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Intersection theorems for Finsler manifolds

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Abstract. In the paper we prove the following theorems generalizing T. Frankel's results on Riemannian manifolds [2].

Theorem A. *If V and W are two compact totally geodesic submanifolds of a real complete connected Finsler manifold M of positive sectional curvature, and* $\dim V + \dim W \geq \dim M$, *then* $V \cap W \neq \emptyset$.

Theorem B. *If V and W are two compact complex analytic submanifolds of a Kähler Finsler manifold M of positive holomorphic bisectional curvature and vanishing* $(1,1)$*-torsion, and* $\dim V + \dim W \geq \dim_{\mathbb{C}} M$, *then* $V \cap W \neq \emptyset$.

1. Introduction

In 1961 T. Frankel [2] proved that two compact, totally geodesic submanifolds V and W of dimension r and s, respectively, of an n-dimensional complete connected Riemannian manifold with positive sectional curvature always have a nonempty intersection provided $r + s \geq n$. Also, if M^n is a complete connected Kählerian manifold of positive sectional curvature, and V and W are two compact complex submanifolds then $V \cap W \neq \emptyset$ provided $r + s \geq n$.

These results have been extended by A. Gray [3] to the case of nearly Kähler spaces, by S. Marchiafava [4] to the case of a quaternionic Kähler

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manifolds, by L. Ornea [5] to the case of locally conformal Kähler manifolds, and by T.Q. Binh, L. Ornea and L. Tamássy [6] to the case of Sasakian manifolds with k-positive bisectional curvature.

The purpose of this paper is to generalize Frankel's theorems to the case of Finsler and Kähler–Finsler manifolds, resp.

Our base reference on Finsler geometry is [1]. At the beginning of the sections we shortly repeat the necessary notions and relations (for details see [1]). Then, we prove the analogons of the cited results of Frankel, first for the real case, and in Section 2, for the complex case.

2. The real case

Let M be a real manifold M of dimension n, and (TM, π, M) the tangent bundle of M. The vertical bundle of the manifold M is the vector bundle $\overline{\pi}\colon \mathcal{V} \to TM$ given by $\mathcal{V} = \ker d\pi \subset T(TM)$. (x^i) will denote local coordinates on an open subset U of M, and (x^i, y^i) the induced coordinates on $\pi^{-1}(U) \subset TM$. The radial vertical vector field ι is locally given by $\iota(x,y) = y^a \frac{\partial}{\partial y^a}$.

A Finsler metric on M is a function $F\colon TM \to \mathbb{R}_+$ satisfying the following properties:

1. F^2 is smooth on $\widetilde{M}$, where $\widetilde{M} = TM \setminus \{0\}$,
2. $F(u) > 0$ for all $u \in \widetilde{M}$,
3. $F(\lambda u) = |\lambda| F(u)$ for all $u \in TM$, $\lambda \in \mathbb{R}$,
4. For any $p \in M$ the indicatrix $I_p = \{u \in T_pM \mid F(u) < 1\}$ is strongly convex.

A manifold M endowed with a Finsler metric F is called a Finsler manifold (M, F).

From the condition 4 it follows the quantities $g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}$ form a positive definite matrix, so a Riemannian metric $\langle\ ,\ \rangle$ can be introduced in the vertical bundle $(\mathcal{V}, \overline{\pi}, TM)$.

On a Finsler manifold there does not exist, in general, a linear metrical connection. The analogue of the Levi–Civita connection lives just in the vertical bundle, however, there are several ones. In this paper we use the Cartan connection, which is a good vertical connection in $\mathcal{V}$, i.e. an $\mathbb{R}$-linear map

$$\nabla^v\colon \mathfrak{X}(\widetilde{M}) \times \mathfrak{X}(\mathcal{V}) \to \mathfrak{X}(\mathcal{V}),$$

having the usual properties of a covariant derivations, metrical with respect to g, and 'good' in the sense that the bundle map $\Lambda\colon T\widetilde{M} \to \mathcal{V}$ defined by $\Lambda(Z) = \nabla^v_Z \iota$ is a bundle isomorphism when ∇^v restricted to $\mathcal{V}$. The latter property induces the horizontal subspaces $H_u = \ker \Lambda$ for all $u \in \widetilde{M}$, which are direct summands of the vertical subspaces $V_u = \operatorname{Ker}(d\pi)_u$:

$$\tau_{\widetilde{M}} = \mathcal{H} \oplus \mathcal{V}.$$

For a tangent vector field X on M we have its vertical lift X^V and its horizontal lift X^H to $\widetilde{M}$.

