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The generalized Berwald P^1-connection with $P^i{}_{jk} = -A_j{}^i{}_k$ and its applications

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Dedicated to Professor Lajos Tamássy on his 70th birthday

0. Introduction

The Berwald connection of a Finsler space has two curvatures, namely the h-and hv-curvatures. Among these curvatures, the hv-curvature vanishes identically if the Finsler space is Riemannian. But a Finsler space with vanishing hv-curvature is not necessarily Riemannian. With respect to the Cartan connection, it seems also to us that a Finsler space with vanishing curvatures except the h-curvature is not necessarily Riemannian.

In this paper, we present an interesting and natural Finsler connection such that a Finsler space with vanishing curvatures except the h-curvature reduces to a Riemannian space and show its applications to the study of Finsler geometry.

Throughout the paper, terminology and notations are those of Matsumoto's monograph [3].

1. The generalized Berwald P^1-connection with $P^i{}_{jk} = -A_j{}^i{}_k$

Let $F^n = (M, L)$ be a Finsler space, where $L(x^i, y^i)$ is the fundamental function, and x^i denotes a point of the underlying manifold M, and y^i denotes a supporting element. The fundamental tensor g_{ij} is given by $g_{ij} = (\partial^2 L^2/\partial y^i \partial y^j)/2$.

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Let $F\Gamma = (F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ be a Finsler connection. Then the h- and v-covariant derivatives of a tensor, for example of $K^h{}_m$, are given by

$$K^h{}_{m|i} = \delta_i K^h{}_m + K^r{}_m F_r{}^h{}_i - K^h{}_r F_m{}^r{}_i,$$
$$K^h{}_m|_i = \dot{\partial}_i K^h{}_m + K^r{}_m C_r{}^h{}_i - K^h{}_r C_m{}^r{}_i,$$

respectively, where $\delta_i = \partial_i - N^r{}_i \dot{\partial}_r$, $\partial_i = \partial/\partial x^i$ and $\dot{\partial}_i = \partial/\partial y^i$.

Differentiating covariantly y^i, we have

$$y^i{}_{|j} = D^i{}_j \ (:= y^r F_r{}^i{}_j - N^i{}_j) \cdots \text{Deflection tensor},$$
$$y^i|_j = \delta^i_j + y^k C_k{}^i{}_j.$$

The $(h)h$-and the $(v)hv$-torsion tensors of the Finsler connection $F\Gamma$ are given by

$$T_j{}^i{}_k = F_j{}^i{}_k - F_k{}^i{}_j, \qquad P^i{}_{jk} = \dot{\partial}_k N^i{}_j - F_k{}^i{}_j,$$

respectively.

In order to construct our Finsler connection of F^n, we quote the following theorem:

Theorem 1. (Aikou and Hashiguchi [1]). *Given (0) p-homogeneous tensors $T_j{}^i{}_k \ (= -T_k{}^i{}_j)$, $P^i{}_{jk}$, there exists a unique Finsler connection $(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ satisfying*

$$L_{|k} = 0, \ D^i{}_k = 0, \ C_j{}^i{}_k = 0,$$

and whose $(h)h$-and $(v)hv$-torsion tensors are the given $T_j{}^i{}_k$, and $P^i{}_{jk}$ respectively, if $T_j{}^i{}_k$ and $P^i{}_{jk}$ satisfy the conditions

$$y^r(\dot{\partial}_k Q_j{}^i{}_r - \dot{\partial}_j Q_k{}^i{}_r) = 0,$$
$$P^i{}_{j0} \ (:= y^k P^i{}_{jk}) = 0,$$

where $Q_j{}^i{}_k$ is a tensor given by $Q_j{}^i{}_k = T_j{}^i{}_k - (P^i{}_{jk} - P^i{}_{kj})$. This Finsler connection is given by

$$N^i{}_k = G^i{}_k - ((T_k{}^i{}_0 + P^i{}_{0k}) + \dot{\partial}_k(T^i{}_{00} + P_{00}{}^i))/2,$$
$$F_j{}^i{}_k = \dot{\partial}_j N^i{}_k - P^i{}_{kj},$$
$$C_j{}^i{}_k = 0,$$

where $G^i{}_k$ is the non-linear connection of the Berwald connection.

In the subsequent considerations, we quote a case of $T_j{}^i{}_k = 0$ in Theorem 1. Such a Finsler connection is called a generalized Berwald P^1-connection by Matsumoto [2]. Then we have

Theorem 2. *There exists a unique generalized Berwald P^1-connection $(F_j{}^i{}_k, N^i{}_k, C_j{}^i{}_k)$ satisfying the following conditions:*

$$\text{(1.1)} \qquad a)\ L_{|k} = 0, \quad b)\ D^i{}_j = 0, \quad c)\ P^i{}_{jk} = -A_j{}^i{}_k, \quad d)\ C_j{}^i{}_k = 0,$$

where $A_j{}^i{}_k = Lg^{ir}\dot{\partial}_r g_{jk}/2$. This Finsler connection is given by

$$\text{(1.2)} \qquad \begin{aligned} &a)\ F_j{}^i{}_k = F(b)_j{}^i{}_k + A_j{}^i{}_k,\\ &b)\ N^i{}_k = y^j F_j{}^i{}_k,\\ &c)\ C_j{}^i{}_k = 0, \end{aligned}$$

where $(F(b)_j{}^i{}_k, y^j F(b)_j{}^i{}_k, 0)$ is the Berwald connection.

