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Stability of a sum form functional equation on open domain

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Abstract. In this note we prove that the sum form functional equation $\sum_{i=1}^{n} \varphi(p_i) = d$, which holds for all complete n-ary ($n \geq 3$ is fixed) probability distributions $(p_1, \ldots, p_n)$ with positive probabilities and for some $d \in \mathbb{R}$, is stable.

1. Introduction

For fixed natural number $n \geq 3$ and real number $c > 0$ define the sets Γ_n^0 and Δ_c by

$$\Gamma_n^0 = \left\{ (p_1, \ldots, p_n) \in \,]\,0,1\,[^n : \sum_{i=1}^{n} p_i = 1 \right\}$$

and

$$\Delta_c = \left\{ (x,y) \in \mathbb{R}^2 : x,\ y,\ x+y \in \,]\,0,c\,[\right\},$$

respectively.

The functional equation

$$\sum_{i=1}^{n} \varphi(p_i) = d, \qquad (p_1, \ldots, p_n) \in \Gamma_n^0 \tag{1}$$

where $n \geq 3$ is fixed integer, $d \in \mathbb{R}$ is fixed and the real-valued unknown function φ is defined on the open unit interval $]0,1[$ is solved by Losonczi in [3] by proving the following

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Theorem 1. *Let $n \geq 3$ be fixed integer and $d \in \mathbb{R}$ be fixed. Suppose that $\varphi :]0,1[\to \mathbb{R}$ satisfies equation (1) for all $(p_1, \ldots, p_n) \in \Gamma_n^0$. Then there exists an additive function A (that is a function $A : \mathbb{R} \to \mathbb{R}$ satisfying the equation $A(x+y) = A(x) + A(y)$ for all $x, y \in \mathbb{R}$) such that*

$$\varphi(p) = A(p) - \frac{A(1) - d}{n}, \qquad p \in]0,1[\,.$$

In this note we prove the stability of (1) in the following sense: If

$$\Big| \sum_{i=1}^{n} \varphi(p_i) - d \Big| \leq \varepsilon, \quad (p_1, \ldots, p_n) \in \Gamma_n^0 \tag{2}$$

for a function $\varphi :]0,1[\to \mathbb{R}$, a natural number $n \geq 3$ and fixed real numbers d and $0 \leq \varepsilon$ then there exists a real number K such that

$$\left| \varphi(p) - A(p) + \frac{A(1) - d}{n} \right| \leq K\varepsilon, \qquad p \in]0,1[$$

with some additive function $A : \mathbb{R} \to \mathbb{R}$. For the general problem of the stability of functional equations in Hyers–Ulam sense we refer to the survey paper of Hyers and Rassias [1].

A similar problem has been solved on closed domain by Maksa in [4] (therein $\overline{\Gamma_n^0}$ and $[0,1]$ were considered instead of Γ_n^0 and $]0,1[$, respectively) and the result was applied by Kocsis–Maksa in [2] to prove that a sum form functional equation arising in a characterization of an information measure is stable. The difficulty of the open domain case lies in the fact that the zero probabilities are excluded.

2. The stability of the Cauchy equation on open square

The stability of the Cauchy equation on restricted open domains has been investigated by Skof in [5] and by Jacek and Józef Tabor in [6].

Let $0 \leq \delta \in \mathbb{R}$ and $I \subset \mathbb{R}$ be an interval of positive length. We say that a function $f : I \to \mathbb{R}$ is δ-additive (see [5] and [6]) if

$$\big| f(x+y) - f(x) - f(y) \big| \leq \delta \tag{3}$$

for all $x, y, x+y \in I$. It is proved in [6] (see Theorem 1) that in the case $0 \in \operatorname{cl} I$ for each $0 \leq \delta$ and for each δ-additve function $f : I \to \mathbb{R}$ there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ such that $\big| f(x) - A(x) \big| \leq \delta$ for all $x \in I$. This is the main tool in proving the following

Lemma. *Let $b, c \in \,]0, \infty[$ and $0 \leq \delta \in \mathbb{R}$ be fixed. Suppose that the function $f :]-2b, 2c[\, \to \mathbb{R}$ satisfies the inequality* (3) *for all $(x, y) \in \,]-b, c[^2$. Then there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ such that*

$$\left|f(x) - A(x)\right| \leq 5\delta$$

holds for all $x \in \,]-2b, 2c[$.

Proof. First we prove that the conditions of the lemma imply that the function f is 5δ-additive on the interval $]-2b, 2c[$ and next we apply Tabor's result.

