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A general Minkowski-type inequality for two variable Gini means

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Abstract. We study the following Minkowski-type inequality

$$(*)\qquad S_{a_0,b_0}(x_1+y_1,x_2+y_2) \le S_{a_1,b_1}(x_1,x_2)+S_{a_2,b_2}(y_1,y_2) \qquad (x_1,x_2,y_1,y_2\in\mathbb{R}_+),$$

where $S_{a,b}$ is the two variable Gini mean defined by

$$S_{a,b}(x,y)=\begin{cases}\left(\dfrac{x^a+y^a}{x^b+y^b}\right)^{\frac{1}{a-b}} & \text{if } a-b\neq 0,\\[2ex] \exp\left(\dfrac{x^a\ln x+y^a\ln y}{x^a+y^a}\right) & \text{if } a-b=0\end{cases}\qquad (a,b\in\mathbb{R},\ x,y>0).$$

The case when $a_0=a_1=a_2$ and $b_0=b_1=b_2$ was investigated by Losonczi–Páles [LP96]. Generalizing their result, we give necessary and sufficient conditions (concerning the parameters $a_i,b_i\in\mathbb{R}$) for the inequality above to hold. As a consequence of this result, it turns out that any inequality of the form $(*)$ is weakening of an analogous inequality where all the participating means are equal to each other.

1. Introduction

Let $a,b\in\mathbb{R}$ be two real numbers. The Gini mean [Gin38] of an

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n-vector $\mathbf{x} = (x_1, \dots, x_n)$ with coordinates in $\mathbb{R}_+ = (0, \infty)$ is defined by

$$(1) \qquad S_{a,b;n}(\mathbf{x}) = \begin{cases} \left(\dfrac{x_1^a + \cdots + x_n^a}{x_1^b + \cdots + x_n^b} \right)^{\frac{1}{a-b}} & \text{if } a - b \neq 0, \\ \exp \left(\dfrac{x_1^a \ln x_1 + \cdots + x_n^a \ln x_n}{x_1^a + \cdots + x_n^a} \right) & \text{if } a - b = 0. \end{cases}$$

Minkowski's inequality for the special Gini mean with $a - b = 1$ was treated by Beckenbach [Bec50]. Concerning the general case

$$(2) \qquad S_{a,b;n}(\mathbf{x} + \mathbf{y}) \leq S_{a,b;n}(\mathbf{x}) + S_{a,b;n}(\mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n,\ n = 2, 3, \dots),$$

Dresher [Dre53] and also Danskin [Dan52] proved that the conditions

$$(3) \qquad 0 \leq \min\{a, b\} \leq 1 \leq \max\{a, b\}$$

are sufficient for (2) to hold. Losonczi [Los71b] showed that the inequality (2) is not only sufficient but it is also necessary for (3) to hold. He also proved that the reverse inequality

$$(4) \qquad S_{a,b;n}(\mathbf{x} + \mathbf{y}) \geq S_{a,b;n}(\mathbf{x}) + S_{a,b;n}(\mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n,\ n = 2, 3, \dots)$$

holds if and only if

$$(5) \qquad \min\{a, b\} \leq 0 \leq \max\{a, b\} \leq 1$$

is satisfied. In [Los77], the inequalities (2), (4) were characterized in the case, where the coordinates of the variables $\mathbf{x}, \mathbf{y}$ vary only in a subinterval (α, β) of $\mathbb{R}_+$.

Another possibility to generalize (2) is that each appearance of $S_{a,b}$ is replaced by a possibly different Gini mean, that is we ask for necessary and sufficient conditions such that

$$(6) \quad S_{a_0,b_0;n}(\mathbf{x} + \mathbf{y}) \leq S_{a_1,b_1;n}(\mathbf{x}) + S_{a_2,b_2;n}(\mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n,\ n = 2, 3, \dots)$$

be valid. The result obtained by Páles [Pál82] states that (6) is valid on the domain indicated if and only if

$$(7) \qquad \begin{array}{rll} \text{(i)} & a_1, a_2, b_1, b_2 \geq 0, & \\ \text{(ii)} & \max\{1, a_0, b_0\} \leq \max\{a_i, b_i\}, & (i = 1, 2), \\ \text{(iii)} & \min\{a_0, b_0\} \leq \min\{1, a_1, b_1, a_2, b_2\}. & \end{array}$$

