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Equality of Cauchy mean values

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Abstract. The Cauchy Mean Value Theorem for divided differences (see e.g. [2]) states the following:

Suppose that $x_1 \le \dots \le x_n$ and $f^{(n-1)}$, $g^{(n-1)}$ exist, with $g^{(n-1)} \neq 0$, on $[x_1, x_n]$. Then there is a $t \in [x_1, x_n]$ (moreover $t \in (x_1, x_n)$ if $x_1 < x_n$) such that

$$\frac{[x_1, \dots, x_n]_f}{[x_1, \dots, x_n]_g} = \frac{f^{(n-1)}(t)}{g^{(n-1)}(t)}$$

where $[x_1, \dots, x_n]_f$ denotes the divided difference of f at the points $x_1, \dots, x_n$.

If the function $\frac{f^{(n-1)}}{g^{(n-1)}}$ is invertible then

$$t = \left(\frac{f^{(n-1)}}{g^{(n-1)}}\right)^{-1} \left(\frac{[x_1, \dots, x_n]_f}{[x_1, \dots, x_n]_g}\right)$$

is a mean value of $x_1, \dots, x_n$. It is called the *Cauchy mean of the numbers $x_1, \dots, x_n$* and will be denoted by $D_{f,g}(x_1, \dots, x_n)$.

Here we solve the equality problem of Cauchy means for $n \ge 3$ i.e. we solve the functional equation

$$D_{f,g}(x_1, x_2, \dots, x_n) = D_{F,G}(x_1, x_2, \dots, x_n) \qquad (x_1, x_2, \dots, x_n \in I)$$

under differentiability conditions.

1. Introduction

As it is well known, the Cauchy mean value theorem of the differential calculus states the following.

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If f, g are continuous real functions on $[x_1, x_2]$ which are differentiable in (x_1, x_2), and $g'(u) \neq 0$ for $u \in (x_1, x_2)$ then there is a point $t \in (x_1, x_2)$ such that

$$\frac{f'(t)}{g'(t)} = \frac{f(x_2) - f(x_1)}{g(x_2) - g(x_1)}.$$

Assuming now that $\frac{f'}{g'}$ is invertible we get

$$t = \left(\frac{f'}{g'}\right)^{-1} \left(\frac{f(x_2) - f(x_1)}{g(x_2) - g(x_1)}\right).$$

This number t is called the *Cauchy mean value of the numbers* x_1, x_2 and will be denoted by $t = D_{f,g}(x_1, x_2)$.

It is possible to define the Cauchy mean value for several variables. To do so we need a mean value theorem for divided differences.

For a function $f : I \to \mathbb{R}$, I being a real interval, the divided differences of f on *distinct* points $x_i \in I$ are usually defined inductively by

$$[x_1]_f := f(x_1),$$

$$[x_1, \dots, x_n]_f := \frac{[x_1, \dots, x_{n-1}]_f - [x_2, \dots, x_n]_f}{x_1 - x_n} \quad (n = 2, 3, \dots)$$

(see e.g. Aumann and Haupt [1] §3.17, their expression contains an extra factor $n - 1$ on the right).

This definition must be modified if two or more points of $[x_1, \dots, x_n]_f$ coincide: if at most r points x_i coincide, the definition is then framed on the assumption that f is $(r-1)$-times differentiable on I. In the case $n = 2$ for example we obtain

$$[x_1, x_2]_f := \begin{cases} \dfrac{f(x_1) - f(x_2)}{x_1 - x_2} & (x_1 \neq x_2), \\ f'(x_1) & (x_1 = x_2). \end{cases}$$

A full definition, as the ratio of two determinants, can be found in Schumaker [7].

Some basic properties of the divided differences are as follows:

(1) A divided difference $[x_1, \dots, x_n]_f$ is independent of the order of its arguments $x_1, \dots, x_n$.

(2) The second line of the above inductive definition remains valid provided only that $x_1 \neq x_n$.