$\Theta\colon \mathcal{V} \to \mathcal{H}$ denotes the horizontal map associated to the horizontal bundle $\mathcal{H}$. Using Θ, first we get the radial horizontal vector field $\chi = \Theta \circ \iota$. In our case $\dot\sigma^H = \chi(\dot\sigma)$. Secondly we can extend the covariant derivation ∇^v of the vertical bundle to the whole tangent bundle of $\widetilde{M}$: Denoting it with ∇, for horizontal vector fields $H \in \mathfrak{X}(\mathcal{H})$ we let

$$\nabla_Z H = \Theta(\nabla^v_Z(\Theta^{-1}(H))) \quad \forall\, Z \in \mathfrak{X}\widetilde{M}.$$

An arbitrary vector field $Y \in \mathfrak{X}\widetilde{M}$ is decomposed into vertical and horizontal parts:

$$\nabla_Z Y = \nabla_Z Y^V + \nabla_Z Y^H.$$

Thus $\nabla\colon \mathfrak{X}\widetilde{M} \times \mathfrak{X}\widetilde{M} \to \mathfrak{X}\widetilde{M}$ is a linear connection on $\widetilde{M}$ induced by a good vertical connections. Its torsion and curvature Ω are defined as usual. Specially the sectional curvature of ∇ at a tangent vector $u \in \widetilde{M}$ is given as follows:

$$R_u(H,K) = \langle \Omega(\chi(u), H)K, \chi(u)\rangle_u$$

for any $H, K \in \mathcal{H}_u$. In [1] this is called the horizontal flag curvature.

Theorem A. *If V and W are two compact totally geodesic submanifolds of a real complete connected Finsler manifold (M,F) of positive sectional curvature, and* $\dim V + \dim W \geq \dim M$, *then* $V \cap W \neq \emptyset$.

Proof. We assume that V and W do not intersect each other. Then there is a shortest geodesic $\sigma(t)$ from V to W with the endpoints $\sigma(a) \in V$, $\sigma(b) \in W$, for V and W are compact.

All quantities from the tangent level are now horizontally lifted to the second tangent level TTM along the tangent curve $\dot\sigma$ of the geodesic σ. Its reason is that the Cartan connection lives there and we want to use the parallel translation of this linear connection. The horizontal lift from $T_{\sigma(a)}M$ and $T_{\sigma(b)}M$ to $H_{\dot\sigma(a)}$ and $H_{\dot\sigma(b)}$, resp. will be simply denoted by the superscript H.

Since σ is the shortest geodesic from V to W it strikes V and W orthogonally by the Gauss lemma: $\dot\sigma^H(a) \perp T^H_{\sigma(a)}V$ and $\dot\sigma^H(b) \perp T^H_{\sigma(b)}W$.

Let $P \subset H_{\dot\sigma(b)}\widetilde{M}$ be the parallel translated of $T^H_{\sigma(a)}V$ with respect to the Cartan connection along $\dot\sigma$ to the point $\dot\sigma(b)$. The parallel translation of the Cartan connection maps horizontal vectors into horizontal ones, $\dot\sigma^H(a)$ into $\dot\sigma^H(b)$, and it is angle-preserving and dimension-preserving. Therefore $P \perp \dot\sigma^H(b)$ and, so $\dim(P + T^H_{\sigma(b)}(W)) \le \dim M - 1$. Then

$$\begin{aligned}\dim\left(P \cap T^H_{\sigma(b)}W\right) &= \dim P + \dim T^H_{\sigma(b)}W - \dim\left(P + T^H_{\sigma(b)}W\right)\\ &\ge \dim V + \dim W - (\dim M - 1) \ge 1.\end{aligned}$$