The reversed inequality

$$(8)\quad S_{a_0,b_0;n}(\mathbf{x}+\mathbf{y}) \geq S_{a_1,b_1;n}(\mathbf{x}) + S_{a_2,b_2;n}(\mathbf{y}) \quad (\mathbf{x},\mathbf{y} \in \mathbb{R}_+^n,\ n = 2,3,\dots)$$

was also characterized in [Pál82]. It holds if and only if

$$(9)\quad \begin{array}{cll} \text{(i)} & 1 \geq a_1, a_2, b_1, b_2, & \\ \text{(ii)} & \min\{0, a_0, b_0\} \geq \min\{a_i, b_i\}, & (i = 1,2), \\ \text{(iii)} & \max\{a_0, b_0\} \geq \max\{0, a_1, b_1, a_2, b_2\}. & \end{array}$$

Further methods and results were obtained by Daróczy and Losonczi [DL70], Losonczi [Los71a], [Los71b], Páles [Pál83] for characterizing inequalities (of quite general form) involving quasiarithmetic means weighted by weightfunctions and by Daróczy [Dar72a], [Dar72b], Losonczi [Los73], Daróczy and Páles [DP82] and Páles [Pál88b] for more general means (deviation and quasideviation means).

In these general results, however, one has to suppose that *the inequalities hold for all* $n = 2, 3, \dots$. Fixing the number of variables n in (2), (4), (6), and (8), we obtain new problems to investigate. The first step in this direction is of course studying the case $n = 2$ and inequalities (2) and (4). This was done in the paper of Losonczi and Páles [LP96]. For brevity of notation, we use $S_{a,b}$ for $S_{a,b;2}$ throughout the paper. Then the main result of [LP96] can be formulated as follows.

Theorem 1 (Losonczi–Páles [LP96]). *Let $a, b \in \mathbb{R}$. Then the inequality*

$$(10)\qquad S_{a,b}(\mathbf{x}+\mathbf{y}) \leq S_{a,b}(\mathbf{x}) + S_{a,b}(\mathbf{y}) \quad (\mathbf{x},\mathbf{y} \in \mathbb{R}_+^2)$$

holds if and only if

$$(11)\qquad 0 \leq \min\{a,b\} \leq 1 \leq a+b.$$

The main aim of the present paper is to characterize the situation when the more general inequality

$$(12)\qquad S_{a_0,b_0}(\mathbf{x}+\mathbf{y}) \leq S_{a_1,b_1}(\mathbf{x}) + S_{a_2,b_2}(\mathbf{y}) \quad (\mathbf{x},\mathbf{y} \in \mathbb{R}_+^2)$$

holds. Our main result is contained in the following theorem.

Theorem 2. *Let $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$. Then* (12) *holds if and only if*

$$\text{(13)}\qquad \begin{aligned} &\text{(i)} && a_1, a_2, b_1, b_2 \geq 0,\\ &\text{(ii)} && \max\{1, a_0 + b_0\} \leq \min\{a_1 + b_1, a_2 + b_2\},\\ &\text{(iii)} && \min\{a_0, b_0\} \leq \min\{1, a_1, b_1, a_2, b_2\}. \end{aligned}$$

The proof of the necessity of conditions (i)–(iii) of this result will be obtained with the help of a sequence of lemmas. The proof of the sufficiency is based on Theorem 1, since, as it will turn out, conditions (i)–(iii) of Theorem 2 are necessary and sufficient for the existence of some parameters $a, b \in \mathbb{R}$ such that (10) is valid and

$$S_{a_0,b_0} \leq S_{a,b}, \qquad S_{a,b} \leq S_{a_1,b_1}, \qquad S_{a,b} \leq S_{a_2,b_2}$$

hold. Thus any inequality of the form (12) is a *weakening* of inequality (10) for some $a, b \in \mathbb{R}$.

