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On the asymptotic behavior of solutions of the second order neutral differential equations

By ŠTEFAN KULCSÁR (Košice)

Abstract. Consider the second order neutral differential equation

$$\big(x(t) - px(t-\tau)\big)'' + q(t)x\big(\sigma(t)\big) = 0,$$

where $\sigma(t) > t$. Sufficient conditions for the convergence to zero of nonoscillatory solutions are presented.

In this paper we deal with the asymptotic behavior of the solutions of the neutral differential equation. We consider the second order differential equation of the form

$$\big(x(t) - px(t-\tau)\big)'' + q(t)x\big(\sigma(t)\big) = 0 \tag{1}$$

under the assumptions

(i) $0 < p < 1$, $\tau > 0$;

(ii) $q, \sigma \in C(R_+, R_+)$, where $R_+ = (0, \infty)$, $\sigma(t) > t$.

We put $z(t) = x(t) - px(t-\tau)$. By a proper solution of Eq. (1) we mean a function $x : [T_x, \infty) \to R$ which satisfies (1) for all sufficiently large t and $\sup\{|x(t)| : t \geq T\} > 0$ for any $T \geq T_x$ so that $z(t)$ is twice continuously differentiable. Such a solution is called oscillatory if it has a sequence of zeros tending to infinity; otherwise it is called nonoscillatory. Recently several authors have studied the asymptotic behavior of

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the solutions of the second order neutral delay differential equations, see for example [2]–[9]. There are only few papers devoted to the asymptotic properties of the solutions of the Eq. (1) with advanced argument. We begin with the following result which is known for differential equations without the neutral term, that is for $p = 0$ (see [1]).

Theorem 1. *Assume that* (i) *and* (ii) *hold and*

$$\int^{\infty} q(s)ds = \infty. \tag{2}$$

Then the nonoscillatory solutions of Eq. (1) *tend to zero as* $t \to \infty$.

Proof. Without loss of generality let $x(t)$ be an eventually positive solution of Eq. (1) and define

$$z(t) = x(t) - px(t - \tau). \tag{3}$$

From Eq. (1) we have $z''(t) < 0$ for all large t, say $t \geq t_0$. If $z'(t) < 0$ eventually, then $\lim\limits_{t\to\infty} z(t) = -\infty$. But $z(t) < 0$ eventually implies that

$$x(t) < px(t - \tau) < p^2 x(t - 2\tau) < \cdots < p^n x(t - n\tau)$$

for all large t, which implies in view of (i) that $\lim_{t\to\infty} x(t) = 0$. This contradicts to $\lim_{t\to\infty} z(t) = -\infty$. Therefore, $z'(t) > 0$ for $t \geq t_0$. There are two possibilities for $z(t)$:

(a) $z(t) > 0$ for $t \geq t_1 \geq t_0$,

(b) $z(t) < 0$ for $t \geq t_1$.

For case (a), Eq. (1) can be written in the form

$$z''(t) + q(t)x\big(\sigma(t)\big) = 0. \tag{4}$$

From (3) one can see that $x(t) > z(t)$ which together with (4) implies

$$z''(t) + q(t)z\big(\sigma(t)\big) \leq 0. \tag{5}$$

Using monotonicity of $z(t)$ and (ii), there exists a constant $c > 0$ such that $z(\sigma(t)) > c$, for $t \geq t_2 \geq t_1$. Then

$$z''(t) + cq(t) \leq 0.$$

Integration of the last inequality from t_2 to t yields

$$z'(t) - z'(t_2) + c\int_{t_2}^{t} q(s)\,ds \le 0,$$

i.e.

$$c\int_{t_2}^{t} q(s)\,ds \le z'(t_2)$$

for $t \ge t_2$. This contradicts the hypothesis (2). For the case (b), as mentioned above, we are led to that $\lim_{t\to\infty} x(t) = 0$. □

Example 1. Consider the following neutral equation

$$\big(x(t) - px(t-\tau)\big)'' + (pe^{\tau} - 1)e^{t}x(2t) = 0.$$

Here $0 < p < 1$, $\tau > 0$ and by Theorem 1 every nonoscillatory solution of given equation tends to zero as $t \to \infty$. One such solution is $x(t) = e^{-t}$.