Thus there is a vector $w^H \in P \cap T^H_{\sigma(b)}W$ with $\langle w^H, w^H\rangle = 1$. Clearly w^H must be the parallel translated along $\dot\sigma$ of some $v^H \in T^H_pV$ with $\langle v^H, v^H\rangle=1$. Let U^H be the unit tangent horizontal vector field along $\dot\sigma$ obtained by parallel translation of v^H. Consider the variation $\Sigma : (-\varepsilon, \varepsilon) \times [a, b] \to M$ of σ with transversal vector field $X = d\pi(U^H)$. Then, by the second variation formula (cf. [1], p. 38) we have

$$\begin{aligned}\frac{d^2\ell_\Sigma}{ds^2}(0) &= \left\langle \nabla_{U^H}U^H, T^H\right\rangle_{\dot\sigma}\Big|_a^b\\ &+ \int_a^b \left[\left\|\nabla_{T^H}U^H\right\|^2_{\dot\sigma} - \left\langle \Omega\left(T^H, U^H\right)U^H, T^H\right\rangle_{\dot\sigma} - \left|\frac{\partial}{\partial t}\left\langle U^H, T^H\right\rangle_{\dot\sigma}\right|^2\right] dt,\end{aligned}$$

where T and U are the tangential and transversal vector fields, resp., of the variation Σ. U^H is parallel along $\dot\sigma$ and $T^H \circ \dot\sigma = \dot\sigma^H$, so $\nabla_{T^H}U^H|_{\dot\sigma} = \nabla_{\dot\sigma^H}U^H = 0$. Thus the first term of the integrand vanishes. So does the last term, for $U^H \perp T^H$ holds along $\dot\sigma$. The term before the integral can be omitted, since we have chosen such variation where all transversal curves are geodesics, therefore $\nabla_{U^H} U^H = 0$. Summarizing we have

$$\frac{d^2\ell_\Sigma}{ds^2}(0) = -\int_a^b \left\langle \Omega\left(T^H, U^H\right) U^H, T^H \right\rangle_{\dot\sigma} \, dt = -\int_a^b R_{\dot\sigma}\left(U^H, U^H\right) \, dt < 0,$$

which contradicts the minimality of σ. □

3. The complex case

From [1] we recall some basic notions and formulae for complex Finsler manifolds.

Let M be a complex manifold of $\dim_{\mathbb{C}} M = n$. Then the complexification $T_{\mathbb{C}}M$ of the real tangent bundle is decomposed as

$$T_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M$ is the holomorphic tangent bundle over M, and $T^{1,0}M$ is the conjugate of $T^{0,1}M$. As is well known $T^{1,0}M$ is also a complex manifold of $\dim_{\mathbb{C}} T^{1,0}M = 2n$, and the projection $\pi_T \colon T^{0,1}M \to M$ is holomorphic. $T^{1,0}M$ and $T^{0,1}M$ are the eigenspaces of the complex structure J belonging to the eigenvalues i and $-i$, respectively.

A complex Finsler metric on a complex manifold M is a continuous function $F \colon T^{1,0}M \to \mathbb{R}$ satisfying

- $G := F^2$ is smooth on $\widetilde{M}$
- $F(v) > 0$ for all $v \in \widetilde{M}$
- $F(\zeta v) = |\zeta| F(v)$ for all $v \in T^{1,0}M$ and $\zeta \in \mathbb{C}$.

F is called strongly pseudoconvex if the Levi matrix $(G_{\alpha\overline{\beta}})$ is positive definite on $\widetilde{M}$ where $G_{\alpha\overline{\beta}} = \frac{\partial G^2}{\partial v^\alpha \partial \overline{v}^\beta}$.

There exists a unique good complex vertical connection which makes the Hermitian structure $(G_{\alpha\overline{\beta}})$ on $\mathcal{V}$ parallel. It can be extended via the horizontal map to a complex linear connection on $\widetilde{M}$. This is called the complex Chern–Finsler connection ∇.

The geodesics σ are characterized by the equation ([1], p. 101):

$$\nabla_{T^H + \overline{T^H}} T^H = 0,$$

where $T = \dot{\sigma}$. The torsions θ, and τ of ∇ are defined as follows:

$$\theta(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$\tau(X,\overline{Y}) = \nabla_X \overline{Y} - \nabla_{\overline{Y}} X - [X,\overline{Y}],$$

the curvature Ω are defined as usual. The holomorphic bisectional curvature is given as follows

$$R(T,U) = \left\langle \Omega(T^H + \overline{T^H}, U^H + \overline{U^H})U^H, T^H \right\rangle \qquad \forall\, T,U \in T^{1,0}M.$$

