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Randers spaces with the h-curvature tensor H dependent on position alone

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Abstract. We give an example of Finsler space having the Berwald h-curvature tensor H which is independent of the direction arguments y^i [3].

1. Introduction

In our previous paper [3] we were concerned with various sets of special kinds of Finsler spaces. Among them we pay attention to the two sets

$B(n)$ … n-dim. Berwald spaces,

$Hx(n)$ … n-dim. spaces having the Berwald h-curvature tensor H dependent on the position alone.

The inclusion relation $B(n) \subset Hx(n)$ is obvious, but any example of a Finsler space belonging to $Hx(n)$ but not to $B(n)$ has not been given in the paper.

The purpose of the present paper is to give an example of such a Finsler space, a *Randers space* $F^n = (M^n, L = \alpha + \beta)$. Its metric L consists of a Riemannian metric α ($\alpha^2 = a_{ij}(x)y^iy^j$) and a differential one-form $\beta = b_i(x)y^i$.

The Riemannian space $R^n = (M^n, \alpha)$ is said to be associated with F^n. Let $\gamma_j{}^i{}_k(x)$ be the Christoffel symbols of R^n. Then we have the Levi-Civita connection $\gamma = \{\gamma_j{}^i{}_k\}$ in R^n and the induced Finsler connection

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$F\gamma = \{\gamma_j{}^i{}_k, \gamma_0{}^i{}_j, 0\}$ in F^n. (Throughout the paper the subscript 0 denotes the transvection by y^i.) The h- and v-covariant differentiations in $F\gamma$ are denoted by $(;\,,\cdot)$ respectively. Let us use the following symbols:

$$r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad r_2 = \frac{1}{2} r_{00}, \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}),$$
$$y_i = a_{ir} y^r, \quad b^i = a^{ir} b_r, \quad s^i{}_j = a^{ir} s_{rj}, \quad s_i = b_r s^r{}_i.$$

We are interested in Randers spaces from the standpoint of not only applications but also pure geometry [1, 1.4]. For instance it is a remarkable result [4], [7] that a Randers space is a *Berwald space*, if and only if $b_{i;j} = 0$. Next a Randers space is a *Douglas space* which has been introduced by the authors [2], if and only if $b_{i;j} - b_{j;i} = 0$, that is, β is a closed form.

We shall adopt here Randers spaces to give an example of Finsler spaces belonging to the set $Hx(n)$. The quantities $G^i(x, y)$ appearing in the equations $d^2x^i/ds^2 + 2G^i(x, dx/ds) = 0$ of geodesic in the Randers space F^n are written as [7]

$$2G^i = \gamma_0{}^i{}_0 + 2B^i \tag{1.1}$$

where the tensor $B^i(x, y)$ is of the form

$$LB^i = B_3{}^i + \alpha B_2{}^i, \tag{1.2}$$

$$B_2{}^i = \beta s^i{}_0 - s_0 y^i, \qquad B_3{}^i = \alpha^2 s^i{}_0 + r_2 y^i. \tag{1.2a}$$

In the following the subscripts $a = 2, \ldots, 9$ denote that the entity is a homogeneous polynomial in y^i of degree a; $B_a{}^i$, $a = 2, 3$, of (1.2a) are homogeneous polynomials in y^i of degree two and three respectively.

2. The h-curvature tensor of a Randers space

We deal with a Randers space $F^n = (M^n, L = \alpha + \beta)$ equipped with the Berwald connection $B\Gamma = \{G^i_j, G_j{}^i{}_k, 0\}$, and denote by $H = (H_i{}^h{}_{jk})$ and $R^1 = (R^h{}_{jk})$ the h-curvature tensor and the $(v)h$-torsion tensor respectively. Then we have well-known relations

$$\text{(i)} \quad R^h{}_{jk} = H_0{}^h{}_{jk}, \qquad \text{(ii)} \quad H_i{}^h{}_{jk} = R^h{}_{jk\cdot i}. \tag{2.1}$$