(3) A divided difference is a linear functional, i.e. we have

$$[x_1, \ldots, x_n]_{af+bg} = a[x_1, \ldots, x_n]_f + b[x_1, \ldots, x_n]_g$$

for arbitrary constants a, b and arbitrary (suitably differentiable) functions f, g.

(4) (Mean value theorem) If f is $(n-1)$-times differentiable on I and $x_i \in I\,(i = 1, \ldots, n)$, then there is a t between the smallest and largest x_i (strictly between if the x_i are not all the same) such that

$$[x_1, \ldots, x_n]_f = \frac{f^{(n-1)}(t)}{(n-1)!}.$$

(5) The "Leibniz rule" for divided differences

$$[x_1, \ldots, x_n]_{fg} = \sum_{i=1}^{n} [x_1, \ldots, x_i]_f \cdot [x_i, \ldots, x_n]_g.$$

(6) The rule of adding an extra point to a divided difference:

$$[x_2, \ldots, x_n]_g = [x_1, \ldots, x_n]_h, \quad h(x) := (x - x_1)g(x).$$

(7) Differentiation with respect to a singly-occurring entry results in a repetition of that entry

$$\frac{d}{dx_k}[x_1, \ldots, x_n]_f = [x_1, \ldots, x_n, x_k]_f \quad (k = 1, \ldots, n).$$

(8) If $f^{(n-1)}$ is continuous then $[x_1, \ldots, x_n]_f$ is a continuous function of $(x_1, \ldots, x_n)$.

(9) If f is analytic then $[x_1, \ldots, x_n]_f$ is analytic in $(x_1, \ldots, x_n)$.

The following mean value theorem (the Cauchy mean value theorem for divided differences) is due to Leach and Sholander [2] (see also Rätz and Russell [6], Páles [4], [5]).

Theorem LS. *Let $x_1 \le \dots \le x_n$ and assume that $f^{(n-1)}$, $g^{(n-1)}$ exist, with $g^{(n-1)}(u) \neq 0$, on $[x_1, x_n]$. Then there is a $t \in [x_1, x_n]$ (moreover $t \in (x_1, x_n)$ if $x_1 < x_n$) such that*

$$\frac{[x_1, \dots, x_n]_f}{[x_1, \dots, x_n]_g} = \frac{f^{(n-1)}(t)}{g^{(n-1)}(t)}.$$

Supposing that the function $u \to \frac{f^{(n-1)}(u)}{g^{(n-1)}(u)}$ is invertible we get that

$$t = \left(\frac{f^{(n-1)}}{g^{(n-1)}}\right)^{-1} \left(\frac{[x_1, \dots, x_n]_f}{[x_1, \dots, x_n]_g}\right)$$

is a mean value of $x_1, \dots, x_n$ which, by property (1), is symmetric in its variables. It is called the *Cauchy (or difference) mean of the numbers* $x_1, \dots, x_n$ and will be denoted by $D_{f,g}(x_1, \dots, x_n)$. This mean value was first defined and examined by Leach and Sholander [2] (they called it *extended* (f, g) mean of $x_1, \dots, x_n$).

The aim of this paper is to solve the functional equation (the equality problem of Cauchy means)

$$D_{f,g}(x_1, \dots, x_n) = D_{F,G}(x_1, \dots, x_n) \qquad (x_1, \dots, x_n \in I)$$

if $n \ge 3$ is fixed.

The equality problem of two variable Cauchy means is much harder than the present one since for $n = 2$ the third derivative of the functional equation does not give an independent equation. This problem is comparable to the two variable equality problem for Gini means, solved by the author [3].