The following Theorem is intended to cover the case when $\int^{\infty} q(s)ds < \infty$.

Theorem 2. *Let* (i), (ii) *hold and moreover* $\sigma(t)$ *is increasing. Further assume that*

$$\limsup_{t\to\infty}\left(\int_{t}^{\sigma(t)}\int_{x}^{\infty} q(s)\,ds\,dx + \int_{\sigma^{-1}(t)}^{t}\int_{\sigma(x)}^{\infty} q(s)\,ds\,dx\right) > 1. \tag{6}$$

Then the nonoscillatory solutions of Eq. (1) *tend to zero as* $t \to \infty$.

Proof. Let $x(t)$ be an eventually positive solution of Eq. (1) and set $z(t) = x(t) - px(t-\tau)$. Then $z''(t) < 0$ eventually. Similarly is in the proof of Theorem 1 we are led to $z'(t) > 0$. There are two possibilities (a) and (b) for $z(t)$ (see proof of Theorem 1). For case (b) the proof is similar as in the proof of Theorem 1 and so it can be omitted. In case (a) we proceede exactly as in the proof of Theorem 1 and we get inequality (5). Integrating (5) from t to ∞, we get

$$z'(t) \ge \int_{t}^{\infty} q(s)z(\sigma(s))\,ds \tag{7}$$

and integrating the last inequality from t to $\sigma(t)$ we have

$$z(\sigma(t)) - z(t) \geq z(\sigma(t)) \int_t^{\sigma(t)} \int_x^\infty q(s)\,ds\,dx, \qquad t \geq t_1, \tag{8}$$

where we have used that $z(t)$ and $\sigma(t)$ are increasing. Now integrating (7) from t_1 to t we obtain

$$\begin{aligned} z(t) &\geq \int_{t_1}^t \int_x^\infty q(s) z(\sigma(s))\,ds\,dx \geq \int_{\sigma^{-1}(t)}^t \int_{\sigma(x)}^\infty q(s) z(\sigma(s))\,ds\,dx \\ &\geq z(\sigma(t)) \int_{\sigma^{-1}(t)}^t \int_{\sigma(x)}^\infty q(s)\,ds\,dx, \qquad t \geq t_1, \end{aligned} \tag{9}$$

where we have used that $z(t)$ and $\sigma(t)$ are increasing. From (8) and (9) we get

$$z(\sigma(t)) \geq z(\sigma(t)) \left(\int_t^{\sigma(t)} \int_x^\infty q(s)\,ds\,dx + \int_{\sigma^{-1}(t)}^t \int_{\sigma(x)}^\infty q(s)\,ds\,dx \right),$$

i.e.

$$\int_t^{\sigma(t)} \int_x^\infty q(s)\,ds\,dx + \int_{\sigma^{-1}(t)}^t \int_{\sigma(x)}^\infty q(s)\,ds\,dx \leq 1.$$

We arrive at a contradiction to (6). The Theorem 2 is proved. □

Example 2. Consider the following neutral equation

$$\big(x(t) - px(t-\tau)\big)'' + \frac{a}{t^2} x\big(\lambda t\big) = 0,$$

where $\lambda > 1$ and a is a positive real number. Then the condition (6) takes the form

$$a\left(1 + \frac{1}{\lambda}\right) \ln \lambda > 1$$

and for example, for $a = 1$, the condition (6) is satisfied for $\lambda = 1{,}933$.

In the sequel we deal with special type of Eq. (1) and further we consider the second order neutral differential equation of the form

$$\big(x(t) - px(t-\tau)\big)'' + q(t)x(t+\alpha) = 0. \tag{10}$$

In the sequel we shall assume that (i) holds. Moreover we assume that

(iii) $q \in C(R_+, R_+)$ and α is a positive number.

Theorem 3. *Let* (i) *and* (iii) *hold and* k *be an integer such that*

$$\alpha - k\tau < 0. \tag{11}$$

Further assume that there exists an integer number $n \geq k$ *such that*

$$\int^{\infty} \left(\frac{p^k(1-p^{n-k+1})}{1-p}(s+\alpha)q(s) - \frac{1}{4(s+\alpha)} \right) ds = \infty. \tag{12}$$

Then the nonoscillatory solutions of Eq. (10) *tend to zero as* $t \to \infty$.