Hereafter, we denote the Finsler connection in Theorem 2 by $(F_j{}^i{}_k, N^i{}_k, 0)$. On the other hand, for the Cartan connection we shall use the symbol (c) such as $(F(c)_j{}^i{}_k, N(c)^i{}_k, C(c)_j{}^i{}_k)$ and use (r) for the Rund connection.

Then we have

$$\text{(1.3)} \qquad \begin{aligned} &F(c)_j{}^i{}_k = F(r)_j{}^i{}_k,\\ &N(c)^i{}_k = N(b)^i{}_k = N(r)^i{}_k = N^i{}_k, \end{aligned}$$

$$\text{(1.4)} \qquad y^i{}_{|j} = 0, \quad y^i|_j = \delta^i_j.$$

The Ricci identities of our Finsler connection are as follows:

$$\begin{aligned} &K^h{}_{i|j|k} - K^h{}_{i|k|j} = K^r{}_i R_r{}^h{}_{jk} - K^h{}_r R_i{}^r{}_{jk} - K^h{}_i|_r R^r{}_{jk},\\ &K^h{}_{i|j}|_k - K^h{}_i|_{k|j} = K^r{}_i P_r{}^h{}_{jk} - K^h{}_r P_i{}^r{}_{jk} - K^h{}_i|_r P^r{}_{jk}, \end{aligned}$$

where

$$\text{(1.5)} \qquad \begin{aligned} &a)\ R^i{}_{jk} = \delta_k N^i{}_j - \delta_j N^i{}_k \cdots (v)h\text{-torsion},\\ &b)\ P^i{}_{jk} = \dot{\partial}_k N^i{}_j - F_k{}^i{}_j \cdots (v)hv\text{-torsion},\\ &c)\ R_h{}^i{}_{jk} = \delta_k F_h{}^i{}_j - \delta_j F_h{}^i{}_k + F_h{}^r{}_j F_r{}^i{}_k - F_h{}^r{}_k F_r{}^i{}_j\\ &\qquad \cdots h\text{-curvature},\\ &d)\ P_h{}^i{}_{jk} = \dot{\partial}_k F_h{}^i{}_j \cdots hv\text{-curvature}. \end{aligned}$$

Applying the Ricci identities to y^i and L, we have from (1.1) and (1.4),

$$\text{(1.6)} \qquad \begin{aligned} &a)\ R_0{}^h{}_{jk} = R^h{}_{jk},\\ &b)\ P_0{}^h{}_{jk} = P^h{}_{jk},\\ &c)\ y_r R^r{}_{jk} = 0. \end{aligned}$$

The Bianchi identities of our Finsler connection are as follows:

$$
(1.7)\qquad
\begin{aligned}
&a)\ R_i{}^h{}_{jk} + R_k{}^h{}_{ij} + R_j{}^h{}_{ki} = 0,\\
&b)\ P_m{}^h{}_{ir}R^r{}_{jk} + P_m{}^h{}_{kr}R^r{}_{ij} + P_m{}^h{}_{jr}R^r{}_{ki}\\
&\qquad\quad + R_m{}^h{}_{ij|k} + R_m{}^h{}_{ki|j} + R_m{}^h{}_{jk|i} = 0,\\
&c)\ P_i{}^h{}_{jk} - P_j{}^h{}_{ik} = 0,\\
&d)\ P_m{}^h{}_{ir}P^r{}_{jk} - P_m{}^h{}_{jr}P^r{}_{ik} + P_m{}^h{}_{jk|i} - P_m{}^h{}_{ik|j}\\
&\qquad\quad + \dot\partial_k R_m{}^h{}_{ij} = 0,\\
&e)\ \dot\partial_i P_m{}^h{}_{kj} - \dot\partial_j P_m{}^h{}_{ki} = 0.
\end{aligned}
$$

For our Finsler connection $(F_j{}^i{}_k, N^i{}_k, 0)$, we have the following main theorem:

Theorem 3. *In a Finsler space with connection (1.2), the following conditions are equivalent:*

(1) $P_i{}^h{}_{jk} = 0$.

(2) $P^h{}_{jk} = 0$.

(3) $F_j{}^i{}_k$ *are functions of position only.*

(4) F^n *is Riemannian.*

Proof. The equivalence of (1) and (3) is given by (1.5)d). From (1.6)b), we have that (1) implies (2). By (1.1)c), (2) implies (4). (4) implies (1) obviously.

2. Special Finsler spaces

First of all, we reveal the relations among the *hv*-curvatures of the typical Finsler connections.

