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On rings with involution equipped with some new product

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Abstract. Let R be a ring with involution $*$. We consider R as a ring equipped with a new product $r \diamond s = rs - sr^*$. The relationship between (ordinary) ideals of R and left and right ideals of R with respect to the product $\diamond$ is studied.

1. Introduction

Let R be an associative $*$-ring, that is, a ring with involution $*$. We introduce a new product, $\diamond$, in R by

$$r \diamond s = rs - sr^*.$$

Clearly, this product is nonassociative in general. We shall say that an additive subgroup L of R is a *left $\diamond$-ideal* of R if $r \diamond x \in L$ for all $r \in R$, $x \in L$. *Right $\diamond$-ideals* are defined analogously. Of course, every ideal of R is also a left $\diamond$-ideal, and every $*$-ideal of R (i.e., an ideal I such that $I^* = I$) is also a right $\diamond$-ideal. The converse of each of both statements is not always true. The question "how far" are left $\diamond$-ideals (right $\diamond$-ideals) from ideals ($*$-ideals, respectively) is the main issue of the paper.

It seems that the product $\diamond$ first appeared in the work of Šemrl [9]. He observed that maps of the form $x \mapsto x \diamond a$ naturally arise in the problem of representing quadratic functionals with sesquilinear functionals. Motivated by Šemrl's work and by the theory of rings (and algebras) equipped

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with a Lie ($[r,s]=rs-sr$) and Jordan ($r\circ s=rs+sr$) product, MOLNÁR recently [7] initiated the systematic treatment of the product $\diamond$. In his main result he showed that a subspace of $\mathcal{B}(H)$, the algebra of all bounded linear operators on a Hilbert space H, is an ideal of $\mathcal{B}(H)$ if and only if it is a left $\diamond$-ideal of $\mathcal{B}(H)$. We shall generalize this result in different ways. Our approach is entirely algebraic and is completely different from Molnár's one. It is based on discovering certain identities that connect the product $\diamond$ with the initial, associative product. The methods that we use are similar to those employed by HERSTEIN in his classical work on Lie and Jordan structure in associative rings (see, e.g., [2]).

In Section 2 we provide several examples of left and right $\diamond$-ideals different from ideals. These examples suggest what kind of results one can(not) expect. In Section 3 we treat left $\diamond$-ideals, and in Section 4 right $\diamond$-ideals.

2. Examples

First we consider left $\diamond$-ideals. We begin with a trivial, but crucial observation.

Example 2.1. Let R be a commutative ring with trivial involution (i.e., $*$ is just the identity map). Then any additive subgroup of R is trivially a left $\diamond$-ideal of R. Moreover, considering a ring which is the direct sum of a commutative ring (with the trivial involution) and a noncommutative ring (with some involution) we are forced to conclude that there are also noncommutative rings containing left $\diamond$-ideals different from ideals. Moreover, such left $\diamond$-ideals are not necessarily contained in the center of the ring.

The next example shows that there are noncommutative rings that can not be decomposed into the direct sum of two ideals, but still contain nontrivial left $\diamond$-ideals. On the other hand, there is some similarity with Example 2.1 – both examples suggest that centers of rings play a special role in the study of left $\diamond$-ideals.

Example 2.2. Let F be a field and let $R=F\langle x,y\rangle$ be a free algebra with unit in two noncommuting indeterminates x and y. Equip R with the standard involution (given by $x^*=x$, $y^*=y$ and $\lambda^*=\lambda$, $\lambda\in F$). Note that for any $p,q\in R$, the element (= polynomial) $p\diamond q$ has the constant term zero. Let G be any subgroup of the additive group F different from F and $\{0\}$, and let L be the set of all elements in R whose constant term lies in G. Then L is a left $\diamond$-ideal of R, but not an ideal. Let us also point out that the element $1 \in R$ cannot be written as a sum of elements of the form $p \diamond q$.

Recall that a ring R is said to be 2-torsionfree if $2r \neq 0$ for any nonzero $r \in R$. For simple (and even prime) rings, this is equivalent to the condition that the characteristic of R is not 2. The significance of the next two examples will become clear later.

