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Transversal Cartan chains in a real hyperquadric

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Abstract. Let Φ be an hermitian quadratic form, of maximal rank and index $(n,1)$, defined over a complex $(n+1)$ dimensional vector space V.

Consider the real hyperquadric defined in the complex projective space P^nV by

$$Q = \{[\zeta] \in P^nV : \Phi(\zeta) = 0\}.$$

Let G be the subgroup of the special linear group which leaves Q invariant, and D the $(2n-2)$ distribution defined by the Cauchy-Riemann structure induced over Q.

We shall study the induced G action on the manifold C^2Q of contact elements of order two and dimension one, proving that the Cartan chains transversal to D are solutions of a differential system defined as a submanifold of C^2Q. This gives a characterization of the transversal Cartan chains as curves whose second order contact elements are singular at all their points.

Moreover, we prove that the Cartan chains are orbits of order 1, induced by the action of a closed subgroup K of G on Q.

1. Introduction

A Cauchy Riemann structure, or an almost complex structure, on a smooth $(2n-1)$-dimensional manifold is defined by a distribution D of $(2n-2)$-dimensional tangent spaces, together with a linear operator I on each subspace D, such that $I^2 = -1$. A codimension 1 real submanifold of a complex analytic manifold has a Cauchy Riemann structure induced by the complex structure.

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These structures were first studied by Poincaré [8], while seeking to generalize Riemann's conformal transformation theorem for domains in $\mathbb{C}^n$, $n \geq 1$. Later, Elie Cartan [2] considered the special case $n = 2$ and introduced the notion of chains, which are certain curves defined by second order ordinary differential equations. These chains were to play a fundamental role in the study of Cauchy Riemann structures.

The objective of this work is to give a coordinate-free characterization of Cartan chains in a hyperquadric Q. We use the induced action of the group G of automorphisms of the quadric on the manifold C^2Q of second order contact elements of Q [11].

We shall define a generalized Frobenius system [10] generated by the singular orbits in C^2Q, proving that the Cartan chains transversal to the distribution D are solutions of this system.

This gives a characterization of these chains as curves whose second order contact elements are singular at all their points.

Moreover, we prove that there exists a closed subgroup $K \subset G$ such that, the Cartan chains are orbits of order 1, [12], induced by the action of K on Q.

2. The real hyperquadric

Let Φ be a hermitian quadratic form of maximal rank and index $(n, 1)$, defined over a $(n+1)$ vector space V. There is a basis $\{f_\alpha\}$ of V, such that Φ is given as

$$\Phi(\zeta) = \zeta^\alpha \overline{\zeta^\alpha} + i(\zeta^n \overline{\zeta^0} - \overline{\zeta^n} \zeta^0), \quad \zeta \in V,$$

and the matrix of its representation is

$$A = \begin{pmatrix} 0 & 0 & i \\ 0 & I_{n-1} & 0 \\ -i & 0 & 0 \end{pmatrix}.$$

If $\langle\ ,\ \rangle$ denotes the bilinear form associated with Φ, then we have

$$\langle \zeta, \nu \rangle = {}^t(\bar{\nu}) A(\zeta) \quad \text{for } \zeta, \nu \in V,$$

where (ν) is the $(n+1)$ matrix of the components of ν in the basis $\{f_\alpha\}$.

Let $G \subset SL(n+1,\mathbb{C})$ be the subgroup which leaves Φ invariant. If we represent $g \in G$ in the standard basis by the matrix (g^α_γ), $0 \leq \alpha, \gamma \leq n$ and its column vector by g_γ, then

$$\begin{aligned} g \in G &\Leftrightarrow (\Phi \circ g)(\zeta) = \Phi(\zeta) \\ &\Leftrightarrow {}^t\bar{\zeta} = {}^t\bar{g}Ag\zeta = {}^t\bar{\zeta}A\zeta \Leftrightarrow (\langle g_\gamma, g_\alpha\rangle) = A. \end{aligned}$$

Hence, for $0 \leq \gamma \leq n$, $1 \leq \alpha$, $\beta \leq n-1$, the following relations are satisfied:

$$\begin{aligned} &\langle g_0, g_\gamma\rangle = -i\delta^\gamma_n, \quad \langle g_\beta, g_\alpha\rangle = -\delta^\alpha_\beta, \\ &\langle g_n, g_\gamma\rangle = i\delta^\gamma_0, \qquad \det g = 1. \end{aligned}$$

The Lie algebra $\mathcal{G}$ of G is given by

$$\mathcal{G} = \left\{ \ell \in T_eG : \ell = \begin{pmatrix} \ell^0_0 & \ell^0_1 & \dots & \ell^0_{n-1} & \ell^0_n \\ \ell^1_0 & & & & -i\overline{\ell^0_1} \\ \vdots & & (\ell^\alpha_\beta) & & \vdots \\ \ell^{n-1}_0 & & & & -i\overline{\ell^0_{n-1}} \\ \ell^n_0 & i\overline{\ell^1_0} & \dots & i\overline{\ell^{n-1}_0} & -\overline{\ell^0_0} \end{pmatrix}; \quad \begin{aligned} &\ell^\alpha_\beta + \overline{\ell^\beta_\alpha} = 0, \\ &\operatorname{tr}\ell = 0, \\ &\ell^0_n, \ell^n_0 \in \mathbb{R} \end{aligned} \right\}.$$