2. The main result

Our main result is

Theorem 1. *Suppose that I is a real interval, $n \ge 3$ is a fixed natural number,*

(i) $f, g, F, G : I \to \mathbb{R}$ *are $n + 2$ times continuously differentiable on I,*

(ii) $g^{(n-1)}(u) \neq 0$, $G^{(n-1)}(u) \neq 0$ *for $u \in I$ and*

(iii) *the functions $\frac{f^{(n-1)}}{g^{(n-1)}}$, $\frac{F^{(n-1)}}{G^{(n-1)}}$ have non-vanishing first derivative on I.*

The functional equation

$$D_{f,g}(x_1, x_2, \ldots, x_n) = D_{F,G}(x_1, x_2, \ldots, x_n) \quad (x_1, x_2, \ldots, x_n \in I) \tag{1}$$

holds if and only if there exist constants α, β, γ, δ *with* $\alpha\delta - \beta\gamma \neq 0$ *such that for all* $x \in I$

$$\begin{cases} f^{(n-1)}(x) = \alpha F^{(n-1)}(x) + \beta G^{(n-1)}(x) \\ g^{(n-1)}(x) = \gamma F^{(n-1)}(x) + \delta G^{(n-1)}(x) \end{cases} \tag{2}$$

is satisfied.

Remark 1. The conditions (ii) and (iii) impose further restrictions on the constants α, β, γ, δ which we do not specify here.

Remark 2. We need to assume (1) only for the values $x_1, x_2 \in [x-\epsilon, x+\epsilon] \cap I$, $x_3 = \cdots = x_n = x$ for all $x \in I$ where ϵ is a positive number.

Remark 3. Condition (iii) implies that the functions $\frac{f^{(n-1)}}{g^{(n-1)}}$, $\frac{F^{(n-1)}}{G^{(n-1)}}$ are invertible on I. The derivative of either ratio cannot have both positive and negative values by the intermediate value property of the derivative. Any of these ratios is either positive or negative on I hence it is strictly monotonic therefore invertible.

To prove the necessity of (2) we deduce a system of differential equations for the unknown functions f, g, F, G and solve it.

Our calculations will be simpler if we use an integral representation of divided differences equivalent to the one given by Steffenson [8, p. 17]:

$$[x_1, \ldots, x_n]_f = \int\limits_{S_{n-1}} f^{(n-1)}(t)\, d\mu$$

where

$$S_{n-1} := \Big\{ \mu = (\mu_1, \ldots, \mu_{n-1}) : \mu_k \geq 0,\ k = 1, \ldots, n-1 \text{ and } \sum_{k=1}^{n-1} \mu_k \leq 1 \Big\}$$

is a simplex in $\mathbb{R}^{n-1}$ and

$$t = x_n + \sum_{k=1}^{n-1} \mu_k (x_k - x_n) = \sum_{k=1}^{n-1} \mu_k x_k + \left(1 - \sum_{k=1}^{n-1} \mu_k\right) x_n.$$

The corresponding formula in [8] uses the variables $t_1, \ldots, t_n$, with $t_k = 1 - \sum_{j=1}^{k} \mu_j$.

Using this we have

$$D_{f,g}(x_1,\ldots,x_n) = \left(\frac{f^{(n-1)}}{g^{(n-1)}}\right)^{-1}\left(\frac{\int\limits_{S_{n-1}} f^{(n-1)}(t)\,d\mu}{\int\limits_{S_{n-1}} g^{(n-1)}(t)\,d\mu}\right). \tag{3}$$

First we calculate some partial derivatives of $D_{f,g}$. Suppressing the dependence from f,g let us introduce the simplified notation

$$M(x_1,x_2,\ldots,x_n) = D_{f,g}(x_1,x_2,\ldots,x_n) = h^{-1}\left(R(x_1,x_2,\ldots,x_n)\right)$$

where $h = \frac{f^{(n-1)}}{g^{(n-1)}}$, $R = \frac{K}{L}$,

$$K(x_1,x_2,\ldots,x_n) = \int\limits_{S_{n-1}} f^{(n-1)}\left(x_n + \sum_{k=1}^{n-1}\mu_k(x_k - x_n)\right)d\mu,$$

$$L(x_1,x_2,\ldots,x_n) = \int\limits_{S_{n-1}} g^{(n-1)}\left(x_n + \sum_{k=1}^{n-1}\mu_k(x_k - x_n)\right)d\mu$$

and $\mu = (\mu_1,\ldots,\mu_{n-1})$.