Concerning the inequality

$$S_{a,b}(\mathbf{x}+\mathbf{y}) \geq S_{a,b}(\mathbf{x}) + S_{a,b}(\mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^2) \tag{14}$$

which is reversed to (10), there are only necessary (but not sufficient) and sufficient (but not necessary) conditions presented in [LP96]. Therefore, the investigation of the inequality reversed to (12) is left as an open problem.

It is interesting to note that the analogous problems, that is, the Minkowski and reversed Minkowski inequalities for the so called Stolarski means can be characterized completely (see Losonczi–Páles [LP98]).

The paper is organized as follows. In Section 2, we recall and extend the result on the comparison of two variable Gini means obtained by Páles [Pál88a] (see also [Pál92]). In Section 3 we establish some asymptotic properties of Gini means that will be useful in proving the necessity of the conditions for (12). In Section 4 we give the complete proof of Theorem 2. Finally, we formulate a generalization of Theorem 2 which can be proved exactly in the same way as Theorem 2.

2. Comparison of two variable Gini means

The comparison problem of two variable Gini means on $\mathbb{R}_+$ was solved by PÁLES [Pál88]. The main result of this paper reads as follows.

Theorem 3. *Suppose that $a, b, c, d \in \mathbb{R}$, $(a-b)(c-d) \neq 0$. Then*

$$S_{a,b}(x,y) \leq S_{c,d}(x,y) \quad (x, y \in \mathbb{R}_+) \tag{15}$$

holds if and only if

$$\begin{aligned} &\text{(i)} \quad a+b \leq c+d, \\ &\text{(ii)} \quad \begin{cases} \min\{a,b\} \leq \min\{c,d\}, & \textit{if } \min\{a,b,c,d\} \geq 0, \\ \max\{a,b\} \leq \max\{c,d\}, & \textit{if } \max\{a,b,c,d\} \leq 0, \\ \dfrac{|a|-|b|}{a-b} \leq \dfrac{|c|-|d|}{c-d}, & \textit{if } \min\{a,b,c,d\} < 0 \\ & \quad 0 < \max\{a,b,c,d\}. \end{cases} \end{aligned} \tag{16}$$

This theorem does not offer conditions when $(a-b)(c-d) = 0$. In order to cover this case as well, we extended Theorem 3 via the following result.

Theorem 4. *Suppose that $a, b, c, d \in \mathbb{R}$. Then* (15) *holds if and only if*

$$\begin{aligned} &\text{(i)} \quad a+b \leq c+d, \\ &\text{(ii)} \quad \begin{cases} \min\{a,b\} \leq \min\{c,d\}, & \textit{if } \min\{a,b,c,d\} \geq 0, \\ \max\{a,b\} \leq \max\{c,d\}, & \textit{if } \max\{a,b,c,d\} \leq 0, \\ \mu(a,b) \leq \mu(c,d) & \textit{if } \min\{a,b,c,d\} < 0 \\ & \quad 0 < \max\{a,b,c,d\}, \end{cases} \end{aligned} \tag{17}$$

where

$$\mu(u,v) := \begin{cases} \dfrac{|u|-|v|}{u-v}, & \textit{if } u \neq v, \\ \operatorname{sgn}(u), & \textit{if } u = v. \end{cases}$$

PROOF. The case $(a-b)(c-d) \neq 0$ is discussed in [Pál88a] in details, hence it suffices to consider the case $(a-b)(c-d) = 0$.

We will use the following auxiliary results:

Lemma 1. *For $a, b \in \mathbb{R}$, we have the identity*

$$S_{a,b}(x,y) = \left[S_{-a,-b}(x^{-1}, y^{-1})\right]^{-1} \quad (x, y \in \mathbb{R}_+). \tag{18}$$

Lemma 2. *The function μ defined in the theorem admits the following properties:*

(i) *μ is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$.*

(ii) *For any fixed real number v, the function $u \mapsto \mu(u,v)$ is increasing on $\mathbb{R}$;*

(iii) *For all $u, v \in \mathbb{R}$, $-1 \leq \mu(u,v) \leq 1$;*

These results can immediately be obtained from the definitions of Gini means and the function μ.