Example 2.3. Let A be a 2-torsionfree commutative ring such that $A^3 = 0$ and $A^2 \neq 0$ (a concrete example is a 2-dimensional algebra A over a field of characteristic not 2, with basis $\{a, b\}$ and multiplication given by $a^2 = b$ and $bA = Ab = \{0\}$). Let $R = M_2(A)$ be the ring of 2×2 matrices over A with transpose involution, and let

$$L = \left\{ \begin{bmatrix} a & \alpha \\ \beta & -a \end{bmatrix} \;\middle|\; a \in A,\, \alpha, \beta \in A^2 \right\}.$$

Then L is a left $\diamond$-ideal of R, which is neither an ideal nor is contained in Z, the center of R. However, $L^2 \subseteq Z$. Note that $x \diamond y = y^* \diamond x$ for all $x, y \in L$.

Example 2.4. Let R be any $*$-ring and let L be the set of all $x \in R$ such that $r \diamond x = 0$, i.e., $rx = xr^*$ for all $r \in R$. Then L is a left $\diamond$-ideal of R, and so is any its additive subgroup. Given $x, y \in L$, we have $rxy = xr^*y = xyr$ for every $r \in R$, so that, as in the previous example, L^2 is contained in the center of R.

Now, let us examine this example in a concrete situation. Let R be the ring of upper triangular 2×2 matrices over a field F, with involution given by $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}^* = \begin{bmatrix} c & b \\ 0 & a \end{bmatrix}$. In this case the set L consists of all strictly upper triangular matrices, and is therefore an ideal of R. However, it is easy to see that any proper additive subgroup of L is a left $\diamond$-ideal of R which is neither contained in the center nor contains a nonzero ideal of R.

We now turn our attention to right $\diamond$-ideals. Again we begin with a trivial but important observation.

Example 2.5. Every additive subgroup of the center of a $*$-ring consisting of symmetric elements is a right $\diamond$-ideal. Moreover, the sum of right $\diamond$-ideals is again a right $\diamond$-ideal, so that one can find noncentral right $\diamond$-ideals which are not $*$-ideals in almost any $*$-ring containing proper $*$-ideals (this includes $\mathcal{B}(H)$ when H is infinite dimensional).

Example 2.5 indicates that the best possible result one can hope for is that, in certain rings, every noncentral right $\diamond$-ideal contains a nonzero $*$-ideal. However, the next three examples show that this may not be true even in the most fundamental rings.

Example 2.6. Let $\mathcal{H}$ be the ring of real quaternions equipped with a standard involution. Then the set of all quaternions of the form $a + bi$ is a right $\diamond$-ideal of $\mathcal{H}$. The same is true for, say, the set of all quaternions of the form $a + bi + cj$.

Regarding quaternions as 2×2 complex matrices, one can easily modify Example 2.6 to obtain

Example 2.7. Let F be a field and let $R = M_2(F)$ with transpose involution. The set of all matrices $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, $a, b \in F$, is then a right $\diamond$-ideal.

Example 2.8. Again let F be a field and $R = M_2(F)$, but now equip R with the symplectic involution, that is, the one given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Then the set of all matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, $a, d \in F$, is a right $\diamond$-ideal.

Obviously, each of right $\diamond$-ideals in Examples 2.6, 2.7 and 2.8 is neither contained in the center of the ring nor contains a nonzero ideal (after all, all the rings under consideration are simple). These examples suggest that the rings of 2×2 matrices, as well as their subrings, should be excluded in a general result on right $\diamond$-ideals. In Section 4 we shall see that, roughly speaking, these are also the only rings that must be excluded.

We conclude with an example concerning characteristic 2. In most of Herstein's theorems on Lie and Jordan ideals of associative rings, there is the requirement that rings must have characteristic different from 2. The characteristic 2 case is indeed different. The basic Herstein's example [2, p. 6] also works in the present setting:

Example 2.9. Let F be a field of characteristic 2 and let $R = M_2(F)$ with transpose involution. The set of all matrices of the form $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$, $a, b \in F$, is simultaneously a left $\diamond$-ideal, a right $\diamond$-ideal, and a subring of R. However, it is neither an ideal nor is contained in the center of R. We also mention that the set of matrices $\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}$, $b \in F$, is a noncentral left $\diamond$-ideal.

3. Left $\diamond$-ideals

We first fix the notation. Throughout this section, R will be a ring with involution $*$. By Z we denote its center and by $[R, R]$ the additive subgroup of R generated by all commutators $[r, s] = rs - sr$, $r, s \in R$. Further, we set $\widehat{R} = R[R, [R, R]]R$, that is, $\widehat{R}$ is an ideal of R consisting of all finite sums of elements of the form $r[s, [t, u]]v$ with $r, s, t, u, v \in R$.