R^1 is defined by

$$R^h{}_{jk} = \partial_k G^h{}_j - \partial_j G^h{}_k - G_j{}^h{}_r G^r{}_k + G_k{}^h{}_r G^r{}_j. \tag{2.2}$$

Further we shall take notice of the relation [5, (18.23)]

$$R^h{}_{jk} = \frac{1}{3}(R^h{}_{0k\cdot j} - R^h{}_{0j\cdot k}), \tag{2.3}$$

for the later use. Then, to find the tensor H, we first construct $R^h{}_{0k}$ and then use (2.3) and (ii) of (2.1).

It follows from (2.2) that

$$R^h{}_{0k} = 2\partial_k G^h - y^j \partial_j G^h{}_k - G^h{}_r G^r{}_k + 2G_k{}^h{}_r G^r.$$

Then, using (1.1) and the h-covariant differentiation $(;)$ in $F\gamma$, we obtain

$$R^h{}_{0k} = \overset{a}{R}_0{}^h{}_{0k} + 2B^h{}_{;k} - B^h{}_{k;0} + 2B_k{}^h{}_r B^r - B^h{}_r B^r{}_k, \tag{2.4}$$

where $\overset{a}{R}$ is the curvature tensor of R^n and we put $B^h{}_k = B^h{}_{\cdot k}$ and $B_k{}^h{}_r = B^h{}_{k\cdot r}$.

In the following we shall quote extensively from the paper [6] the procedure in order to obtain $R^h{}_{0k}$ of the Randers space belonging to the set $Hx(n)$.

First, from (1.2) we have

$$L^2\alpha B^i{}_j = C_4{}^i{}_j + \alpha C_3{}^i{}_j, \tag{2.5}$$

$$\begin{cases} C_3{}^i{}_j = \alpha^2 B_2{}^i{}_{\cdot j} + \beta B_3{}^i{}_{\cdot j} - B_3{}^i b_j, \\ C_4{}^i{}_j = \alpha^2(\beta B_2{}^i{}_{\cdot j} + B_3{}^i{}_{\cdot j}) + B_2{}^i(\beta y_j - \alpha^2 b_j) - B_3{}^i y_j, \end{cases} \tag{2.5a}$$

$$L^3\alpha^3 B_j{}^i{}_k = D_6{}^i{}_{jk} + \alpha D_5{}^i{}_{jk}, \tag{2.6}$$

$$\begin{cases} D_5{}^i{}_{jk} = \alpha^2(\beta C_3{}^i{}_{j\cdot k} + C_4{}^i{}_{j\cdot k}) - 2\alpha^2 C_3{}^i{}_j b_k - 3C_4{}^i{}_j y_k, \\ D_6{}^i{}_{jk} = \alpha^2(\alpha^2 C_3{}^i{}_{j\cdot k} + \beta C_4{}^i{}_{j\cdot k}) - 2\alpha^2 C_3{}^i{}_j y_k \\ \qquad\qquad -C_4{}^i{}_j(\beta y_k + 2\alpha^2 b_k). \end{cases} \tag{2.6a}$$

Then we have

$$(2.7) \qquad L^4\alpha^2 B^i{}_r B^r{}_j = E_8{}^i{}_j + \alpha E_7{}^i{}_j,$$

$$(2.7a) \qquad \begin{cases} E_7{}^i{}_j = C_4{}^i{}_r C_3{}^r{}_j + C_3{}^i{}_r C_4{}^r{}_j, \\ E_8{}^i{}_j = C_4{}^i{}_r C_4{}^r{}_j + \alpha^2 C_3{}^i{}_r C_3{}^r{}_j, \end{cases}$$

$$(2.8) \qquad L^3\alpha B^i{}_r B^r = F_7{}^i + \alpha F_6{}^i,$$

$$(2.8a) \qquad F_6{}^i = C_3{}^i{}_r B_3{}^r + C_4{}^i{}_r B_2{}^r, \quad F_7{}^i = C_4{}^i{}_r B_3{}^r + \alpha^2 C_3{}^i{}_r B_2{}^r,$$

