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On a theorem of H. Daboussi

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Abstract. The main result is a generalization of Daboussi's theorem: If f is a uniformly summable multiplicative function with a void Bohr–Fourier spectrum, and if g is a q-multiplicative function with $|g(n)| = 1$ for all n, then we have

$$\sum_{n\le x} f(n)g(n) = o(x) \qquad (x \to \infty).$$

1. Introduction

Let $e(\alpha) = \exp(2\pi i\alpha)$.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the set of natural numbers, integers, real and complex numbers, respectively.

Furthermore, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $q \ge 2$ and let $n = \sum \varepsilon_j(n)q^j$ be the q-ary expansion of $n \in \mathbb{N}_0$ with digits $\varepsilon_j(n) \in \mathbb{A} = \{0, 1, \dots, q-1\}$. A function $g : \mathbb{N}_0 \to \mathbb{C}$ is called q-multiplicative if $g(0) = 1$, and

$$g(n) = \prod_{j=0}^{\infty} g(\varepsilon_j(n)q^j).$$

Let $\bar{\mathcal{M}}_q$ be the class of q-multiplicative functions with modulus 1: i.e. $g \in \bar{\mathcal{M}}_q$, if g is q-multiplicative and $|g(n)| = 1$ $(n \in \mathbb{N}_0)$.

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Similarly, $f : \mathbb{N}_0 \to \mathbb{R}$ is called q-additive if $f(0) = 0$, and

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n)q^j).$$

A sequence x_n $(n = 1, 2, \dots)$ of real numbers is said uniformly distributed mod 1, if

$$\lim_{M\to\infty} \frac{1}{M} \# \{n \le M \mid \{x_n\} \subseteq (\alpha, \beta]\} = \beta - \alpha,$$

for all $0 \le \alpha < \beta \le 1$, where $\{y\}$ denotes the fractional part of y.

A classical theorem of H. Weyl asserts that x_n is uniformly distributed mod 1 if and only if for every $k \in \mathbb{Z}$,

$$\frac{1}{M} \sum_{n=1}^{M} e(kx_n) \to 0 \quad \text{as} \quad M \to \infty.$$

A function $f : \mathbb{N} \to \mathbb{C}$ is called uniformly summable, if

$$C(K) := \sup_{x\ge 1} \frac{1}{x} \sum_{\substack{n\le x\\ |f(n)|\ge K}} |f(n)| \to 0 \quad \text{as} \quad K \to \infty.$$

The notion of uniformly summable arithmetical functions was introduced and studied by K.-H. Indlekofer in [11]. The space of uniformly summable arithmetical functions can be considered as the closure of the l_1 space.

Let f be a uniformly summable function. We say that $\alpha \in \mathbb{R}$ belongs to its Bohr–Fourier spectrum, if

$$\limsup_{x\to\infty} \frac{1}{x} \left| \sum_{n\le x} f(n)e(-n\alpha) \right| > 0.$$

This notion originally was introduced for the space of almost periodic (arithmetical) functions and later extended to wider spaces.

According to a nice theorem of H. Daboussi [1], if f is a multiplicative function, $|f(n)| \le 1$, then

$$x^{-1} \sum_{n\le x} f(n)e(n\alpha) \to 0 \qquad (x \to \infty) \tag{1.1}$$

for each irrational α.

There are several generalizations of this theorem. (See e.g. [2–9].)

Let $\mathcal{T}$ be that class of arithmetical functions t, for which for each $K > 0$ there exist suitable prime numbers $p_1 < p_2 < \cdots < p_R$ such that $\sum_{j=1}^{R} 1/p_j > K$, and

$$\frac{1}{x}\sum_{m<x} e(t(p_i m) - t(p_j m)) \to 0 \quad (x \to \infty) \tag{1.2}$$

for every $i \neq j$.