Theorem 3.1. *Let R be an arbitrary $*$-ring and let L be a left $\diamond$-ideal of R. Then $4\widehat{R}L \subseteq L$ and $4L\widehat{R} \subseteq L$.*

Proof. Noting that $r \diamond (s \diamond x) + (rs) \diamond x = 2rsx - rxs^* - sxr^*$ we see that, by the very definition of a left $\diamond$-ideal, $2rsx - rxs^* - sxr^* \in L$ for all $r, s \in R$, $x \in L$. Replacing the roles of r and s in this relation we get $2srx - sxr^* - rxs^* \in L$. Comparing the last two relations we arrive at $2[R, R]L \subseteq L$. Now, the identity $4r[s, [t, u]] = 4[s, r[t, u]] - (2[s, r])(2[t, u]) \in 2[R, R] + (2[R, R])^2$ implies that $4R[R, [R, R]]L \subseteq L$. Consequently, from $r[s, [t, u]]v = r[sv, [t, u]] - rs[v, [t, u]] \in R[R, [R, R]]$ we finally conclude that $4\widehat{R}L \subseteq L$. Clearly, if $a \in R$ is such that $aL \subseteq L$, then $La^* \subseteq aL + a \diamond L \subseteq L + L = L$. Since $\widehat{R}$ is invariant under $*$, it follows that $4L\widehat{R} \subseteq L$.

Corollary 3.2. *Let R be any $*$-ring such that $R = 4\widehat{R}$. Then a set $L \subseteq R$ is an ideal of R if and only if L is a left $\diamond$-ideal of R.*

In particular, Corollary 3.2 applies to noncommutative simple rings of characteristic not 2. Indeed, if R is such a ring, then it is well-known and easy to see (cf. the argument at the end of the proof of Corollary 3.6 below) that $[R, [R, R]] \neq \{0\}$. Therefore, $4\widehat{R} = 4R = R$. So we have

Corollary 3.3. *Let R be a noncommutative simple $*$-ring of characteristic not 2. Then 0 and R are the only left $\diamond$-ideals of R. That is, R is left $\diamond$-simple.*

As Examples 2.1 and 2.9 show, the conditions that R is noncommutative and of characteristic not 2 cannot be removed. Thus, Corollary 3.3 is somehow the best possible result for simple rings. On the other hand, Example 2.2 indicates that one cannot expect definitive results for some more general classes of rings, such as primitive (or even prime) rings. But there is another interesting class of rings in which ideals and left $\diamond$-ideals coincide. Recall that a subset $\{e_{11}, e_{12}, e_{21}, e_{22}\}$ of a ring R with unit 1 is said to be a set of matrix units if $e_{11} + e_{22} = 1$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$, $i, j, k, l = 1, 2$.

Corollary 3.4. *Let R be a $*$-ring containing 1 and the element $\frac{1}{2}$ (i.e., the element $1 + 1$ is invertible in R). Suppose that R contains a set of matrix units $\{e_{11}, e_{12}, e_{21}, e_{22}\}$. Then a set $L \subseteq R$ is an ideal of R if and only if L is a left $\diamond$-ideal of R.*

Proof. Set $r = \frac{1}{2}(e_{12}+e_{21})$, $s = e_{11}-e_{22}$, and note that $[r, [r, s]] = s$ is invertible in R (indeed, $s^2 = 1$). Hence $\widehat{R} = R$ and so the result follows from Corollary 3.2.

The last two corollaries yield the result of Molnár [7, Theorem]:

Corollary 3.5. *Let H be a real or complex Hilbert space of dimension at least 2, and let $\mathcal{B}(H)$ be the algebra of all bounded linear operators on H. Then a set $L \subseteq \mathcal{B}(H)$ is an ideal of $\mathcal{B}(H)$ if and only if L is a left $\diamond$-ideal of R.*

Proof. If H is finite dimensional, then the result follows from Corollary 3.3. If H is infinite dimensional, then H is isomorphic to $H \oplus H$, and so $\mathcal{B}(H)$ is isomorphic to $M_2(\mathcal{B}(H))$. Therefore, the conditions of Corollary 3.4 are fulfilled in this case.

We remark that Corollary 3.5 holds for any involution in $\mathcal{B}(H)$, but one usually has in mind the one arising from the Hilbert space adjoint.

As a matter of fact, there is a slight difference between Corollary 3.5 and [7, Theorem]: we do not assume that a left $\diamond$-ideal is a subspace of $\mathcal{B}(H)$ (as in [7]), but only an additive subgroup. That is why we had to exclude the one-dimensional case.