Let us denote the partial derivatives of $M(x_1,x_2,\ldots,x_n)$ with respect to x_1, x_1x_1 etc. at the point $(x,x,\ldots,x)$ by $M_{x_1}, M_{x_1^2}$ etc. respectively. Then we have

Lemma 1. *Suppose that* (i), (ii), (iii) *and* (1) *hold then we have*

$$nM_{x_1} = 1,$$

$$\frac{n^2(n+1)}{n-1}M_{x_1^2} = \frac{h''}{h'} + 2\frac{g^{(n)}}{g^{(n-1)}},$$

$$\begin{aligned} n^3(n+1)(n+2)M_{x_1^2x_2} = {} & (n^2-3n-2)\frac{h'''}{h'} + (-n^2+n+6)\left(\frac{h''}{h'}\right)^2 \\ & + (-6n+12)\frac{h''}{h'}\,\frac{g^{(n)}}{g^{(n-1)}} + (-2n^2+2n+12)\left(\frac{g^{(n)}}{g^{(n-1)}}\right)^2 \\ & + (2n^2-8n)\frac{g^{(n+1)}}{g^{(n-1)}} \end{aligned}$$

where all derivatives of h, g are taken at the point x.

Proof. We start with the formulae

$$\begin{aligned}
M_{x_1} &= \left(h^{-1}\right)'(R)R_{x_1},\\
M_{x_1^2} &= \left(h^{-1}\right)''(R)R_{x_1}^2 + \left(h^{-1}\right)'(R)R_{x_1^2},\\
M_{x_1^2x_2} &= \left(h^{-1}\right)'''(R)R_{x_2}R_{x_1}^2 + 2\left(h^{-1}\right)''(R)R_{x_1}R_{x_1x_2}\\
&\quad + \left(h^{-1}\right)''(R)R_{x_2}R_{x_1^2} + \left(h^{-1}\right)'(R)R_{x_1^2x_2}
\end{aligned}$$

where R and its partial derivatives R_{x_1}, R_{x_2} etc. are taken at the point $(x, x, \dots, x)$. Since

$$\begin{aligned}
\left(h^{-1}\right)' &= \frac{1}{h'(h^{-1})},\\
\left(h^{-1}\right)'' &= -\frac{h''(h^{-1})}{(h'(h^{-1}))^3},\\
\left(h^{-1}\right)''' &= -\frac{h'''(h^{-1})}{(h'(h^{-1}))^4} + 3\frac{(h''(h^{-1}))^2}{(h'(h^{-1}))^5}
\end{aligned}$$

we have

$$\text{(4)}\qquad \begin{cases}
M_{x_1} = \dfrac{1}{h'}R_{x_1},\\[2mm]
M_{x_1^2} = -\dfrac{h''}{(h')^3}R_{x_1}^2 + \dfrac{1}{h'}R_{x_1^2},\\[2mm]
M_{x_1^2x_2} = \left(-\dfrac{h'''}{(h')^4} + 3\dfrac{(h'')^2}{(h')^5}\right)R_{x_2}R_{x_1}^2\\[2mm]
\qquad\qquad - \dfrac{h''}{(h')^3}\left(2R_{x_1}R_{x_1x_2} + R_{x_1^2}R_{x_2}\right) + \dfrac{1}{h'}R_{x_1^2x_2}
\end{cases}$$

where the derivatives of h have to be taken at $h^{-1}(R(x, x, \dots, x)) = M(x, x, \dots, x) = x$.

As $R = \frac{K}{L}$, the derivatives R_{x_1}, R_{x_2}, $R_{x_1^2}$, $R_{x_1x_2}$, $R_{x_1^2x_2}$ are rational functions of K, L and their partial derivatives. For example

$$\text{(5)}\qquad R_{x_1^2} = \frac{K_{x_1^2}L^2 - KLL_{x_1^2} - 2K_{x_1}LL_{x_1} + 2KL_{x_1}^2}{L^3}$$

Using the integral formulae of Section 2 and differentiating behind the integral sign (k times with respect to x_1 and l times with respect to x_2) we have

$$K_{x_1^k x_2^l}(x, x, \dots, x) = f^{(n+k+l-1)}(x) \int\limits_{S_{n-1}} \mu_1^k \mu_2^l \, d\mu.$$