Let $x, y, x + y \in \,]-2b, 2c[$. Then $\frac{x}{2}, \frac{y}{2}, \frac{x+y}{2} \in \,]-b, c[$ and

$$\begin{aligned} \left|f(x+y) - f(x) - f(y)\right| &\leq \left|f(x+y) - 2f\left(\frac{x+y}{2}\right)\right| \\ &+ \left|2f\left(\frac{x+y}{2}\right) - f\left(\frac{x}{2}\right) - f\left(\frac{y}{2}\right)\right| \\ &+ \left|2f\left(\frac{x}{2}\right) - f(x)\right| + \left|2f\left(\frac{y}{2}\right) - f(y)\right| \leq 5\delta. \end{aligned}$$

Applying Theorem 1 in [6] to the function f with $I = \,]-2b, 2c[$ and $G = \mathbb{R}$ we get the statement of the lemma. $\square$

3. The main result

Theorem 2. *Let $n \geq 3$ be a fixed integer and $0 \leq \varepsilon \in \mathbb{R}$, $d \in \mathbb{R}$ be fixed. Suppose that the inequality* (2) *holds for all $(p_1, \dots, p_n) \in \Gamma_n^0$. Then there exists a real number K and an additive function $A : \mathbb{R} \to \mathbb{R}$ such that*

$$\left|\varphi(p) - A(p) + \frac{A(1) - d}{n}\right| \leq K\varepsilon$$

for all $p \in \,]0, 1[$.

Proof. With the notation $\psi(p) = \varphi(p) - \frac{d}{n}$, $p \in \,]0, 1[$ (2) reduces to

$$\left|\sum_{i=1}^{n} \psi(p_i)\right| \leq \varepsilon, \quad (p_1, \dots, p_n) \in \Gamma_n^0. \tag{4}$$

Define the function h on $]0,1[$ by $h(x) = \psi(x) - 2\psi(\frac{1}{2n})$. We show that

$$(5) \qquad \bigl|h(x+y) - h(x) - h(y)\bigr| \le L\varepsilon, \quad (x,y) \in \bigl]0, \tfrac{1}{2}\bigr[^2,$$

where

$$L = \begin{cases} \dfrac{17}{8} & \text{if } n = 3, \\ 16 & \text{if } n > 3. \end{cases}$$

The case $n = 3$. Let $(x,y) \in \Delta_1$. Substituting $p_1 = x$, $p_2 = y$, $p_3 = 1 - x - y$ in (4) we get

$$(6) \qquad \bigl|\psi(x) + \psi(y) + \psi(1-x-y)\bigr| \le \varepsilon.$$

With $x = \frac{1}{2}$ and with $y = \frac{1}{2}$ (6) implies

$$(7) \qquad \left|\psi\left(\frac{1}{2}\right) + \psi(y) + \psi\left(\frac{1}{2} - y\right)\right| \le \varepsilon, \qquad y \in \bigl]0, \tfrac{1}{2}\bigr[$$

and

$$(8) \qquad \left|\psi(x) + \psi\left(\frac{1}{2}\right) + \psi\left(\frac{1}{2} - x\right)\right| \le \varepsilon, \qquad x \in \bigl]0, \tfrac{1}{2}\bigr[,$$

respectively. Adding the inequalities (6), (7) and (8) up and applying the triangle inequality we have that

$$\left|\psi(1-x-y) - \psi\left(\frac{1}{2} - x\right) - \psi\left(\frac{1}{2} - y\right) - 2\psi\left(\frac{1}{2}\right)\right| \le 3\varepsilon, \qquad (x,y) \in \bigl]0, \tfrac{1}{2}\bigr[^2.$$

Replacing here x and y by $\frac{1}{2} - x$ and $\frac{1}{2} - y$, respectively we obtain

$$(9) \qquad \left|\psi(x+y) - \psi(x) - \psi(y) - 2\psi\left(\frac{1}{2}\right)\right| \le 3\varepsilon, \quad (x,y) \in \bigl]0, \tfrac{1}{2}\bigr[^2.$$

With the substitutions $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{2}$, $p_3 = \frac{1}{3}$ and $p_1 = p_2 = p_3 = \frac{1}{3}$ in (4) we get that $|\psi(\frac{1}{6}) + \psi(\frac{1}{2}) + \psi(\frac{1}{3})| \le \varepsilon$ and $3|\psi(\frac{1}{3})| \le \varepsilon$, respectively. Thus, by the triangle inequality,

$$(10) \qquad \left|\psi\left(\frac{1}{6}\right) + \psi\left(\frac{1}{2}\right)\right| \le \frac{4}{3}\varepsilon$$

Now (5) follows from (9),(10) and the definition of h with $n = 3$.