Is it possible to say anything about left $\diamond$-ideals in arbitrary (or almost arbitrary) $*$-rings? While it is clear from the examples above that it is very easy to find noncentral left $\diamond$-ideals different from ideals, at least one could conjecture that, under appropriate assumptions, noncentral left $\diamond$-ideals must necessarily contain nonzero ideals. Herstein obtained results of that kind for Lie and Jordan ideals in 2-torsionfree semiprime rings [2, pp. 3–4]. Their analogue for left $\diamond$-ideals can now be easily obtained.

Corollary 3.6. *Let R be a 2-torsionfree semiprime $*$-ring. If L is a left $\diamond$-ideal of R, then either $L \subseteq Z$ or L contains a nonzero ideal of R.*

Proof. Theorem 3.1 implies that L contains the ideal $(4\widehat{R})L(4\widehat{R})$. Thus, all we have to show is that $(4\widehat{R})L(4\widehat{R}) = \{0\}$ implies $L \subseteq Z$. Suppose, therefore, that $(4\widehat{R})L(4\widehat{R}) = \{0\}$. Since R is 2-torsionfree, this gives $\widehat{R}L\widehat{R} = \{0\}$. That is, $\widehat{R}L$ is a left ideal whose square is zero. The semiprimeness of R yields $\widehat{R}L = 0$, i.e., $R[R,[R,R]]RL = 0$. In particular, $(R[R,[R,L]])^2 = 0$, and so $[R,[R,L]] = 0$. Given $r,s \in R$ and $x \in L$, we thus have $[r,x][s,x] = [r,[xs,x]] - x[r,[s,x]] = 0$. But then $[r,x]s[r,x] = [r,x][sr,x] - [r,x][s,x]r = 0$ for all $r,s \in R$, $x \in L$. Since R is semiprime it follows $[r,x] = 0$, that is, $x \in Z$ for any $x \in L$.

A different approach gives some result for arbitrary rings.

Theorem 3.7. *Let R be an arbitrary $*$-ring and let L be a left $\diamond$-ideal of R. Then L contains the ideal of R generated by all elements of the form $x \diamond y - y^* \diamond x$, $x, y \in L$. Moreover, if R is 2-torsionfree, then this ideal is nonzero, unless $L^2 \subseteq Z$.*

Proof. The key identity

$$(x \diamond y - y^* \diamond x)r^* = r \diamond (y^* \diamond x) + rx \diamond y - ry^* \diamond x$$

shows that $(x \diamond y - y^* \diamond x)R \subseteq L$ for all $x, y \in L$. Consequently, $R(x \diamond y - y^* \diamond x) \subseteq L$ since

$$r(x \diamond y - y^* \diamond x) = r \diamond (x \diamond y - y^* \diamond x) + (x \diamond y - y^* \diamond x)r^* \in L.$$

Finally,

$$s(x \diamond y - y^* \diamond x)r = s \diamond (x \diamond y - y^* \diamond x)r + (x \diamond y - y^* \diamond x)rs^* \in L$$

gives $R(x \diamond y - y^* \diamond x)R \subseteq L$.

Now assume that R is 2-torsionfree and that $x \diamond y - y^* \diamond x = 0$ for all $x, y \in L$. That is, $2xy = y^*x + yx^*$ for all $x, y \in L$. Hence it follows $2y^*x^* = (2xy)^* = (y^*x + yx^*)^* = x^*y + xy^* = 2yx$, and so $y^*x^* = yx$ for all $x, y \in L$. Replacing x by $r \diamond x = rx - xr^*$ with $x \in L$, $r \in R$, in this relation we get $y^*x^*r^* - y^*rx^* = yrx - yxr^*$. Using $y^*x^* = yx$ we can rewrite the last relation as $2yxr^* = y^*rx^* + yrx$, $x, y \in L, r \in R$. This further implies

$$2yxr^* = y^*rx^* + yrx = (x^*r^*y^* + xr^*y)^* = (2xyr)^* = 2r^*y^*x^* = 2r^*yx.$$

Thus, $2[yx, R] = 0$, and so $yx \in Z$ for all $x, y \in R$.

Examples 2.3 and 2.4 prove that the condition $x \diamond y = y^* \diamond x$, $x, y \in L$, is not sufficient for concluding $L \subseteq Z$. Moreover, Example 2.4 shows that there exist left $\diamond$-ideals which are neither contained in the center nor contain nonzero ideals. Therefore, the conclusion $L^2 \subseteq Z$ is somehow the best possible.