It is easy to check that for any continuous function p

$$\begin{aligned}\int\limits_{S_{n-1}} p(\mu_1, \mu_2)\, d\mu &= \int\limits_0^1 \int\limits_0^{1-\mu_1} \cdots \int\limits_0^{1-\mu_1-\cdots-\mu_{n-2}} p(\mu_1, \mu_2)\, d\mu_{n-1} \dots d\mu_2\, d\mu_1 \\ &= \int\limits_0^1 \int\limits_0^{1-\mu_1} p(\mu_1, \mu_2) \frac{(1-\mu_1-\mu_2)^{n-3}}{(n-3)!} d\mu_2\, d\mu_1.\end{aligned}$$

Hence we obtain that

$$K(x, x, \dots, x) = \frac{f^{(n-1)}(x)}{(n-1)!}, \qquad K_{x_1}(x, x, \dots, x) = \frac{f^{(n)}(x)}{n!},$$

$$K_{x_2}(x, x, \dots, x) = \frac{f^{(n)}(x)}{n!}, \qquad K_{x_1 x_2}(x, x, \dots, x) = \frac{f^{(n+1)}(x)}{(n+1)!},$$

$$K_{x_1^2}(x, x, \dots, x) = 2\frac{f^{(n+1)}(x)}{(n+1)!}, \quad K_{x_1^2 x_2}(x, x, \dots, x) = 2\frac{f^{(n+2)}(x)}{(n+2)!}.$$

We obtain analogous expressions for L and its derivatives. Substituting these and $f^{(n-1)} = hg^{(n-1)}$, $f^{(n)} = h'g^{(n-1)} + hg^{(n)}$ etc. into (5) we obtain that

$$R_{x_1^2} = \frac{2}{n(n+1)} h'' + \frac{2(n-1)}{n^2(n+1)} h' \frac{g^{(n)}}{g^{(n-1)}}.$$

In a similar way we find that

$$R_{x_1} = R_{x_2} = \frac{1}{n} h',$$

$$R_{x_1 x_2} = \frac{1}{n(n+1)} h'' - \frac{2}{n^2(n+1)} h' \frac{g^{(n)}}{g^{(n-1)}},$$

$$R_{x_1^2x_2} = \frac{2}{n(n+1)(n+2)}h''' - \frac{2n^2-8n}{n^3(n+1)(n+2)}\left(h''\frac{g^{(n)}}{g^{(n-1)}} + h'\frac{g^{(n+1)}}{g^{(n-1)}}\right)$$
$$+ \frac{-2n^2+2n+12}{n^3(n+1)(n+2)}h'\left(\frac{g^{(n)}}{g^{(n-1)}}\right)^2.$$

Substituting these expressions into (4) we obtain the statement of Lemma 1.

The calculations presented in Lemma 1 were partially checked by the software package Maple V. □

Lemma 2. *Suppose that* (i), (ii), (iii) *and* (1) *hold then for all* $u \in J := \{H(x) : x \in I\}$ *we have*

$$\frac{\tilde{h}''(u)}{\tilde{h}'(u)} + 2\frac{\tilde{g}'(u)}{\tilde{g}(u)} = 2\frac{\tilde{G}'(u)}{\tilde{G}(u)} \tag{6}$$

$$\begin{cases} (n^2-3n-2)\dfrac{\tilde{h}'''(u)}{\tilde{h}'(u)} + (-n^2+n+6)\left(\dfrac{\tilde{h}''(u)}{\tilde{h}'(u)}\right)^2 \\ \quad +(-6n+12)\dfrac{\tilde{h}''(u)}{\tilde{h}'(u)}\dfrac{\tilde{g}'(u)}{\tilde{g}(u)} \\ \quad +(-2n^2+2n+12)\left(\dfrac{\tilde{g}'(u)}{\tilde{g}(u)}\right)^2 + (2n^2-8n)\dfrac{\tilde{g}''(u)}{\tilde{g}(u)} \\ = (-2n^2+2n+12)\left(\dfrac{\tilde{G}'(u)}{\tilde{G}(u)}\right)^2 + (2n^2-8n)\dfrac{\tilde{G}''(u)}{\tilde{G}(u)} \end{cases} \tag{7}$$

where

$$\tilde{h} := h\left(H^{-1}\right), \quad \tilde{g} := g^{(n-1)}\left(H^{-1}\right), \quad \tilde{G} = G^{(n-1)}\left(H^{-1}\right). \tag{8}$$