Sufficiency. Assume first that a, b, c, d are nonnegative numbers such that (17)(i) and

$$\min\{a,b\} \leq \min\{c,d\} \tag{19}$$

hold. We have to show that (15) is valid. Define

$$a_n := a, \qquad b_n := b + \frac{1}{n}, \qquad c_n := c + \frac{1}{n}, \qquad d_n := d + \frac{2}{n}.$$

It is clear that $a_n \neq b_n$ and $c_n \neq d_n$ for n large enough. Furthermore, by (16)(i) and (19),

$$a_n + b_n \leq c_n + d_n, \quad \min\{a_n, b_n\} \leq \min\{c_n, d_n\}, \quad \min\{a_n, b_n, c_n, d_n\} \geq 0.$$

Therefore,

$$S_{a_n,b_n}(x,y) \leq S_{c_n,d_n}(x,y) \quad (x, y \in \mathbb{R}_+).$$

The inequality (15) now follows by taking the limit $n \to \infty$ and using the continuity of Gini means with respect to their parameters.

Using Lemma 1, the sufficiency of the conditions of (17) for the case $a, b, c, d \leq 0$ can directly be obtained from the previous one.

Now consider the case $\min\{a,b,c,d\} < 0 < \max\{a,b,c,d\}$. Assume that (17)(i) and $\mu(a,b) \leq \mu(c,d)$ hold. Define

$$a_n := a - \frac{1}{n}, \qquad b_n := b, \qquad c_n := c, \qquad d_n := d + \frac{1}{n}.$$

By Lemma 2 and (17)(i),

$$\mu(a_n, b_n) = \mu(a_n, b) \leq \mu(a, b) \leq \mu(c, d) \leq \mu(c, d_n) = \mu(c_n, d_n).$$

If n is large enough, then $a_n \neq b_n$, $c_n \neq d_n$ and we also have $a_n + b_n \leq c_n + d_n$. Therefore we can apply Theorem 3 again and the argument can similarly be completed as in the first case.

Thus the proof of the sufficiency is complete.

A simple consequence of the sufficiency is that $S_{a,b}$ is an increasing function of its parameters, that is, we have

Lemma 3. *If $a, b_1, b_2 \in \mathbb{R}$, $b_1 \leq b_2$, then*

$$S_{a,b_1}(x, y) \leq S_{a,b_2}(x, y) \quad (x, y \in \mathbb{R}_+).$$

Necessity. We will again distinguish the same three cases. In all cases, due to the symmetry, we can assume that $a \leq b$ and $c \leq d$.

First, let all the parameters be nonnegative numbers. Assuming (15), we have to show that (17)(i) and (19) hold. In the case $a = b = 0$ there is nothing to prove. Thus we can suppose that $b > 0$. Define

$$a_n := \begin{cases} a - \dfrac{1}{n}, & \text{if } a > 0, \\ a = 0, & \text{if } a = 0, \end{cases} \qquad b_n := \begin{cases} b, & \text{if } a > 0, \\ b - \dfrac{1}{n}, & \text{if } a = 0, \end{cases}$$

$$c_n := c, \qquad d_n := d + \frac{1}{n}.$$

Applying Lemma 3 and (15), we get

$$S_{a_n,b_n}(x, y) \leq S_{a,b}(x, y) \leq S_{c,d}(x, y) \leq S_{c_n,d_n}(x, y), \quad (x, y \in \mathbb{R}_+),$$

and $(a_n - b_n)(c_n - d_n) \neq 0$ for large n. So by Theorem 3, we obtain

$$a_n + b_n \leq c_n + d_n, \quad \min\{a_n, b_n\} \leq \min\{c_n, d_n\}.$$

Performing the limit $n \to \infty$, we get (17).

The statement for the case $a, b, c, d \leq 0$ follows from the first case with the application of Lemma 1.