In our paper [7] we proved

Theorem A. *Let f be an arbitrary uniformly summable multiplicative function, $t \in \mathcal{T}$. Then*

$$\lim \frac{1}{x}\sum_{n\le x} f(n)e(t(n)) = 0.$$

In a recent paper [10] we proved the following theorem which we quote now as

Lemma 1. *Let $1 \le a < b$ $(a,b) = 1$ $(ab,q) = 1$, $g \in \bar{\mathcal{M}}_q$.
If*

$$\overline{\lim_{x\to\infty}}\left|\frac{1}{x}\sum_{n<x} g(an)\bar{g}(bn)\right| > 0,$$

then there exists such an $r \in \mathbb{N}$ for which

$$\sum_{j=0}^{\infty}\sum_{c\in\mathbb{A}} \mathrm{Re}\left(1 - e\left(\frac{-rcq^j}{b-a}\right) g(cq^j)\right) < \infty.$$

Hence, and from Theorem A we deduce

Theorem 1. *Assume that f is a uniformly summable multiplicative function, $g \in \bar{\mathcal{M}}_q$, and that*

$$\limsup_{x} \frac{1}{x}\left|\sum_{n\le x} f(n)g(n)\right| > 0.$$

Then $g(n)$ can be written as $g(n) = e\left(\frac{r}{D}\right) h(n)$ with a suitable rational number $\frac{r}{D}$ and with a function $h \in \bar{\mathcal{M}}_q$ for which

$$\sum_{j=0}^{\infty} \sum_{c \in \mathbb{A}} \operatorname{Re}(1 - h(cq^j)) < \infty \tag{1.4}$$

holds.

If the Bohr–Fourier spectrum of f is empty, then

$$\frac{1}{x} \sum_{n \le x} f(n) g(n) \to 0$$

for each $g \in \bar{\mathcal{M}}_q$.

Remark. Since $e(\alpha n) \in \bar{\mathcal{M}}_q$ for each $\alpha \in \mathbb{R}$, Theorem 1 contains the theorem of Daboussi.

2. Proof of Theorem 1

Let us write $g(n)$ as $e(t(n))$ where $t(cq^j) \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, and is extended as a q-additive function. For $x \in \mathbb{R}$ let $\|x\|$ the distance of x to the closest integer.

If $p_1 \ne p_2$ primes, $(p_1 p_2, q) = 1$, then either (1.2) holds, or by Lemma 1 there exists an integer $r = r(p_1, p_2)$, $|r| \le |p_2 - p_1|$, such that

$$\sum_{j=1}^{\infty} \sum_{c \in \mathbb{A}} \left\| \frac{rcq^j}{p_2 - p_1} - t(cq^j) \right\|^2 < \infty.$$

It is clear that no more than one rational number $\frac{k}{l}$ may exist in $[0, 1]$ for which

$$\sum_{j=1}^{\infty} \sum_{c \in \mathbb{A}} \left\| \frac{k}{l} cq^j - t(cq^j) \right\|^2 < \infty. \tag{2.1}$$

Thus, either (1.2) holds for each prime pairs $p_1, p_2 > q$, $p_1 \ne p_2$, or (2.1) holds. Then (1.4) holds with $h(n) := e\left(-\frac{k}{l} n\right) g(n)$.

Assume that

$$\limsup_{x \to \infty} \frac{1}{x} \left| \sum_{n \le x} f(n) e\left(\frac{k}{l} n\right) h(n) \right| > 0. \tag{2.2}$$

Let $R \geq 1$ be an arbitrary integer,

$$h_R(n) = \prod_{j=0}^{R} h(\varepsilon_j(n)q^j), \quad s_R(n) = \prod_{j=R+1}^{\infty} h(\varepsilon_j(n)q^j).$$

Let $\lambda(n)$ be defined as a q-additive function, where $\lambda(cq^l)$ is defined as the fractional part of $t(cq^l) - \frac{k}{l}cq^l$.

Let

$$M_{R,N} := \frac{1}{q} \sum_{j=R}^{R+N-1} \sum_{c \in \mathbb{A}} \lambda(cq^j)$$

$$D_{R,N}^2 = \frac{1}{q} \sum_{j=R}^{R+N-1} \sum_{c \in \mathbb{A}} \lambda^2(cq^j),$$

$\xi_{R,N} := e(M_{R,N})$. Since $|1 - e(\eta)| \leq c_1|\eta|$, we have

$$\sum_{n < q^{R+N}} |1 - \bar{\xi}_{R,N} s_R(n)|^2 \leq cq^R \sum_{\nu < q^N} (\lambda(\nu q^R) - M_{R,N})^2.$$