$$(2.9) \qquad L^4\alpha^3 B_j{}^i{}_r B^r = G_9{}^i{}_j + \alpha G_8{}^i{}_j,$$

$$(2.9a) \qquad \begin{cases} G_8{}^i{}_j = D_5{}^i{}_{jr} B_3{}^r + D_6{}^i{}_{jr} B_2{}^r, \\ G_9{}^i{}_j = D_6{}^i{}_{jr} B_3{}^r + \alpha^2 D_5{}^i{}_{jr} B_2{}^r. \end{cases}$$

Next we get

$$(2.10) \qquad L^2 B^i{}_{;j} = H_4{}^i{}_j + \alpha H_3{}^i{}_j,$$

$$(2.10a) \qquad \begin{cases} H_3{}^i{}_j = \beta B_2{}^i{}_{;j} + B_3{}^i{}_{;j} - B_2{}^i(r_{0j} + s_{0j}), \\ H_4{}^i{}_j = \alpha^2 B_2{}^i{}_{;j} + \beta B_3{}^i{}_{;j} - B_3{}^i(r_{0j} + s_{0j}), \end{cases}$$

$$(2.11) \qquad L^3\alpha B^i{}_{j;0} = I_6{}^i{}_j + \alpha I_5{}^i{}_j,$$

$$(2.11a) \qquad \begin{cases} I_5{}^i{}_j = \beta C_3{}^i{}_{j;0} + C_4{}^i{}_{j;0} - 4r_2 C_3{}^i{}_j, \\ I_6{}^i{}_j = \alpha^2 C_3{}^i{}_{j;0} + \beta C_4{}^i{}_{j;0} - 4r_2 C_4{}^i{}_j. \end{cases}$$

Substituting (2.10), (2.11), (2.9) and (2.7) in (2.4), we get $R^h{}_{0k}$ in the form

$$(2.12) \qquad L^4\alpha^3(R^h{}_{0k} - \overset{a}{R}_0{}^h{}_{0k}) = 2L^2\alpha^3(H_4{}^h{}_k + \alpha H_3{}^h{}_k)$$
$$-L\alpha^2(I_6{}^h{}_k + \alpha I_5{}^h{}_k) + 2(G_9{}^h{}_k + \alpha G_8{}^h{}_k) - \alpha(E_8{}^h{}_k + \alpha E_7{}^h{}_k).$$

Since we have $(B^i{}_r B^r)_{\cdot j} = B_j{}^i{}_r B^r + B^i{}_r B^r{}_j$, (2.8), (2.9) and (2.7) yield the relation

$$
(2.13)\qquad \begin{cases} G_8{}^i{}_j + E_8{}^i{}_j = \alpha^2(\beta F_6{}^i{}_{\cdot j} + F_7{}^i{}_{\cdot j}) - 3\alpha^2 F_6{}^i b_j - 4F_7{}^i y_i, \\ G_9{}^i{}_j + \alpha^2 E_7{}^i{}_j = \alpha^2(\alpha^2 F_6{}^i{}_{\cdot j} + \beta F_7{}^i{}_{\cdot j}) - 3\alpha^2 F_6{}^i y_i \\ \qquad\qquad - F_7{}^i(\beta y_j + 3\alpha^2 b_j). \end{cases}
$$

3. The condition for Randers spaces to belong to $Hx(n)$

It is observed that (2.12) is obtained from (2.1) of the paper [6] by the

$$
(3.1)\qquad \text{change: } (K, \overset{a}{R}) \longrightarrow (0, \overset{a}{R} - R).
$$