To complete the proof, we consider now the case $\min\{a,b,c,d\} < 0 < \max\{a,b,c,d\}$. Define

$$a_n := a - \frac{1}{n}, \qquad d_n := d + \frac{1}{n}.$$

Then, for large n, we have $a_n \neq b$, $c \neq d_n$, and

$$\min\{a_n, b, c, d_n\} < 0 < \max\{a_n, b, c, d_n\}.$$

Furthermore, by (15) and Lemma 3,

$$S_{a_n,b}(x,y) \le S_{a,b}(x,y) \le S_{c,d}(x,y) \le S_{c,d_n}(x,y) \quad (x, y \in \mathbb{R}_+),$$

that is, by Theorem 3,

$$a_n + b \le c + d_n \quad \text{and} \quad \mu(a_n, b) \le \mu(c, d_n). \tag{20}$$

Taking the limit $n \to \infty$ in the first inequality, we get $a + b \le c + d$, that is (17)(i). The inequality

$$\mu(a,b) \le \mu(c,d)$$

also follows from (20), by Lemma 2, if $(a,b) \neq (0,0)$ and $(c,d) \neq (0,0)$. We cannot have $a = b = c = d = 0$, hence we have to consider only the cases when either $a = b = 0$, $c < 0 < d$, or $a < 0 < b$, $c = d = 0$. For symmetry reasons (e.g. using Lemma 1), it suffices to consider the first case. We already have (17)(i). Thus, $0 \le c + d$. On the other hand, $\mu(0,0) = 0$, hence $\mu(a,b) \le \mu(c,d)$ is equivalent to $c + d \ge 0$. Therefore (17)(ii) also holds.

The proof of the theorem is complete. □

3. Asymptotic properties of Gini means

In this section, we list a number of asymptotic properties (as the variables tend to zero, or infinity) of Gini means. The first result is known also for any homogeneous means (cf. [ALP87], [AP88], [BC87], [HN85]).

Lemma 4. *Assume that $a, b \in \mathbb{R}$. Then*

$$\lim_{y\to\infty} \big(S_{a,b}(x_1+y, x_2+y) - y\big) = S_{0,1}(x_1, x_2) \quad (x_1, x_2 \in \mathbb{R}_+).$$

Proof. Using the homogeneity of $S_{a,b}$ and replacing y by $1/t$, we get

$$\lim_{y\to\infty} \big(S_{a,b}(x_1+y, x_2+y) - y\big) = \lim_{t\to 0+} \frac{S_{a,b}(tx_1+1, tx_2+1) - 1}{t}$$
$$= \frac{\partial}{\partial t} S_{a,b}(tx_1+1, tx_2+1)|_{t=0} = \frac{x_1+x_2}{2}. \qquad \square$$

Lemma 5. *Suppose that a, b are positive real numbers. Then*

$$\lim_{\substack{x_1\to 0+\\ x_2\to y}} S_{a,b}(x_1, x_2) = y, \quad (y \in \mathbb{R}_+).$$

Proof. The statement easily follows from the definition of Gini means. $\square$

Lemma 6. *Assume that $a, b > 1$. Then*

$$\lim_{y\to\infty} \big(S_{a,b}(x_1, x_2+y) - y\big) = x_2. \quad (x_1, x_2 \in \mathbb{R}_+).$$

Proof. Using the homogeneity of $S_{a,b}$ and replacing y by $1/t$, we get

$$\lim_{y\to\infty} \big(S_{a,b}(x_1, x_2+y) - y\big) = \lim_{t\to 0} \frac{S_{a,b}(tx_1, tx_2+1) - 1}{t}.$$

Due to Lemma 5, the numerator on the right hand side goes to 0, hence we can apply L'Hospital's rule again to obtain the statement. $\square$

4. Proof of the generalized Minkowski inequality for Gini means

After these preparations, we are ready to prove Theorem 2. Because of the symmetry, we can assume that

$$a_0 \le b_0, \qquad a_1 \le b_1, \qquad a_2 \le b_2. \tag{21}$$

Sufficiency. Define

$$a := \min\{a_1, a_2, 1\}, \qquad b := \min\{a_1 + b_1, a_2 + b_2\} - a.$$

We are going to prove the following three statements.

(I) $S_{a,b}$ satisfies the Minkowski inequality (10),

(II) $S_{a_0,b_0} \le S_{a,b}$,

(III) $S_{a,b} \le S_{a_i,b_i}$ $(i = 1, 2)$.