It is easy to derive that in the case of the Chern–Finsler connection this takes the form

$$R(T,U) = \left\langle \Omega(T^H, \overline{U^H})U^H, T^H \right\rangle - \left\langle \Omega(U^H, \overline{T^H})U^H, T^H \right\rangle.$$

A strongly pseudoconvex Finsler metric F is called Kähler if its $(2,0)$-torsion θ satisfies

$$\theta(H,\chi) = 0 \qquad \forall\, H \in \mathcal{H}.$$

The horizontal $(1,1)$-torsion is defined by

$$\tau^{\mathcal{H}}(X,\overline{Y}) = \Theta(\tau(X,\overline{Y}))$$

where Θ is the horizontal map. The symmetric product $\langle\!\langle \cdot,\cdot \rangle\!\rangle\colon \mathcal{H}\times\mathcal{H} \to \mathbb{C}$ is locally given by

$$\langle\!\langle H,K \rangle\!\rangle_v = G_{\alpha\beta}(v) H^\alpha H^\beta \qquad \forall\, H,K \in \mathcal{H}_v,\ v \in \widetilde{M}.$$

It is clearly globally well defined and satisfies $\langle\!\langle H,\chi \rangle\!\rangle = 0$ for all $H \in \mathcal{H}$.

In the proof of our second theorem the second variation formula will play a crucial role. Let $F\colon T^{1,0}M \to \mathbb{R}$ be a Kähler Finsler metric on a complex manifold M. Take a geodesic $\sigma_0\colon [a,b] \to M$ with $F(\dot{\sigma}_0) = 1$, and a regular variation $\Sigma\colon (-\varepsilon,\varepsilon)\times[a,b] \to M$ of σ_0. Then it is known [1] (p. 102) that

$$\begin{aligned}\frac{d^2\ell_\Sigma}{ds^2}(0) &= \operatorname{Re}\left\langle \nabla_{U^H+\overline{U^H}} U^H, T^H \right\rangle_{\dot{\sigma}_0} \Big|_a^b \\ &\quad + \int_a^b \left\{ \left\| \nabla_{T^H+\overline{T^H}} U^H \right\|^2_{\dot{\sigma}_0} - \left| \frac{\partial}{\partial t} \operatorname{Re}\left\langle U^H, T^H \right\rangle_{\dot{\sigma}_0} \right|^2 \right.\end{aligned}$$

$$
-\operatorname{Re}\left[\left\langle \Omega(T^H, \overline{U^H})U^H, T^H \right\rangle_{\dot\sigma_0} - \left\langle \Omega(U^H, \overline{T^H})U^H, T^H \right\rangle_{\dot\sigma_0} \right.
$$
$$
\left.\left. + \left\langle\!\left\langle \tau^{\mathcal{H}}(U^H, \overline{T^H}), U^H \right\rangle\!\right\rangle_{\dot\sigma_0} - \left\langle\!\left\langle \tau^{\mathcal{H}}(T^H, \overline{U^H}), U^H \right\rangle\!\right\rangle_{\dot\sigma_0} \right]\right\} dt.
$$

Theorem B. *If V and W are two compact complex analytic submanifolds of a Kähler Finsler manifold (M, F) of positive holomorphic bisectional curvature and vanishing horizontal $(1,1)$-torsion, and $\dim V + \dim W \geq \dim_{\mathbb{C}} M$, then $V \cap W \neq \emptyset$.*

Proof. We use here Frankel's method again. Suppose that $V \cap W = \emptyset$. Then, there exists a minimazing geodesic $\sigma\colon [a,b] \to M$ with $\sigma(a) \in V$, $\sigma(b) \in W$, σ is orthogonal to V and W in $\sigma(a)$ and $\sigma(b)$, resp.

We construct a regular variation $\Sigma\colon (-\varepsilon, \varepsilon) \times [a,b] \to M$ of σ such that $\nabla_{T^H + \overline{T^H}} U^H = 0$. Let $P \subset H_{\dot\sigma(b)} T^{1,0} M$ be the parallel translated of $T^H_{\sigma(a)}(V)$ with respect to the Chern–Finsler connection along $\dot\sigma$ to the point $\dot\sigma(b)$. Considering the horizontal lifts to $\widetilde{M}$ along $\dot\sigma$, analogously to the real case we get

$$
\begin{aligned}
\dim\left(P \cap (T^H_{\sigma(b)} W)\right) &= \dim P + \dim(T^H_{\sigma(b)} W) - \dim\left(P + (T^H_{\sigma(b)} W)\right) \\
&\geq \dim V + \dim W - (\dim_{\mathbb{C}} M - 1) \geq 1.
\end{aligned}
$$