The *hv*-curvature tensors of the Rund connection and the Berwald connection are given by

$$(2.1)\qquad P(r)_h{}^i{}_{jk} = P(c)_h{}^i{}_{jk} + C(c)_h{}^i{}_{k\underset{(c)}{|}j} - C(c)_h{}^i{}_r P(c)^r{}_{jk},$$

$$(2.2)\qquad P(b)_h{}^i{}_{jk} = P(r)_h{}^i{}_{jk} + C(c)_j{}^i{}_{h|0\cdot k}, \quad \left(\cdot k = \frac{\partial}{\partial y^k}\right),$$

respectively [3]. Accordingly, from (2.1), (2.2) and (1.5)d), the *hv*-curvature tensor $P_h{}^i{}_{jk}$ of our Finsler connection is expressed as follows:

$$
(2.3)\quad
\begin{aligned}
P_h{}^i{}_{jk} &= P(c)_h{}^i{}_{jk} + C(c)_h{}^i{}_{k \underset{(c)}{|} j} - C(c)_h{}^i{}_r P(c)^r{}_{jk} \\
&\qquad + C(c)_j{}^i{}_{h|0\cdot k} + A_h{}^i{}_{j\cdot k} \\
&= P(b)_h{}^i{}_{jk} + A_h{}^i{}_{j\cdot k} \\
&= P(r)_h{}^i{}_{jk} + (L^{-1}A_j{}^i{}_h)_{|0\cdot k} + A_h{}^i{}_{j\cdot k}.
\end{aligned}
$$

Using (2.3), from Theorem 3, we obtain

Theorem 4. *The following conditions are equivalent:*
(1) A Finsler space F^n is Riemannian.
(2) $P(c)_h{}^i{}_{jk} = -C(c)_h{}^i{}_{k \underset{(c)}{|} j} + C(c)_h{}^i{}_r P(c)^r{}_{jk} - C(c)_j{}^i{}_{h|0\cdot k} - A_h{}^i{}_{j\cdot k}$.
(3) $P(b)_h{}^i{}_{jk} = -A_h{}^i{}_{j\cdot k}$.
(4) $P(r)_h{}^i{}_{jk} = -(L^{-1}A_j{}^i{}_h)_{|0\cdot k} - A_h{}^i{}_{j\cdot k}$.

The next theorem and corollary concern Landsberg spaces.

Theorem 5. *A Finsler space with connection (1.2) is a Landsberg space, if and only if*

$$
(2.4)\quad R^h{}_{0j\cdot k} + R_k{}^h{}_{j0} + R^h{}_{jk} = 0.
$$

Proof. Contracting (1.7)d) by $y^i y^m$, we have

$$
(2.5)\quad R_m{}^h{}_{ij\cdot k} y^i y^m - A_j{}^h{}_{k|0} = 0.
$$

On the other hand, differentiating the equation $R^h{}_{0j} = R_m{}^h{}_{ij} y^i y^m$ with respect to y^k, we have

$$
(2.6)\quad R^h{}_{0j\cdot k} = R_m{}^h{}_{ij\cdot k} y^i y^m - R_k{}^h{}_{j0} - R^h{}_{jk}.
$$

From (2.5) and (2.6), we obtain that (2.4) is equivalent to $A_j{}^h{}_{k|0} = 0$. On the other hand, we know that a Landsberg space is characterized by $A_j{}^h{}_{k|0} = 0$.

The following corollary is easily obtained from (2.5):

Corollary 6. *(1) If the h-curvature tensor $R_i{}^h{}_{jk}$ of connection (1.2) vanishes identically, then the Finsler space is a Landsberg space. (2) If the components of the h-curvature tensor $R_i{}^h{}_{jk}$ of connection (1.2) are functions of position only, then the Finsler space is a Landsberg space.*

The remaining theorems concern Berwald and locally Minkowski spaces.

Theorem 7. *A Finsler space with connection (1.2) is a Berwald space, if and only if*

$$
(2.7)\quad P_h{}^i{}_{jk} = -P^i{}_{hj\cdot k}.
$$

PROOF. We know that a Finsler space is a Berwald space, if and only if $P(b)_i{}^h{}_{jk} = 0$. From (2.3), the condition $P(b)_i{}^h{}_{jk} = 0$ is equivalent to (2.7).

Theorem 8. *A Finsler space with connection (1.2) is a locally Minkowski space, if and only if*

$$(2.8) \qquad a)\ R^h{}_{jk} = 0, \qquad b)\ P_i{}^h{}_{jk} = -P^h{}_{ij\cdot k}.$$

PROOF. We know that a Finsler space is a locally Minkowski space, if and only if

$$(2.9) \qquad a)\ R(c)^i{}_{jk} = 0, \qquad b)\ C(c)_{hij \underset{(c)}{|} k} = 0.$$

From (1.3) and (1.5)a), (2.9)a) is equivalent to (2.8)a). (2.9)b) is the well-known condition for a Finsler space to be a Berwald one. Moreover, by Theorem 7, (2.9)b) is equivalent to (2.8)b).

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