We shall prove that the right hand side is less than $c_2 q^{R+N} D_{R,N}^2$. If we consider $\lambda(\nu q^R) - M_{R,N}$ as a random variable defined on $\nu \in \{0,1,\ldots,q^N-1\}$, then it is the sum of the independent random variables η_l $(l = 0, \ldots, N-1)$, where

$$P(\eta_l = \lambda(cq^{l+R}) - m_l) = 1/q \quad (c \in \mathbb{A}), \qquad m_l = \frac{1}{q} \sum_{c \in \mathbb{A}} \lambda(cq^{l+R}).$$

Thus the right hand side is less than $c_2 q^{R+N}$ times $\sum D^2 \eta_l \leq c_2 D_{R,N}^2$. Here c_2 is an absolute constant.

Since $D_{R,N}^2 \to 0$, if $R \to \infty$, $N \geq 1$, the inequality

$$\limsup_{x \to \infty} \frac{1}{x} \left| \sum_{n \leq x} f(n) e\left(\frac{k}{l} n\right) h_R(n) \right| > 0 \tag{2.3}$$

holds, if R is large enough.

Let us fix an R for which (2.3) holds. The function $h_R(n)$ is periodic mod q^N, therefore it can be expanded in a finite Fourier series:

$$h_R(n) = \sum_{j=0}^{q^R-1} d_j e\left(\frac{jn}{q^R}\right).$$

Then

$$\limsup_{x\to\infty} \frac{1}{x}\left|\sum_{n\le x} f(n)e\left(\left(\frac{k}{l}+\frac{j}{q^R}\right)n\right)\right| > 0$$

for some $j \in \{0,\dots,q^R-1\}$.

The theorem is proved.

3. Further remarks

From a theorem of Delange we know that for $g \in \bar{\mathcal{M}}_q$ the mean value

$$\frac{1}{x}\sum_{n<x} g(n)$$

tends to zero if and only if either

$$\sum_{c\in\mathbb{A}} g(cq^j) = 0$$

for some j, or

$$\sum_{j=0}^{\infty}\sum_{c\in\mathbb{A}} \operatorname{Re}(1-g(cq^j)) = \infty.$$

Hence, by using Weyl's criterion, the following assertion which we state now as Lemma 2 follows easily:

Lemma 2. *A q-additive function $\varphi : \mathbb{N}_0 \to \mathbb{R}$ is uniformly distributed* mod 1 *if and only if either for every $k \in \mathbb{N}$, there exists such a j for which*

$$\sum_{c\in\mathbb{A}} e(k\varphi(cq^j)) = 0,$$

or

$$\sum_{j=0}^{\infty}\sum_{c\in\mathbb{A}} \|\varphi(cq^j)\|^2 = \infty. \tag{3.1}$$

Hence we obtain

Lemma 3. *For a q-additive function* φ *the sequence* $\varphi(nq^R)$ $(n \in \mathbb{N}_0)$ *is uniformly distributed* mod 1 *for every* $R \in \mathbb{N}_0$, *if and only if the sum*

$$\sum_{j=0}^{\infty} \sum_{c \in \mathbb{A}} \|\varphi(cq^j)\|^2 \tag{3.2}$$

is divergent.

Proof. The divergence of (3.2) implies the uniform distribution mod 1 of $\varphi(nq^R)$ for every $R \in \mathbb{N}_0$.

Assume that (3.2) is convergent. Since $\|\varphi(cq^j)\| \to 0$ $(j \to \infty)$, therefore

$$\sum_{c \in \mathbb{A}} e(\varphi(cq^j)) = 0$$

cannot hold if $j \geq R$, R is large enough.

For such an R $\varphi(nq^R)$ $(n \in \mathbb{N}_0)$ due to Lemma 2 cannot be uniformly distributed mod 1. □

From Theorem 1 we obtain immediately

Theorem 2. *Assume that* φ *is q-additive and* $\varphi(nq^R)$ *is uniformly distributed* mod 1 *for every* $R \in \mathbb{N}_0$. *Then for each additive function* $F(n)$, *the sequence*

$$F(n) + \varphi(nq^R) \quad (n \in \mathbb{N})$$

is uniformly distributed mod 1 *for every* $R \in \mathbb{N}_0$.

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