The case $n > 3$. Let $c \in]0,1[$ and $(x, y) \in \Delta_c$. With the substitutions $p_1 = x$, $p_2 = y$, $p_3 = c-x-y$, $p_4 = \cdots = p_n = \frac{1-c}{n-3}$ and $p_1 = p_2 = p_3 = \frac{c}{3}$, $p_4 = \cdots = p_n = \frac{1-c}{n-3}$ in (4) we get that

$$\left|\psi(x) + \psi(y) + \psi(c-x-y) + (n-3)\psi\left(\frac{1-c}{n-3}\right)\right| \le \varepsilon,$$

and

$$\left|3\psi\left(\frac{c}{3}\right) + (n-3)\psi\left(\frac{1-c}{n-3}\right)\right| \le \varepsilon,$$

respectively. Applying the triangle inequality we have

$$\left|\psi(x) + \psi(y) + \psi(c-x-y) - 3\psi\left(\frac{c}{3}\right)\right| \le 2\varepsilon, \quad (x, y) \in \Delta_c, \tag{11}$$

while with the substitutions $p_1 = x+y$, $p_2 = c-x-y$, $p_3 = \cdots = p_n = \frac{1-c}{n-2}$ and $p_1 = p_2 = \frac{c}{2}$, $p_3 = \cdots = p_n = \frac{1-c}{n-2}$ we get that

$$\left|\psi(x+y) + \psi(c-x-y) + (n-2)\psi\left(\frac{1-c}{n-2}\right)\right| \le \varepsilon$$

and

$$\left|2\psi\left(\frac{c}{2}\right) + (n-2)\psi\left(\frac{1-c}{n-2}\right)\right| \le \varepsilon,$$

respectively. Applying the triangle inequality again we have

$$\left|\psi(x+y) - \psi(c-x-y) - 2\psi\left(\frac{c}{2}\right)\right| \le 2\varepsilon, \quad (x, y) \in \Delta_c. \tag{12}$$

The inequalities (11) and (12) imply that

$$\left|\psi(x+y) - \psi(x) - \psi(y) - 2\psi\left(\frac{c}{2}\right) + 3\psi\left(\frac{c}{3}\right)\right| \le 4\varepsilon, \quad (x, y) \in \Delta_c. \tag{13}$$

Now we show that the inequality (13) holds for all $(x, y) \in \Delta_1$ with 12ε instead of 4ε on the right hand side.

Let $d \in]0,1[$. Then, by (13),

$$\left|\psi(x+y) - \psi(x) - \psi(y) - 2\psi\left(\frac{d}{2}\right) + 3\psi\left(\frac{d}{3}\right)\right| \le 4\varepsilon, \quad (x, y) \in \Delta_d.$$

moreover for a fixed $(x,y) \in \Delta_{\min\{c,d\}}$ the triangle inequality implies that

$$\left|2\psi\left(\frac{c}{2}\right) - 3\psi\left(\frac{c}{3}\right) - 2\psi\left(\frac{d}{2}\right) + 3\psi\left(\frac{d}{3}\right)\right| \le 8\varepsilon, \quad c,d \in \left]0,1\right[. \tag{14}$$

Now let $(x,y) \in \Delta_1$. Then there exists $d \in \left]0,1\right[$ such that $(x,y) \in \Delta_d$. Thus by (13) and (14), we obtain that

$$\begin{aligned} &\left|\psi(x+y) - \psi(x) - \psi(y) - 2\psi\left(\frac{c}{2}\right) + 3\psi\left(\frac{c}{3}\right)\right| \\ &\quad\le \left|\psi(x+y) - \psi(x) - \psi(y) - 2\psi\left(\frac{d}{2}\right) + 3\psi\left(\frac{d}{3}\right)\right| \\ &\qquad + \left|2\psi\left(\frac{d}{2}\right) - 3\psi\left(\frac{d}{3}\right) - 2\psi\left(\frac{c}{2}\right) + 3\psi\left(\frac{c}{3}\right)\right| \\ &\quad\le 12\varepsilon, \qquad (x,y) \in \Delta_1. \end{aligned} \tag{15}$$

With $p_1 = \cdots = p_n = \frac{1}{n}$ and with $p_1 = \frac{1}{2n}, p_2 = \frac{3}{2n}, p_3, \ldots, p_n = \frac{1}{n}$ (4) implies that

$$\left|n\psi\left(\frac{1}{n}\right)\right| \le \varepsilon \quad \text{and} \quad \left|\psi\left(\frac{1}{2n}\right) + \psi\left(\frac{3}{2n}\right) + (n-2)\psi\left(\frac{1}{n}\right)\right| \le \varepsilon, \tag{16}$$

respectively, that is,

$$\left|\psi\left(\frac{1}{2n}\right) + \psi\left(\frac{3}{2n}\right)\right| \le \left(1 + \frac{n-2}{n}\right)\varepsilon. \tag{17}$$