There is another useful identity connecting the product $\diamond$ with the associative product, namely

$$(y \diamond x)r^* = (ry) \diamond x - r \diamond (yx)$$

Arguing as at the beginning of Theorem 3.7, one shows easily, using this identity, that a left $\diamond$-ideal L of R, which is at the same time a subring of R, contains the ideal of R generated by all $x \diamond y$, $x, y \in L$. It is possible that $L \nsubseteq Z$, but this ideal is still zero. An example is the first set in Example 2.9.

This identity has another application, as we shall see. Given an ideal I of a $*$-ring R, we let $R \diamond I$ to be the additive subgroup of R generated by all $r \diamond x$, $r \in R$, $x \in I$. Of course, $R \diamond I \subseteq I$. The question arises when the equality holds, that is, when is every element in I equal to a finite sum of elements of the form $r \diamond x$, $r \in R$, $x \in I$? Molnár showed that this holds true for any ideal I of $\mathcal{B}(H)$ [7, Proposition]. We now turn to algebraic aspects of this problem. Clearly, $R \diamond I$ is a left $\diamond$-ideal of R. But actually we have

Proposition 3.8. *Let I be an ideal of a $*$-ring R. Then $R \diamond I$ is also an ideal of R.*

Proof. Let $r, s \in R$ and $x \in I$. Using the identity observed above, $(s \diamond x)r^* = (rs) \diamond x - r \diamond (sx)$, we see that $R \diamond I$ is a right ideal of R. But then $r(s \diamond x) = r \diamond (s \diamond x) + (s \diamond x)r^*$ shows that it is also a left ideal.

Thus, $R \diamond I$ is always an ideal contained in I. In order to conlude that $R \diamond I = I$, we have to impose some conditions. Recall that matrix units $e_{11}, e_{12}, e_{21}, e_{22}$ in a $*$-ring are called $*$-matrix units if $e_{ij}^* = e_{ji}$, $i, j = 1, 2$.

Corollary 3.9. *Suppose that a $*$-ring R contains a set of $*$-matrix units $\{e_{11}, e_{12}, e_{21}, e_{22}\}$. Then $R \diamond I = I$ for any ideal I of R.*

Proof. Let I be an ideal of R and let $x \in I$. Then

$$x = -(e_{11}(e_{21} \diamond x)e_{21} + e_{22}(e_{12} \diamond x)e_{12} + e_{11}(e_{22} \diamond x) + e_{22}(e_{11} \diamond x))$$

lies in $R \diamond I$ since $R \diamond I$ is an ideal of R. Hence $R \diamond I = I$.

We remark that the algebra $\mathcal{B}(H)$ has such a set of $*$-matrix units whenever H is infinite (or even) dimensional. In the finite dimensional case, however, the algebra is simple and so in that case Molnár's result follows from the next corollary.

Corollary 3.10. *Let R be a simple $*$-ring. Then $R \diamond R = R$, unless R is a field with trivial involution.*

Proof. Proposition 3.8 implies that either $R \diamond R = R$ or $R \diamond R = \{0\}$. If the latter occurs, then we have $rs = sr^*$ for all $s, r \in R$. Consequently, $[r, st] = (rs)t - s(tr) = sr^*t - sr^*t = 0$ for all $r, s, t \in R$. That is, $[R, R^2] = 0$. Since $R^2 = R$, this means that R is commutative, and therefore a field. Now $rs = sr^*$ for all $r, s \in R$ clearly implies $r = r^*$ for all $r \in R$.

Examples 2.1 and 2.2 show that $R \diamond R = R$ is not fulfilled in any $*$-ring R. Moreover, these examples somehow indicate that $R \diamond R = R$ can hold only if the set of skew-symmetric elements is "big enough". To clarify this, we state

Corollary 3.11. *Let R be a $*$-ring with 1 such that $2R = R$. Further, assume that the ideal of R generated by all skew-symmetric elements is equal to R. Then $R \diamond R = R$.*

Proof. For any skew-symmetric element $k \in R$ we have $2k = k \diamond 1 \in R \diamond R$. But then, since $R \diamond R$ is an ideal by Proposition 3.8, our assumptions imply that $R \diamond R = R$.

4. Right $\diamond$-ideals

First we fix the notation for this section. Throughout, R will be a *prime* $*$-ring with center Z and characteristic not 2. By S and K we denote the set of all its symmetric and skew-symmetric elements, respectively.