Proof. Without loss of generality let $x(t)$ be an eventually positive solution of Eq. (10). Further analogously as in the proof of the Theorem 1 we get for case (a) that Eq. (10) we can written in the form

$$z''(t) + q(t)x(t+\alpha) = 0.$$

Using (3) we have

$$z''(t) + q(t)z(t+\alpha) + pq(t)x(t+\alpha-\tau) = 0.$$

Repeating this procedure we arrive at

$$z''(t) + q(t)\sum_{i=0}^{n} p^i z(t+\alpha-i\tau) + p^{n+1}q(t)x(t+\alpha-(n+1)\tau) = 0,$$

i.e. by (i) and (iii) we have

$$z''(t) + q(t)\sum_{i=0}^{k-1} p^i z(t+\alpha-i\tau) + q(t)\sum_{i=k}^{n} p^i z(t+\alpha-i\tau) \leq 0.$$

Using (i) and (iii) we get

$$z''(t) + q(t)\sum_{i=k}^{n} p^i z(t+\alpha-i\tau) \leq 0.$$

Denote $a_n(t) = \sum_{i=k}^{n} p^i z(t+\alpha-i\tau)$. Then

$$z''(t) + a_n(t)q(t) \leq 0. \tag{13}$$

Define

$$v(t)=\frac{(t+\alpha)\sum_{i=k}^{n}p^i}{a_n(t)}z'(t),\quad t\ge t_1.$$

Then $v(t)>0$. Observe that

$$\begin{aligned}v'(t)&=\frac{\sum_{i=k}^{n}p^i}{a_n(t)}z'(t)+\frac{(t+\alpha)\sum_{i=k}^{n}p^i}{a_n(t)}z''(t)\\&\quad-\frac{(t+\alpha)\sum_{i=k}^{n}p^i}{a_n(t)}z'(t)\,\frac{\sum_{i=k}^{n}p^iz'(t+\alpha-i\tau)}{a_n(t)}.\end{aligned}$$

Since $z'(t)$ is decreasing, one gets that $z'(t+\alpha-i\tau)\ge z'(t+\alpha-k\tau)$ for $i\ge k$ and by (11) we have $z'(t+\alpha-k\tau)\ge z'(t)$. Thus by (13) we get

$$v'(t)\le\frac{1}{t+\alpha}\big(v(t)-v^2(t)\big)-(t+\alpha)q(t)\sum_{i=k}^{n}p^i.$$

It is easy to see that the polynomial $P(v)=v-v^2\le\frac14$. Thus

$$v'(t)\le\frac{1}{4(t+\alpha)}-(t+\alpha)q(t)\sum_{i=k}^{n}p^i.$$

Then integrating the last inequality from t_1 to t, we are led to

$$v(t)\le v(t_1)-\int_{t_1}^{t}\left(\frac{p^k(1-p^{n-k+1})}{1-p}(s+\alpha)q(s)-\frac{1}{4(s+\alpha)}\right)ds.$$

Letting $t\to\infty$ we have in view of (12) that $v(t)\to-\infty$, a contradiction.

For case (b), as mentioned before, we are led to $\lim_{t\to\infty}x(t)=0$. □

Corollary 1. *Assume that* (i), (iii) *hold and let k be an integer such that* (11) *holds. Let*

$$\liminf_{t\to\infty}(t+\alpha)^2q(t)>\frac{1-p}{4p^k}.$$

Then the nonoscillatory solutions of Eq. (1) *tend to zero as $t\to\infty$.*

Proof. Denote $a=\liminf_{t\to\infty}(t+\alpha)^2q(t)$. Let an integer n be chosen such that

$$a-\epsilon>\frac{1-p}{4p^k\big(1-p^{n-k+1}\big)},$$

where $\epsilon > 0$ is small enough. Then there exists a t_1 (large enough) that

$$q(t)(t+\alpha)^2 - \frac{1-p}{4p^k\bigl(1-p^{n-k+1}\bigr)} > \epsilon, \quad t \geq t_1. \tag{14}$$

Noting that (14) implies (12) we complete the proof. □

Example 3. Consider the neutral equation

$$\bigl(x(t) - px(t-\tau)\bigr)'' + (pe^{\tau} - 1)e^{\ln 3}x(t + \ln 3) = 0.$$

For $0 < p < 1$, $\tau > \ln 2$ and $k > 1$ the solution $x(t) = e^{-t}$ tend to zero as $t \to \infty$. Now we give an analogy of Theorem 3 for the case $p = 1$.

