i} \cong R_{(e;i,\lambda)}$. Since $R_{(e;i,\lambda)}$ is a subring, we know that it is semilocal (semiprimary, right perfect, left perfect). Lemma 5 implies that all the Q_{L_λ} are semilocal (semiprimary, right perfect, left perfect), too, and so the same can be said of Q. This completes the proof. □

PROOF of Theorem 3. Set $R_0 = 0$. Evidently, $R = \sum_{s\in S^0} R_s$ is a structural S^0-sum. Given that S is semisimple, Lemma 4 tells us that S^0 has a finite ideal chain

$$0 = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = S^0$$

such that each factor S_{i+1}/S_i, where $0 \le i \le n-1$, is isomorphic to a finite Rees matrix semigroup $M^0(\{e\}; I, \Lambda; P)$. We show by induction on $k = 0, 1, \dots, n$ that all rings R_{S_k} are semilocal (semiprimary, right perfect, left perfect).

The induction basis is trivial again. Suppose that $k > 0$ and $R_{S_{k-1}}$ has the desired property. As above in view of Lemma 5 it suffices to show that $R_{S_k}/R_{S_{k-1}}$ belongs to our class.

Clearly, $R_{S_k}/R_{S_{k-1}} = \sum_{s\in S_k\setminus S_{k-1}} R_s$ is a structural S_k/S_{k-1}-sum. Given that S_k/S_{k-1} is isomorphic to a finite Rees matrix semigroup, since every structural S_k/S_{k-1}-sum is an S_k/S_{k-1}-sum, it follows from Theorem 3 that $R_{S_k}/R_{S_{k-1}}$ is semilocal (semiprimary, right perfect, left perfect). This completes our proof. □

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