If we deal with the Randers space F^n belonging to $Hx(n)$, then (i) of (2.1) gives

$$
R^h{}_{0k} - \overset{a}{R}_0{}^h{}_{0k} = (H_r{}^h{}_{sk}(x) - \overset{a}{R}_r{}^h{}_{sk}(x))y^r y^s,
$$

homogeneous polynomials in y^i of degree two. Consequently the discussions in §2–4 of [6] can be applied without modification. The conclusion in [6] is that

Lemma 3. *Randers spaces of dimension more than two and of constant curvature K are classified as follows*

(I) *RCG-space:* $r_{ij} = 2c(a_{ij} - b_i b_j)$, $s_{ij} = 0$, $K + c^2 = 0$,
(II) *RCT-space:* $r_{ij} = 0$, $s_i = 0$, $c = 0$, $s_{ij;k} = K(a_{ik}b_j - a_{jk}b_i)$.

On these conditions the remarkable form of $\overset{a}{R}$ was given by (5.3) with (5.4) of [6].

By the change (3.1) we then have the conclusion as follows:

A Randers space F^n, $n > 2$, belongs to the set $Hx(n)$, if and only if

(I) G-type: $r_{ij} = s_{ij} = 0$,
(II) T-type: $r_{ij} = 0$, $s_i = 0$, $s_{ij;k} = 0$.

In any case we have $c = 0$ and $r_{ij} = 0$, and hence (5.3) with (5.4) of [6] leads to

$$
(3.2)\qquad \begin{aligned} R^h{}_{0k} = {} & \overset{a}{R}_0{}^h{}_{0k} - (s^r s_r)y^h y_k - 3s^h{}_0 s_{0k} + s^h{}_r s^r{}_0 y_k \\ & - \alpha^2 s^h{}_r s^r{}_k + y^h s_{0r} s^r{}_k + \{(s^r s_r)\alpha^2 - s_{0r}s^r{}_0\}\delta^h{}_k. \end{aligned}
$$

We have $b_{i;j} = 0$ for F^n of G-type, and consequently F^n is a Berwald space where $G_j{}^i{}_k = \gamma_j{}^i{}_k(x)$ and the h-curvature tensor H obviously coincides with the Riemannian $\overset{a}{R}$.

On the other hand, for F^n of T-type we have

$$\begin{aligned} r_{ij} = 0:\ & b_{i;j} = s_{ij} \text{ (skew-symmetric)}, \\ s_i = 0:\ & s_i = b^r b_{r;i} = \frac{1}{2}(a^{rs} b_r b_s)_{;i} = 0. \end{aligned}$$

The former shows that b_i is a Killing vector in R^n, and the latter indicates that the length of b_i is constant in R^n. Therefore b_i is the so-called translation. Further $s_{ij;k} = 0$ together with

$$s_{i;j} = (b^r b_{r;i})_{;j} = b^r{}_{;j} b_{r;i} + b^r b_{r;i;j} = s^r{}_j s_{ri} + b^r s_{ri;j},$$

leads to $s^r{}_j s_{ri} = 0$. Therefore (3.2) is reduced to

$$R^h{}_{0k} = \overset{a}{R}_0{}^h{}_{0k} - 3 s^h{}_0 s_{0k}. \tag{3.3}$$

Then (3.3) gives

$$R^h{}_{0k\cdot j} = \overset{a}{R}_j{}^h{}_{0k} + \overset{a}{R}_0{}^h{}_{jk} - 3(s^h{}_j s_{0k} + s^h{}_0 s_{jk}),$$

and (2.3) yields

$$R^h{}_{jk} = \frac{1}{3}\left(\overset{a}{R}_j{}^h{}_{0k} - \overset{a}{R}_k{}^h{}_{0j} + 2\overset{a}{R}_0{}^h{}_{jk}\right) - s^h{}_j s_{0k} + s^h{}_k s_{0j} - 2 s^h{}_0 s_{jk}.$$

On account of the well-known identities satisfied by $\overset{a}{R}_{hijk}$ we have

$$\overset{a}{R}_{jh0k} - \overset{a}{R}_{kh0j} = -\overset{a}{R}_{hj0k} - \overset{a}{R}_{kh0j} = \overset{a}{R}_{jk0h} = \overset{a}{R}_{0hjk}.$$