Once we have proved (I)–(III), we can obtain (15) in the following way:

$$S_{a_0,b_0}(\mathbf{x}+\mathbf{y}) \le S_{a,b}(\mathbf{x}+\mathbf{y}) \le S_{a,b}(\mathbf{x}) + S_{a,b}(\mathbf{y}) \le S_{a_1,b_1}(\mathbf{x}) + S_{a_2,b_2}(\mathbf{y}).$$

In order to prove (I), we have to verify that (11) holds. By the definition of a and (13)(i), we get that $0 \le a \le 1$. By (13)(ii), we also have

$$\min\{a_1 + b_1, a_2 + b_2\} \ge 1 \ge a.$$

Thus

$$b = \min\{a_1 + b_1, a_2 + b_2\} - a \ge 0,$$

whence $0 \le \min\{a, b\} \le 1$. Using again the definition of a, b, and (13)(ii), we obtain

$$a + b = \min\{a_1 + b_1, a_2 + b_2\} \ge 1. \tag{22}$$

Therefore $S_{a,b}$ satisfies the Minkowski inequality (10).

In order to prove (II), we distinguish two cases.

If $a_0 < 0$, then $S_{a_0,b_0} \le S_{a,b}$ holds if and only if

$$a_0 + b_0 \le a + b, \quad \mu(a_0, b_0) \le \mu(a, b). \tag{23}$$

The first inequality follows from (13)(ii):

$$a_0 + b_0 \le \max\{1, a_0 + b_0\} \le \min\{a_1 + b_1, a_2 + b_2\} = a + b.$$

Due to (22), $\max\{a, b\} > 0$. Therefore, we have that $\mu(a, b) = 1$. Thus, by Lemma 2, the second inequality in (23) is obvious.

If $a_0 \ge 0$ then $S_{a_0,b_0} \le S_{a,b}$ holds if and only if

$$a_0 + b_0 \le a + b, \quad \min\{a_0, b_0\} \le \min\{a, b\}. \tag{24}$$

The proof of the first inequality coincides with that of the previous case. To obtain the second inequality, we show that $a_0 \le a$ and $a_0 \le b$. Since then

$$\min\{a_0, b_0\} = a_0 \le \min\{a, b\}.$$

By (13)(iii),

$$a_0 = \min\{a_0, b_0\} \le \min\{1, a_1, b_1, a_2, b_2\} = \min\{1, a_1, a_2\} = a.$$

In order to obtain $a_0 \le b$ we need to show that

$$a_0 \le \min\{a_1 + b_1, a_2 + b_2\} - \min\{a_1, a_2, 1\},$$

which is equivalent to the inequalities

$$a_0 + \min\{a_1, a_2, 1\} \le a_i + b_i \quad (i = 1, 2).$$

On the other hand, by (13)(iii) and (21)

$$a_0 + \min\{a_1, a_2, 1\} \le a_0 + a_i \le a_i + a_i \le a_i + b_i \quad (i = 1, 2).$$

Thus we have proved (II).

To obtain (III), we have to show that

$$a + b \le a_i + b_i, \quad \min\{a, b\} \le \min\{a_i, b_i\} \quad (i = 1, 2). \tag{25}$$

The first inequality obviously follows from the definition of b. On the other hand,

$$\min\{a, b\} \le a = \min\{a_1, a_2, 1\} \le a_i = \min\{a_i, b_i\} \quad (i = 1, 2),$$

therefore the second inequality of (25) is also valid. Thus the proof of (III) is also complete.