Proof. By Lemma 1 the equation (1) implies that for all $x \in I$

$$\frac{h''(x)}{h'(x)} + 2\frac{g^{(n)}(x)}{g^{(n-1)}(x)} = \frac{H''(x)}{H'(x)} + 2\frac{G^{(n)}(x)}{G^{(n-1)}(x)} \tag{9}$$

$$
(10)\quad \begin{cases}
(n^2-3n-2)\dfrac{h'''(x)}{h'(x)}+(-n^2+n+6)\left(\dfrac{h''(x)}{h'(x)}\right)^2 \\
\qquad +(-6n+12)\dfrac{h''(x)}{h'(x)}\,\dfrac{g^{(n)}(x)}{g^{(n-1)}(x)} \\
\qquad +(-2n^2+2n+12)\left(\dfrac{g^{(n)}(x)}{g^{(n-1)}(x)}\right)^2 \\
\qquad +(2n^2-8n)\dfrac{g^{(n+1)}(x)}{g^{(n-1)}(x)} \\
=(n^2-3n-2)\dfrac{H'''(x)}{H'(x)}+(-n^2+n+6)\left(\dfrac{H''(x)}{H'(x)}\right)^2 \\
\qquad +(-6n+12)\dfrac{H''(x)}{H'(x)}\,\dfrac{G^{(n)}(x)}{G^{(n-1)}(x)} \\
\qquad +(-2n^2+2n+12)\left(\dfrac{G^{(n)}(x)}{G^{(n-1)}(x)}\right)^2 \\
\qquad +(2n^2-8n)\dfrac{G^{(n+1)}(x)}{G^{(n-1)}(x)}
\end{cases}
$$

From (8) we obtain that

$$
\begin{aligned}
\frac{\tilde{h}''}{\tilde{h}'} &= \frac{1}{H'(H^{-1})}\left[\frac{h''(H^{-1})}{h'(H^{-1})}-\frac{H''(H^{-1})}{H'(H^{-1})}\right],\\
\frac{\tilde{h}'''}{\tilde{h}'} &= \frac{1}{\left(H'(H^{-1})\right)^2}\left[\frac{h'''(H^{-1})}{h'(H^{-1})}-3\frac{h''(H^{-1})}{h'(H^{-1})}\frac{H''(H^{-1})}{H'(H^{-1})}\right.\\
&\qquad\qquad\left.-\frac{H'''(H^{-1})}{H'(H^{-1})}+3\left(\frac{H''(H^{-1})}{H'(H^{-1})}\right)^2\right],\\
\frac{\tilde{g}''}{\tilde{g}} &= \frac{1}{\left(H'(H^{-1})\right)^2}\left[\frac{g^{(n+1)}(H^{-1})}{g^{(n-1)}(H^{-1})}-\frac{g^{(n)}(H^{-1})}{g^{(n-1)}(H^{-1})}\frac{H''(H^{-1})}{H'(H^{-1})}\right],\\
\frac{\tilde{G}''}{\tilde{G}} &= \frac{1}{\left(H'(H^{-1})\right)^2}\left[\frac{G^{(n+1)}(H^{-1})}{G^{(n-1)}(H^{-1})}-\frac{G^{(n)}(H^{-1})}{G^{(n-1)}(H^{-1})}\frac{H''(H^{-1})}{H'(H^{-1})}\right].
\end{aligned}
$$

Substituting these expressions into (6) and (7), multiplying the first equation by $H'(H^{-1})$ the second by $\big(H'(H^{-1})\big)^2$ we obtain, after some calculations, exactly (9) and (10) at the point $x = H^{-1}(u)$. We remark that in case of (7) we have to use, during the calculations, also (9). This shows that (6), (7) and (9), (10) are equivalent by the transformation (8) (and its inverse), proving Lemma 2. $\square$

Now we can prove our main result.