Theorem 4. *Assume that* $p = 1$ *and* (11) *holds. Then the nonoscillatory solutions of Eq.* (10) *are bounded provided there exists an integer* $n \geq k$ *such that*

$$\int^{\infty} \left((n-k+1)(s+\alpha)q(s) - \frac{1}{4(s+\alpha)} \right) ds = \infty. \tag{15}$$

Proof. For case (a) the proof follows the same line as the proof of Theorem 3 and so it can be omitted. If $z(t) < 0$ eventually, then $x(t) < x(t-\tau)$, which implies that $x(t)$ is bounded. □

Corollary 2. *Assume that* $p = 1$. *Let*

$$\liminf_{t \to \infty} (t+\alpha)^2 q(t) > 0.$$

Then the nonoscillatory solutions of Eq. (10) *are bounded.*

Proof. Denote $a = \liminf_{t\to\infty} 4(t+\alpha)^2 q(t)$. Let an integer n be chosen such that $a - \epsilon > \frac{1}{n-k+1}$, where $\epsilon > 0$ is small enough. Then there exists a t_1 (large enough) such that

$$4(t+\alpha)^2 q(t) - \frac{1}{n-k+1} > \epsilon, \quad t \geq t_1. \tag{17}$$

Note that (17) implies (15). The proof is complete. □

Example 4. Consider the neutral equation

$$\big(x(t) - px(t-\tau)\big)'' + (e^{\tau} - 1)e^{-(t+\alpha)} = 0.$$

For $\tau > 0$, $\alpha > 0$ and $k = [\frac{\alpha}{\tau}] + 1$ the solution $x(t) = e^{-t}$ tend to zero as $t \to \infty$.

In finally we give an analogy of Theorem 3 and 4 for the case $p > 1$.

Theorem 5. *Assume that* $p > 1$ *and* (11) *holds. Further assume that there exists an integer* $n \geq k$ *such that*

$$\int^{\infty} \left(\frac{p^k(p^{n-k+1} - 1)}{p-1}(s+\alpha)q(s) - \frac{1}{4(s+\alpha)} \right) ds = \infty. \tag{18}$$

Then every nonoscillatory solution $x(t)$ *of Eq.* (10) *satisfies* $x(t)<px(t-\tau)$.

Proof. We begin equally that in the proof of Theorem 3 and we get three possibilities as in the proof of Theorem 1. For case $z'(t) > 0$, $z(t) > 0$ the proof runs exactly as in the proof of Theorem 3 and so it can be omitted. For cases $z'(t) > 0$ or $z'(t) < 0$ we have assumed that $z(t) < 0$, then $x(t) < px(t-\tau)$ is obvious. □

Example 5. Consider the neutral equation

$$\big(x(t) - px(t-\tau)\big)'' + 6(t+\alpha)^2 \left(\frac{2}{(t-\tau)^4} - \frac{1}{t^4} \right) x(t+\alpha) = 0.$$

For $\tau > 0$, $\alpha > 0$ the condition (18) is fulfilled and the solution $x(t) = t^{-2}$ of this equation has demanded properties.

Remark. When considering more general neutral differential equations with function $p(t)$ instead of a constant p,

$$\big(x(t) - p(t)x(t-\tau)\big)'' + q(t)x\big(\sigma(t)\big) = 0, \tag{19}$$

$$\big(x(t) - p(t)x(t-\tau)\big)'' + q(t)x(t+\alpha) = 0, \tag{19'}$$

then it is usual to impose the condition $p_1 < p(t) < p_2$ on the function $p(t)$. From the proofs of the abovementioned results one can see that the technique presented in this paper can be applied to Eqs. (19) and (19').

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ŠTEFAN KULCSÁR
DEPARTMENT OF MATHEMATICAL ANALYSIS
FACULTY OF SCIENCES
ŠAFÁRIK UNIVERSITY
JESENNÁ 5, 041 54 KOŠICE
SLOVAKIA

E-mail: pkulcsar@duro.science.upjs.sk

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