Recall that R is said to satisfy S_4, the standard polynomial identity of degree 4, if $\sum_{\pi\in S_4}(-1)^\pi r_{\pi(1)}r_{\pi(2)}r_{\pi(3)}r_{\pi(4)} = 0$ for all $r_1, r_2, r_3, r_4 \in R$, where S_4 is the permutation group. From the classical theory of rings with polynomial identities (see, e.g., [8]) it is well-known that R satisfies S_4 if and only if it embeds in $M_2(F)$ for some field F. As noted in Section 2, this class of rings deserves a special attention when treating right $\diamond$-ideals.

The present section still uses rather elementary methods, but it is not as self-contained as Section 4. Let us now gather together a few results, mostly easy and known to specialists, that shall be needed in the proof of the main theorem. For some of them we shall provide a short proof, while for the others we shall just give appropriate references.

Remark 4.1. Suppose that $a, b, c \in R$ are such that $a \neq 0$ and $arb + cra = 0$ for all $r \in R$. We claim that then $b = -c$. Indeed, our assumption implies $arbsa = -c(ras)a = ar(asb) = -arcsa$, so that $aR(b+c)Ra = \{0\}$, which clearly yields $b = -c$. Actully, a well-known result of Martindale [5] tells us that much more is true, namely, that b and a are linearly dependent over the extended centroid of R, but this fact will not be needed.

Remark 4.2. If $a \in R$ is such that $[a,[a,r]] = 0$ for all $r \in R$, then $a \in Z$. This is well-known and easy to see. Indeed, first applying $2[a,r][a,s] = [a,[a,rs]] - r[a,[a,s]] - [a,[a,r]]s$ we conclude that $[a,r][a,s] = 0$ for all $r, s \in R$. Replacing s by sr in this identity and using $[a,sr] = [a,s]r + s[a,r]$ we then get $[a,r]R[a,r] = \{0\}$ for all $r \in R$. But then $[a,r] = 0$ by the primeness of R.

Remark 4.3. Suppose that there exists a nonzero element $a \in R$ such that $ak + ka = 0$ for all $k \in K$. (A concrete example when this occurs is when $R = M_2(F)$ with transpose involution and a is any matrix of the form $\begin{bmatrix} b & c \\ c & -b \end{bmatrix}$, $b, c \in F$.) We claim that in this case R satisfies S_4. First observe that our assumption implies that a commutes with each element in K^2. In particular, $a[k,l] = [k,l]a$ for all $k, l \in K$. But on the other hand, since $[k,l]$ is also skew-symmetric, we must have $a[k,l] + [k,l]a = 0$. Since R is assumed to have characteristic different from 2, it follows that

$[K,K]a = \{0\}$. Consequently, $[[K,K],K]a = \{0\}$ which together with $[K,K]a = \{0\}$ implies $[K,K]Ka = \{0\}$. But then [1, Lemma 2.2] tells us that $[K,K] = \{0\}$. Now it follows easily from [6, Lemma 5.5] that R must satisfy S_4.

Remark 4.4. Suppose that $k^2 \in Z$ for any $k \in K$. Then it follows from [4, Lemma 10] that R satisfies S_4. We remark that this situation occurs in $M_2(F)$ with either transpose or symplectic involution, as well as in the ring of quaternions.

The last auxiliary result, due to Lee and Lee [3], is deeper.

Remark 4.5. Suppose that $a_1, \dots, a_n \in S$ and $b_1, \dots, b_n \in R$ are such that $\sum_{i=1}^n a_i s b_i = 0$ for all $s \in S$. Then $\sum_{i=1}^n a_i r b_i = 0$ for all $r \in R$ [3, Theorem 2]. It is clear from the proof (as well as from the statement of [3, Theorem 1]) that this conclusion holds true if some of the summands in $\sum_{i=1}^n a_i s b_i$ is equal to sb_i, even when R does not contain 1.

We are now ready to consider right $\diamond$-ideals.

Theorem 4.6. *Let R be a prime $*$-ring of characteristic not 2. Suppose that R contains a right $\diamond$-ideal J which is neither contained in the center of R nor contains a nonzero $*$-ideal of R. Then R satisfies the standard polynomial identity S_4.*

Proof. A straightforward verification shows that the following identities

$$\begin{aligned} r([x^*,y]+[y^*,x]) &= x\diamond ry + y\diamond rx + x\diamond yr^* + y\diamond xr^* \\ &\quad - (x\diamond r)\diamond y - (y\diamond r)\diamond x, \\ ([x^*,y]+[y^*,x])r &= -x\diamond y^*r - y\diamond x^*r - x\diamond r^*y^* - y\diamond r^*x^* \\ &\quad + (x\diamond r^*)\diamond y^* + (y\diamond r^*)\diamond x^* \end{aligned}$$

hold true. This is our clue, as we shall see.