Proof. Necessity. If (1) holds then by Lemma 2 (6) and (7) are satisfied. We shall solve this system of equations. Using (6) we get

$$\begin{aligned}\frac{\tilde G''}{\tilde G} &= \left(\frac{\tilde G'}{\tilde G}\right)' + \left(\frac{\tilde G'}{\tilde G}\right)^2 = \left(\frac{\tilde h''}{2\tilde h'} + \frac{\tilde g'}{\tilde g}\right)' + \left(\frac{\tilde h''}{2\tilde h'} + \frac{\tilde g'}{\tilde g}\right)^2 \\ &= \frac{1}{2}\frac{\tilde h'''}{\tilde h'} - \frac{1}{4}\left(\frac{\tilde h''}{\tilde h'}\right)^2 + \frac{\tilde h''}{\tilde h'}\frac{\tilde g'}{\tilde g} + \frac{\tilde g''}{\tilde g}.\end{aligned}$$

Substituting this into (7) we obtain that

$$2\frac{\tilde h'''}{\tilde h'} - 3\left(\frac{\tilde h''}{\tilde h'}\right)^2 = 0. \tag{11}$$

With the notation $w(u) = \frac{\tilde h''(u)}{\tilde h'(u)}$ $(u \in J)$ we can rewrite the last equation in the form

$$2w' - w^2 = 0.$$

The solutions of this separable equation are $w(u) = -\frac{2}{u+c}$ where c is an arbitrary constant and $w(u) = 0$. From

$$\frac{\tilde h''(u)}{\tilde h'(u)} = (\ln|h'(u)|)' = w(u) = \begin{cases} -\dfrac{2}{u+c} \\ 0 \end{cases}$$

by integration

$$\tilde h(u) = \begin{cases} \dfrac{e}{u+c} + d \\ au + b \end{cases} = \frac{Au + B}{Cu + D} \qquad (u \in J) \tag{12}$$

where A, B, C, D are arbitrary constants.

Rewriting equation (6) in the form

$$\left(\ln\left|\frac{\tilde{G}(u)}{\tilde{g}(u)}\right|\right)' = \frac{1}{2}\frac{\tilde{h}''(u)}{\tilde{h}'(u)} = \frac{1}{2}w(u) = \begin{cases} -\dfrac{C}{Cu+D} & \text{if } C \neq 0 \\ 0 & \text{if } C = 0 \end{cases}$$

we get by integration that

$$\frac{\tilde{G}(u)}{\tilde{g}(u)} = \frac{P}{Cu+D} \quad (u \in J) \tag{13}$$

where $P \neq 0$ is a constant. From (12), (13) by $h = \frac{f^{(n-1)}}{g^{(n-1)}}$, $H = \frac{F^{(n-1)}}{G^{(n-1)}}$ and (8) we obtain that

$$\frac{f^{(n-1)}(x)}{g^{(n-1)}(x)} = \frac{AH(x)+B}{CH(x)+D} = \frac{AF^{(n-1)}(x)+BG^{(n-1)}(x)}{CF^{(n-1)}(x)+DG^{(n-1)}(x)},$$

$$\frac{G^{(n-1)}(x)}{g^{(n-1)}(x)} = \frac{P}{CH(x)+D} = \frac{PG^{(n-1)}(x)}{CF^{(n-1)}(x)+DG^{(n-1)}(x)}$$

From the last equation

$$g^{(n-1)}(x) = \frac{C}{P}F^{(n-1)}(x) + \frac{D}{P}G^{(n-1)}(x) = \gamma F^{(n-1)}(x) + \delta G^{(n-1)}(x)$$

and using this

$$f^{(n-1)}(x) = \frac{A}{P}F^{(n-1)}(x) + \frac{B}{P}G^{(n-1)}(x) = \alpha F^{(n-1)}(x) + \beta G^{(n-1)}(x)$$

proving (2).