The inequality (15) with $c = \frac{3}{n}$, (16) and the triangle inequality yield

$$\begin{aligned} &\left|\psi(x+y) - \psi(x) - \psi(y) - 2\psi\left(\frac{3}{2n}\right)\right| \\ &\quad\le 12\varepsilon + 3\left|\psi\left(\frac{1}{n}\right)\right| \\ &\quad\le \left(12 + \frac{3}{n}\right)\varepsilon, \qquad (x,y) \in \Delta_1. \end{aligned} \tag{18}$$

The inequalities (17) and (18) imply that

$$\begin{aligned}
&\left|\psi(x+y)-\psi(x)-\psi(y)+2\psi\left(\frac{1}{n}\right)\right| \\
&\quad\le \left|\psi(x+y)-\psi(x)-\psi(y)-2\psi\left(\frac{3}{2n}\right)\right| + 2\left|\psi\left(\frac{3}{2n}\right)+\psi\left(\frac{1}{2n}\right)\right| \\
&\quad\le \left(12+\frac{3}{n}+2+2\frac{n-2}{n}\right)\varepsilon \le 16\varepsilon
\end{aligned}$$

for all $(x,y)\in\Delta_1\supset\,]\,0,\frac{1}{2}\,[^2$, that is, (5) holds also for $n>3$.

Define the function g on $]-\frac{1}{n},1-\frac{1}{n}[$ by $g(t)=\psi(t+\frac{1}{n})$. We show that

$$\text{(19)}\qquad \left|g(\xi+\eta)-g(\xi)-g(\eta)\right|\le 3L\varepsilon,\quad (\xi,\eta)\in\left]-\frac{1}{2n},\frac{1}{2}-\frac{1}{2n}\right[^2.$$

It follows from (5) that

$$\text{(20)}\quad \left|h\left(\xi+\eta+\frac{1}{n}\right)-h\left(\xi+\frac{1}{2n}\right)-h\left(\eta+\frac{1}{2n}\right)\right|\le L\varepsilon,$$
$$(\xi,\eta)\in\left]-\frac{1}{2n},\frac{1}{2}1-\frac{1}{2n}\right[^2.$$

With $\eta=0$, $\xi=0$ (20) yields

$$\left|h\left(\xi+\frac{1}{n}\right)-h\left(\xi+\frac{1}{2n}\right)-h\left(\frac{1}{2n}\right)\right|\le L\varepsilon$$

and

$$\left|h\left(\eta+\frac{1}{n}\right)-h\left(\frac{1}{2n}\right)-h\left(\eta+\frac{1}{2n}\right)\right|\le L\varepsilon,$$

respectively. The last three inequalities and the triangle inequality imply that

$$\left|h\left(\xi+\eta+\frac{1}{n}\right)-h\left(\xi+\frac{1}{n}\right)-h\left(\eta+\frac{1}{n}\right)+2h\left(\frac{1}{2n}\right)\right|\le 3L\varepsilon,$$

that is,

$$\left|\psi\left(\xi+\eta+\frac{1}{n}\right)-\psi\left(\xi+\frac{1}{n}\right)-\psi\left(\eta+\frac{1}{n}\right)\right|\le 3L\varepsilon,$$
$$(\xi,\eta)\in\left]-\frac{1}{2n},\frac{1}{2}-\frac{1}{2n}\right[^2.$$

Thus, by the definition of g, we obtain (19). Applying our Lemma to the function g in (19) we get that there exists an additive function $A:\mathbb{R}\to\mathbb{R}$ such that

$$\big|g(x)-A(x)\big|\le 15L\varepsilon,\quad x\in\left]-\frac{1}{n},1-\frac{1}{n}\right[. \tag{21}$$

Finally let $x\in\,]0,1[\,$. Then there exists $(\xi,\eta)\in\,]-\frac{1}{2n},1-\frac{1}{2n}[^2$ such that $x=\xi+\eta+\frac{1}{n}$. By the definition of the function ψ, h and g and by (21) we have that

$$\left|\varphi(x)-A(x)+\frac{A(1)-d}{n}\right|=\left|\psi(x)-A(x)+\frac{A(1)}{n}\right|$$
$$=\left|\psi\left(\xi+\eta+\frac{1}{n}\right)-A(\xi+\eta)\right|=\big|g(\xi+\eta)-A(\xi+\eta)\big|\le 15L\varepsilon. \qquad \square$$

Remark. It is clear from the proof that the inequality in Theorem 2 holds if $K=\frac{255}{8}$ in the case $n=3$ and if $K=220$ in the case $n>3$. It would be interesting to know the smallest possible value of K.

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