Pick $x, y \in J$ and set $\alpha = [x^*,y]+[y^*,x]$. Note that $\alpha \in S$. The above identities show that $R\alpha \subseteq J$ and $\alpha R \subseteq J$. Therefore, $z\alpha r = z\diamond \alpha r + \alpha r z^*$, $z \in J$, $r \in R$, implies $J\alpha R \subseteq J$. This, together with $r\alpha z = r\alpha \diamond z + z\alpha r^*$, yields $R\alpha J \subseteq J$, which in turn implies $R\alpha J\alpha R \subseteq J\alpha R \subseteq J$. Thus, J contains the ideal $R\alpha J\alpha R$. Our goal, however, is to find a $*$-ideal inside J.

We claim that $R\alpha(J+J^*)\alpha R \subseteq J$. Indeed, $z^*\alpha r = z^*\alpha\diamond r + r\alpha z$ shows that $J^*\alpha R \subseteq J$, which, together with $R\alpha J \subseteq J$, readily gives $R\alpha J^*\alpha R \subseteq J$.

According to our assumption, the $*$-ideal $R\alpha(J+J^*)\alpha R$ contained in J, must be zero. In particular, $\alpha J\alpha = 0$.

Pick $k \in K$ and $r \in R$. Then $k\alpha r + r\alpha k = k\alpha\diamond r \subseteq R\alpha\diamond R \subseteq J\diamond R \subseteq J$. Therefore, $\alpha(k\alpha r + r\alpha k)\alpha = 0$, that is, $(\alpha k\alpha)r\alpha + \alpha r(\alpha k\alpha) = 0$ for all $r \in R$, $k \in K$. But then Remark 4.1 implies that $\alpha k\alpha = -\alpha k\alpha$ and so $\alpha K\alpha = \{0\}$.

Further, $\alpha J\alpha = 0$ implies $\alpha(z\diamond r)\alpha = 0$ for all $z \in J$, $r \in R$. That is, $\alpha zr\alpha - \alpha rz^*\alpha = 0$. Again applying Remark 4.1 we infer that $\alpha z = z^*\alpha$ for all $z \in J$. Letting z to be $k\alpha s + s\alpha k$ with $k \in K$, $s \in S$ (namely, as noted above, every such element belongs to J) and using $\alpha k\alpha = 0$, we arrive at $(\alpha s\alpha)k + k(\alpha s\alpha) = 0$. We are now in a position to use Remark 4.3. Thus, either R satisfies S_4, which is the desired conclusion, or $\alpha s\alpha = 0$ for all $s \in S$. Note that the latter together with $\alpha K\alpha = \{0\}$ implies $\alpha = 0$.

Therefore, without loss of generality we may assume that $\alpha = 0$, that is, $[x^*, y] + [y^*, x] = 0$ for any $x, y \in J$ (incidentally, this is clearly equivalent to the condition that every element x in J is normal, i.e., it commutes with x^*). Pick any $a \in S \cap J$. Then $a \diamond r \in J$ for any $r \in R$, and so $[(a\diamond r)^*, a] + [a, a\diamond r] = 0$. That is, $[a, [a, r + r^*]] = 0$ for all $r \in R$, meaning that $[a, [a, S]] = \{0\}$. Now, Remark 4.5 yields $[a, [a, R]] = \{0\}$, which in turn implies (Remark 4.2) that $a \in Z$. Thus, we proved that $S \cap J \subseteq Z$.

Suppose that $K \cap J = \{0\}$. If there exists a nonzero $a \in S \cap J \subseteq Z$, then $x \diamond a$ lies in $K \cap J$ for any $x \in J$ and so it must be 0. That is, $a(x - x^*) = 0$. Since nonzero central elements in a prime ring cannot be zero divisors, it follows that $x = x^*$ for every $x \in J$, that is, $J = S\cap J \subseteq Z$, contrary to the assumption. On the other hand, if $S\cap J$ is $\{0\}$ too, then for any $x \in J$ we have $K \cap J \ni x\diamond s = 0$ for every $s \in S$, and $S \cap J \ni x\diamond k = 0$ for any $k \in K$. Hence $x \diamond r = 0$ for all $r \in R$, that is, $xr = rx^*$. But then $[x, r]s = x(rs) - r(xs) = rsx^* - rsx^* = 0$ for all $r, s \in R$, which, since R is prime, yields $x \in Z$. Moreover, $xr = rx^*$ now implies $x = x^*$ and so $x \in S \cap J = \{0\}$. Thus, in this case, $J = \{0\}$, which, of course, also contradicts the assumption. Therefore, $K \cap J \neq \{0\}$.