The canonical form ω over G, with components ω^α_γ with respect to the standard basis $I^\alpha_\gamma \in \mathcal{G}(n+1,\mathbb{C})$, satisfies the relations

$$\omega_x(v) = \sum_0^n \omega^\alpha_{\gamma x}(v) I^\alpha_\gamma, \text{ for } v \in T_xG; \quad d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta = 0.$$

Let Q be the $(2n-1)$-dimensional real hyperquadric, defined in the complex projective space P^nV by the equation

$$Q = \{[\zeta] \in P^nV,\ \Phi(\zeta) = 0\}.$$

The group G acts on P^nV by $g \cdot [\zeta] = [g \cdot \zeta]$, and the quadric Q is invariant by the action of G on P^nV. Moreover, G acts transitively on Q, see [4].

Now, given $p_0 = [(1,0,0)]$, the isotropy group G^0 at p_0 is

$$G^0 = \{g \in G : g^\alpha_0 = 0,\ 1 \leq \alpha \leq n\},$$

and its Lie algebra $\mathcal{G}^0$ is given by

$$\mathcal{G}^0 = \left\{ \ell \in \mathcal{G} : \ \ell = \begin{pmatrix} \ell_0^0 & \cdots & \ell_n^0 \\ 0 & (\ell_\beta^\alpha) & \vdots \\ 0 & 0 & -\overline{\ell_0^0} \end{pmatrix} \right\}.$$

The map $\psi^0 : g \in G \mapsto g \cdot p_0 \in Q$ defines an isomorphism: $G/G^0 \simeq Q$.

The Cartan forms $\{\omega_0^\alpha\}$ of the group G vanish on $T_e G^0$, allowing us to define a basis $\{\widetilde{\omega}_0^\alpha\}$ of $T^*_{p_0}Q$ as follows (see [3]):

Given $\widetilde{v} \in T_{p_0}Q$, let $\widetilde{\omega}_0^\alpha(\widetilde{v}) = \omega_0^\alpha{}_e(v_e)$, where $v \in \mathcal{G}$ and $T_e\psi_e^0(v) = \widetilde{v}$.

These forms are well defined, since for $u, v \in \mathcal{G}$

$$\begin{aligned} T_e\psi_e^0(u_e) = T_e\psi_e^0(v_e) &\Leftrightarrow T_e\psi_e^0(u_e - v_e) = 0 \Leftrightarrow u_e - v_e \in T_e G^0 \simeq \mathcal{G} \\ &\Leftrightarrow \omega_0^\alpha{}_e(u_e - v_e) = 0, \end{aligned}$$

and so $\widetilde{\omega}_0^\alpha(\widetilde{u}) = \widetilde{\omega}_0^\alpha(\widetilde{v})$.

Since the dimension of $T^*_{p_0}Q$ is $2n-1$, the forms $\{\widetilde{\omega}_0^\alpha\}$ define a basis of $T^*_{p_0}Q$.

Now, the Lie algebra $\mathcal{G}^0$ acts on T^*Q and its coordinate expression is given by the following

Proposition 1. *The Lie algebra $\mathcal{G}^0$ acts on $T^*_{p_0}Q$ as follows:*

$$(\ell, \widetilde{\omega}) \in \mathcal{G}^0 \times T^*_{p_0}Q \to \ell \cdot \widetilde{\omega},$$

$$\textit{where} \quad \ell \cdot \widetilde{\omega}(\widetilde{v}) = -d\omega \mid_e (\ell_e, v_e) \quad \textit{with } v \in \mathcal{G}, T_e\psi^0(v_e) = \widetilde{v},$$

and is given in coordinates as:

$$\begin{aligned} \ell \cdot \omega_0^\alpha &= \ell_\alpha^\gamma \omega_0^\gamma + (-\ell_0^0 + \ell_\alpha^\alpha)\omega_0^\alpha - i\overline{\ell_\alpha^0}\omega_0^n, && 1 \le \alpha, \gamma \le n-1. \\ \ell \cdot w_0^n &= -2\,\mathrm{Re}\,\ell_n^0 w_0^n, && \gamma \ne \alpha. \end{aligned}$$

Proof. Let us write

$$\ell.\omega_0^\alpha = \left(a_0^\beta + i b_0^\beta\right)\omega_0^\beta + c\omega_0^n, \qquad a_0^\beta, b_0^\beta \in \Re,$$

and consider a basis of vectors in T_eG dual to the basis ω_0^α. Applying $\ell.\omega_0^\alpha$ to such a basis, and using the relation

$$\omega_x(v) = \sum_0^n \omega_{\gamma x}^\alpha(v) I_\gamma^\alpha \quad v \in T_xG, \qquad \mathrm{d}\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma = 0,$$

we obtain the result. □

We observe that the subspace $D_{p_0} \subset T_{p_0} = Q$ defined by $\widetilde{\omega}_0^u = 0$ is invariant by G^0, since $\mathcal{G}^0$ transforms ω_0^u in a multiple of itself, and G^0 is connected.