Thus we get

$$R^h{}_{jk} = \overset{a}{R}_0{}^h{}_{jk} - s^h{}_j s_{0k} + s^h{}_k s_{0j} - 2 s^h{}_0 s_{jk},$$

and finally (ii) of (2.1) gives

$$H_i{}^h{}_{jk} = \overset{a}{R}_i{}^h{}_{jk} - s^h{}_j s_{ik} + s^h{}_k s_{ij} - 2 s^h{}_i s_{jk}, \quad s_{ij} = b_{i;j}. \tag{3.4}$$

Theorem 1. *A Randers space F^n, $n > 2$, has the h-curvature tensor H of the Berwald connection which depends on the position alone, if and only if* (I) $b_{i;j} = 0$, *F^n being a Berwald space, or* (II) *b_i is a translation in the associated Riemannian space, that is, $b_{i;j} + b_{j;i} = 0$ and $b^r b_{r;i} = 0$, and that satisfies $b_{i;j;k} = 0$.*

In the case (II) *we have $G^i = \gamma_0{}^i{}_0/2 + \alpha b^i{}_{;0}$ and the tensor H is written in the form* (3.4).

4. The two-dimensional case

A Randers space of dimension two is an exceptional case in [6], because "Lemma 1" (p. 256) needs the restriction $n > 2$. However the condition (II) of Theorem 1 may be applicable to the case $n = 2$. Thus this last section is devoted to the consideration of a two-dimensional Randers space F^2 satisfying

$$b_{i;j} + b_{j;i} = 2r_{ij} = 0, \quad s_i = 0, \quad s_{ij;k} = 0. \tag{4.1}$$

Thus (1.2a) gives

$$B_2{}^i = \beta s^i{}_0, \quad B_3{}^i = \alpha^2 s^i{}_0, \quad B_2{}^i{}_{;j} = s^i{}_0 s_{0j}, \quad B_3{}^i{}_{;j} = 0,$$
$$B_2{}^i{}_{\cdot j} = s^i{}_0 b_j + \beta s^i{}_j, \quad B_3{}^i{}_{\cdot j} = 2s^i{}_0 y_j + \alpha^2 s^i{}_j.$$

Then (2.5a) yields

$$C_3{}^i{}_j = 2\beta(s^i{}_0 y_j + \alpha^2 s^i{}_j), \quad C_4{}^i{}_j = (\alpha^2 + \beta^2)(s^i{}_0 y_j + \alpha^2 s^i{}_j).$$

Next (2.10a) gives $H_3 = H_4 = 0$ and $\beta_{;0} = 0$ leads to $C_{3;0} = C_{4;0} = 0$. Thus (2.11a) gives $I_5 = I_6 = 0$. Further we have $F_6 = F_7 = 0$ from (2.8a) and hence (2.13) leads to $G_8 = -E_8$ and $G_9 = -\alpha^2 E_7$. (2.7a) gives

$$E_8{}^i{}_j = \alpha^2(\alpha^4 + 6\alpha^2\beta^2 + \beta^4) s^i{}_0 s_{0j},$$
$$E_7{}^i{}_j = 4\alpha^2\beta(\alpha^2 + \beta^2) s^i{}_0 s_{0j}.$$

Therefore (2.12) is written as $L^4\alpha^3(R^h{}_{0k} - \overset{a}{R}_0{}^h{}_{0k}) = -3L^4\alpha^3 s^h{}_0 s_{0k}$ which is nothing but (3.3).

Consequently we have

Theorem 2. *Let F^2 be a Randers space of dimension two. If F^2 satisfies $b_{i;j} + b_{j;i} = 0$, $b^r b_{r;i} = 0$ and $b_{i;j;k} = 0$, then the h-curvature tensor H of the Berwald connection depends on the position alone, written in the form* (3.4).

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