Now pick a nonzero $b \in K \cap J$. Then $bk + kb = b \diamond k \in S \cap J \subseteq Z$ for any $k \in K$. But then

$$(bk + kb)k^2 = (bk + kb)k^2 - k(bk + kb)k + k^2(bk + kb) = bk^3 + k^3b \in Z$$

since k^3 also lies in K. Consequently, $(bk + kb)[k^2, R] = \{0\}$ for all $k \in K$. Again using the fact that nonzero central elements in a prime ring are not zero divisors it follows that for each $k \in K$, either $k^2 \in Z$ or $bk + kb = 0$. We claim that one of these two conditions is fulfilled for all $k \in K$. If this were not true, there would be $k, l \in K$ such that $k^2 \notin Z$ and $bl + lb \neq 0$. Hence $bk + kb = 0$ and $l^2 \in Z$. Now consider $k + l, k - l \in K$. Clearly, $b(k + l) + (k + l)b \neq 0$ and $b(k - l) + (k - l)b \neq 0$, so that both $(k + l)^2 = k^2 + (kl + lk) + l^2$ and $(k - l)^2 = k^2 - (kl + lk) + l^2$ belong to Z. But then, since $l^2 \in Z$, it follows that $2k^2 \in Z$, which in turn implies $k^2 \in Z$, contrary to the assumption. Thus we proved that either $bk + kb = 0$ for all $k \in K$ or $k^2 \in K$ for all $k \in K$. In view of Remarks 4.3 and 4.4, each of the two conditions implies that R satisfies S_4, proving the theorem (as a matter of fact, since $b \in K$, it is easy to see that the first condition cannot really occur, but this does not effect the proof).

Assuming that a prime ring R is noncommutative, it is possible to slightly sharpen the conclusion of Theorem 4.6. Namely, in that case any right $\diamond$-ideal J of R which is contained in Z must consist of symmetric elements only. Indeed, given $x \in J$, we have $r(x - x^*) = x \diamond r \in J \subseteq Z$ from which we easily infer that $x = x^*$. Hence we have

Corollary 4.7. *Let R be a noncommutative simple $*$-ring of characteristic not 2. Suppose that R does not satisfy S_4. If J is a right $\diamond$-ideal of R, then either $J = R$ or $J \subseteq Z \cap S$.*

In view of our discussion in Section 3, it is now natural to pose the following question: Under which condition a $*$-ideal I of a $*$-ring R satisfies $I \diamond R = I$? Molnár showed that this holds true when $R = \mathcal{B}(H)$ [7, Proposition]. Of course, this is not true in every ring, since even a simpler question whether $R \diamond R$ is equal to R does not always have a positive answer. But actually, at least in unital rings, these two questions are equivalent.

Proposition 4.8. *Let R be a $*$-ring with 1. Suppose that $R \diamond R = R$. Then $I \diamond R = I$ for any $*$-ideal I of R.*

Proof. According to our assumption, we have $1 = \sum_{i=1}^{n} r_i \diamond s_i$ for some $r_i, s_i \in R$. Let I be any ideal of R and pick $x \in I$. Then

$$x = x1 = \sum_{i=1}^{n} x(r_i \diamond s_i) = \sum_{i=1}^{n} (xr_i \diamond s_i - x \diamond s_i r_i^*) \in I \diamond R.$$

Thus, $I \subseteq I \diamond R$ holds for any ideal I. If I is a $*$-ideal, then the converse, $I \diamond R \subseteq I$, is trivial.

We recall that some sufficient conditions for $R \diamond R$ to be equal to R are given in Corollaries 3.10 and 3.11. In particular, these results show that $\mathcal{B}(H)$ (with H at least 2-dimensional) has this property.

Finally we remark that the "left analogue" of Proposition 4.8 does not hold. Consider a concrete example given in Example 2.4. It is easy to check that $R \diamond R = R$, but the $*$-ideal L is such that $R \diamond L = \{0\} \neq L$.

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