So we can choose a non-vanishing vector $U^H \in P \cap (T^H_{\sigma(b)} W)$. Its parallel translated along $\dot\sigma$ will be denoted by U^H, too. Since U^H is orthogonal to $\dot\sigma$ at the endpoint $\dot\sigma(b)$, it remains orthogonal along the entire curve $\dot\sigma(t)$ by the metrical property of the Chern–Finsler connection. We consider the regular variation of σ with transversal vector field U.

In this case the second variation formula reduces to the following form:

$$
\begin{aligned}
\frac{d^2 \ell_\Sigma}{ds^2}(0) &= \operatorname{Re}\left\langle \nabla_{U^H + \overline{U^H}} U^H, T^H \right\rangle_{\dot\sigma} \Big|_a^b \\
&\quad + \int_a^b \left\{ \left\| \nabla_{T^H + \overline{T^H}} U^H \right\|_{\dot\sigma}^2 - \left| \frac{\partial}{\partial t} \operatorname{Re}\left\langle U^H, T^H \right\rangle_{\dot\sigma} \right|^2 - \operatorname{Re}\left[R_{\dot\sigma}(T, U) \right] \right\} dt,
\end{aligned}
$$

since the horizontal $(1,1)$-torsion $\tau^{\mathcal{H}}$ vanishes.

The first term of the integrand vanishes, for U^H is parallel along σ, and therefore, by the hypothesis on the holomorphic bisectional curvature the remaining two terms of the integrand will be negative except the first one at most. We consider also the variation belonging to the transversal vector field JU^H, and prove that the initial terms belonging to U^H, and JU^H cannot be positive at the same time. This will give the contradiction.

Therefore we calculate $\nabla_{JU^H+\overline{JU^H}}JU^H$.

$$\nabla_{JU^H+\overline{JU^H}}JU^H = J\nabla_{JU^H+\overline{JU^H}}U^H = J\left(\nabla_{JU^H}U^H + \nabla_{\overline{JU^H}}U^H\right).$$

Using the torsion we have

$$\nabla_{JU^H}U^H = \nabla_{U^H}JU^H + \left[JU^H, U^H\right] + \theta\left(JU^H, U^H\right).$$

The last term $\theta(JU^H, U^H)$ vanishes because F is a Kähler Finsler metric, and U^H is a radial vector field. Since the horizontal $(1,1)$-torsion is zero,

$$\begin{aligned}\nabla_{\overline{JU^H}}U^H &= \nabla_{U^H}\overline{JU^H} - [U^H, \overline{JU^H}] \\ &= J\left[\nabla_{\overline{U^H}}U^H + [U^H, \overline{U^H}]\right] - [U^H, \overline{JU^H}] \\ &= J\nabla_{\overline{U^H}}U^H + J[U^H, \overline{U^H}] - [U^H, \overline{JU^H}].\end{aligned}$$

It follows now

$$\begin{gathered}\nabla_{JU^H+\overline{JU^H}}JU^H \\ J\left(\nabla_{U^H}JU^H + [JU^H, U^H] + J\nabla_{\overline{U^H}}U^H + J[U^H, \overline{U^H}] - [U^H, \overline{JU^H}]\right) \\ = -\nabla_{U^H+\overline{U^H}}U^H + J[JU^H, U^H] - J[U^H, \overline{U^H}] - [U^H, \overline{JU^H}].\end{gathered}$$

Now V and W are complex submanifolds, U^H is a horizontal lift, and tangent to $T^H_{\sigma(a)}V$ and $T^H_{\sigma(b)}W$ at $\dot\sigma(a)$ and $\dot\sigma(b)$, respectively. Since the horizontal space is a complex linear space, and we use the Chern Finsler connection, all Lie brackets above are horizontal vectors, and are orthogonal to T^H at $\sigma(a)$ and $\sigma(b)$. So

$$\operatorname{Re}\left\langle \nabla_{JU^H+\overline{JU^H}}JU^H, T^H\right\rangle = -\operatorname{Re}\left\langle \nabla_{U^H+\overline{U^H}}U^H, T^H\right\rangle.$$

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