Necessity. Assume now that the Minkowski inequality (12) holds. Substitute $y_1 = y_2 = y$ in (12). We obtain that

$$S_{a_0,b_0}(x_1 + y, x_2 + y) - y \le S_{a_1,b_1}(x_1, x_2) \quad (x_1, x_2, y \in \mathbb{R}_+). \tag{26}$$

Taking the limit $y \to \infty$ and using Lemma 4, we get

$$\begin{gathered} \frac{x_1 + x_2}{2} \le S_{a_1,b_1}(x_1, x_2), \quad \text{that is,} \\ S_{0,1}(x_1, x_2) \le S_{a_1,b_1}(x_1, x_2) \quad (x_1, x_2 \in \mathbb{R}_+). \end{gathered} \tag{27}$$

Consequently, by Theorem 3, $0+1 \le a_1+b_1$, and analogously, $0+1 \le a_2+b_2$, therefore

$$1 \le \min\{a_1+b_1, a_2+b_2\}. \tag{28}$$

Using (21), we get that $b_1, b_2 > 0$. If a_1 were negative, then (27) would also yield that $\mu(1,0) \le \mu(a_1,b_1)$, that is, $1 \le \frac{|a_1|-|b_1|}{a_1-b_1}$. This inequality however implies $a_1 \ge 0$. The contradiction obtained shows that $a_1 \ge 0$ and similarly $a_2 \ge 0$. Thus (13)(i) is proved.

In order to prove (13)(ii), take the limit $y \to 0$ for both sides of (26). Then we get that

$$S_{a_0,b_0}(x_1,x_2) \le S_{a_1,b_1}(x_1,x_2) \quad (x_1,x_2 \in \mathbb{R}_+). \tag{29}$$

Thus, by Theorem 4, the inequality $a_0+b_0 \le a_1+b_1$ holds. Analogously, we can obtain $a_0+b_0 \le a_2+b_2$. These inequalities together with (28) yield (13)(ii).

To obtain (13)(iii), we show first that

$$\min\{a_0,b_0\} \le \min\{a_1,b_1,a_2,b_2\}. \tag{30}$$

In the case $\min\{a_0,b_0\} < 0$, (30) is obvious because the right hand side is nonnegative. In case of $\min\{a_0,b_0\} \ge 0$, (29) and Theorem 4 yield

$$\min\{a_0,b_0\} \le \min\{a_1,b_1\}.$$

Similarly

$$\min\{a_0,b_0\} \le \min\{a_2,b_2\}.$$

Hence (30) is valid.

To complete the proof, we have only to show that $\min\{a_0,b_0\} \le 1$. On the contrary, suppose that $a_0, b_0 > 1$. We know, by (30), that in this case $a_1,b_1,a_2,b_2 > 1$ (and consequently $a_1,b_1,a_2,b_2 > 0$). Taking the limit $y_1 \to 0$ in (12) and applying Lemma 5, we obtain

$$S_{a_0,b_0}(x_1,x_2+y_2) \le S_{a_1,b_1}(x_1,x_2) + y_2 \quad (x_1,x_2,y_2 \in \mathbb{R}_+).$$

Thus

$$\lim_{y_2\to\infty} \big(S_{a_0,b_0}(x_1,x_2+y_2) - y_2\big) \le S_{a_1,b_1}(x_1,x_2) \quad (x_1,x_2 \in \mathbb{R}_+). \tag{31}$$

By Lemma 6, the limit of the left hand side is x_2, that is,

$$x_2 \leq S_{a_1,b_1}(x_1, x_2),$$

for all positive x_1 and x_2. If $x_1 < x_2$, then the inequality obtained contradicts the mean value property of $S_{a,b}$.

Thus the proof of Theorem 2 is completed.

Using the ideas followed in the paper, one can get the following generalization of Theorem 2.

Theorem 5. *Let $k \geq 2$ and $a_0, a_1, \dots, a_k, b_0, b_1, \dots, b_k \in \mathbb{R}$. Then*

$$S_{a_0,b_0}(\mathbf{x}_1 + \dots + \mathbf{x}_k) \leq S_{a_1,b_1}(\mathbf{x}_1) + \dots + S_{a_k,b_k}(\mathbf{x}_k) \quad (\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}_+^2)$$

holds if and only if

(i) $\qquad a_1, \dots, a_k, b_1, \dots, b_k \geq 0,$

(ii) $\qquad \max\{1, a_0 + b_0\} \leq \min\{a_1 + b_1, \dots, a_k + b_k\},$

(iii) $\qquad \min\{a_0, b_0\} \leq \min\{1, a_1, b_1, \dots, a_k, b_k\}.$

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