Here the constants have to satisfy $\alpha\delta - \beta\gamma \neq 0$ otherwise $\frac{f^{(n-1)}}{g^{(n-1)}}$ would be a constant and not invertible.

To prove the sufficiency of (2) we notice that

$$h = \frac{f^{(n-1)}}{g^{(n-1)}} = \frac{\alpha F^{(n-1)} + \beta G^{(n-1)}}{\gamma F^{(n-1)} + \delta G^{(n-1)}} = \frac{\alpha H + \beta}{\gamma H + \delta}$$

therefore
$$h^{-1}(u) = H^{-1}\left(\frac{\delta u - \beta}{\alpha - \gamma u}\right).$$

Thus, using the representation (2) we get with $u = \frac{K}{L}$

$$D_{f,g}(x_1, x_2, \ldots, x_n) = h^{-1}(u) = H^{-1}\left(\frac{\delta \frac{K}{L} - \beta}{\alpha - \gamma \frac{K}{L}}\right) = H^{-1}\left(\frac{\delta K - \beta L}{\alpha L - \gamma K}\right)$$

and

$$\frac{\delta K - \beta L}{\alpha L - \gamma K} = \frac{\int\limits_{S_{n-1}} (\delta f^{(n-1)}(t) - \beta g^{(n-1)}(t)) d\mu}{\int\limits_{S_{n-1}} (\alpha g^{(n-1)}(t) - \gamma f^{(n-1)}(t)) d\mu} = \frac{\int\limits_{S_{n-1}} F^{(n-1)}(t)\, d\mu}{\int\limits_{S_{n-1}} G^{(n-1)}(t)\, d\mu}.$$

Hence

$$\begin{aligned} D_{f,g}(x_1, x_2, \ldots, x_n) &= H^{-1}\left(\frac{\int\limits_{S_{n-1}} F^{(n-1)}(t)\, d\mu}{\int\limits_{S_{n-1}} G^{(n-1)}(t)\, d\mu}\right) \\ &= D_{F,G}(x_1, x_2, \ldots, x_n). \end{aligned}$$

$\square$

3. Closing remarks

It is worth to specify our result for $n = 3$ since in this case the regularity conditions of Theorem 1 are the weakest.

Theorem 2. *Suppose that I is a real interval,*

(i) *$f, g, F, G : I \to \mathbb{R}$ are five times continuously differentiable,*

(ii) *$g''(u) \neq 0, G''(u) \neq 0$ and*

(iii) *the functions $\frac{f''}{g''}$, $\frac{F''}{G''}$ have non-vanishing first derivative on I.*

The functional equation

$$D_{f,g}(x_1, x_2, x_3) = D_{F,G}(x_1, x_2, x_3) \qquad (x_1, x_2, x_3 \in I) \tag{14}$$

holds if and only if there exist constants α, β, γ, δ with $\alpha\delta - \beta\gamma \neq 0$ such that for all $x \in I$

$$\begin{aligned} f''(x) &= \alpha F''(x) + \beta G''(x) \\ g''(x) &= \gamma F''(x) + \delta G''(x) \end{aligned} \tag{15}$$

is satisfied.

Since (15) implies (2) it follows that if two Cauchy means are equal for $n = 3$ variables then they are equal for $n \geq 3$ variables too (provided that conditions (i)–(iii) of Theorem 1 and Theorem 2 are satisfied).

If two Cauchy means are equal for $n = 3$ variables then they are equal for $n = 2$ variables as well if we assume the additional conditions

$$\begin{cases} f'(c) = \alpha F'(c) + \beta G'(c) \\ g'(c) = \gamma F'(c) + \delta G'(c) \end{cases}$$

at some point $c \in I$ (and the existence of the two variable Cauchy means). Namely these, together with (15) ensure the validity of (2) (hence also (1)) for $n = 2$.

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