The transitivity of the action of G on Q allows us to define a $(2n-2)$-dimensional distribution over Q, as follows:

$$D : p \in Q \to (l_g)_*(D_{p_0}), \quad \text{where } g \in G \quad \text{and } l_g(p_0) = g \cdot p_0 = p.$$

To study the real curves in Q, it is natural to consider two cases: the curves tangent to the distribution D at all its points, and the curves transversal to D at all its points. In this paper we consider the Cartan chains transversal to D. We will use the following theorem about Lie groups [1]:

Let G be a Lie group that acts on a smooth manifold M and $\mathcal{G}$ its Lie algebra. Let $\chi(M)$ denote the smooth (C^∞) vector fields on M, and let F be the map

$$F : \mathcal{G} \longrightarrow \chi(M), \quad \ell \longmapsto F_\ell, \quad \text{with } F_\ell(x) = \frac{d}{dt}\,|_{t=0}\, (\exp((t\ell)\dot{x})),$$

then [1] *we have:*

a) *The integral curve $y(x)$ of the field F_ℓ at the point $x \in M$ is contained in the orbit $G(x)$ of x.*

b) *The action of G on M is transitive if and only if for each $x \in M$ and for any $v \in T_xM$ there exists $\ell \in \mathcal{G}$ such that $F_\ell(x) = v$.*

3. Orbits of contact elements of Q

Two imbedded submanifolds $\Gamma, \Gamma_1 \subset Q$ of dimension one have contact of order $s \geq 1$ at $q \in \Gamma \cap \Gamma_1$, if there exist local parametrizations of Γ and Γ_1 given by imbeddings

$$\gamma, \gamma_1 : U \subset \mathbb{R} \to Q,$$

and a local coordinate system $(V,(x,y^j))$, $1 \leq j \leq 2n-1$ such that:

i) $\gamma(0) = \gamma_1 = (0)q$,

ii) $x \circ \gamma = x \circ \gamma_1 = r$ is the canonical coordinate in $\mathbb{R}$,

iii) the partial derivates at 0 of $(y^j \circ \gamma)$ and $(y^j \circ \gamma_1)$ are equal up to order s.

The equivalence class of contact elements of order s and dimension one at $q \in \Gamma$ is called a contact element of order s at $q \in \Gamma$.

Clearly, $C_q^1\Gamma = C_q^1\Gamma_1$ if and only if $T_q\Gamma = T_q\Gamma_1$.

Let C_q^sQ be the manifold of contact elements of order s and dimension 1 at $q \in Q$, and C^sQ the manifold of all contact elements C_q^sQ with $q \in Q$ (see [5], [6], [11]).

Consider the canonical projection

$$\pi_k^s : C^sQ \to C^kQ, \quad C_q^s\Gamma \mapsto C_q^k\Gamma, \quad k \leq s,$$

and the canonical immersions

$$i^s : \Gamma \longrightarrow C^sQ, \qquad\qquad q \in \Gamma \mapsto C_q^s\Gamma,$$

$$i^{1,s} : C^{s+1}Q \to C^1(C^sQ), \qquad C_q^{s+1}\Gamma \mapsto C^1_{C_q^s\Gamma}C^s\Gamma.$$

The action $\alpha : G \times Q \to Q$, $\alpha(g,q) = g.q$ induces an action

$$\alpha^s : G \times C^sQ \to C^sQ, \qquad g.C_q^s\Gamma = C_{g.q}^s g.\Gamma.$$

Let $\mathcal{H}^1$ be the fiber of the contact elements of order 1 transversal to D which project onto p_0, i.e.

$$\mathcal{H}^1 = \left\{X^1 \in C^1_{p_0}Q : \ \widetilde{\omega}_0^n|X^1 \neq 0\right\},$$

where $\widetilde{\omega}_0^n|X^1$ denotes the restriction of $\widetilde{\omega}_0^n$ to the 1-dimensional subspace defined by the contact element X^1.

Consider on $\mathcal{H}^1$ the following coordinates, defined as in [7]:

$$\widetilde{\omega}_0^\alpha|X^1 = \lambda_0^\alpha\widetilde{\omega}_0^n|X^1, \qquad \lambda_0^\alpha \in \mathbb{C},\ 1 \leq \alpha \leq n-1,$$

and express X^1 in coordinates as

$$X^1 = (\lambda_0^1 \dots \lambda_0^{n-1}).$$

Denote by $\widetilde{C}^1Q$ all the contact elements of order 1, transversal to D.

Proposition 2. *The group G acts transitively on $\widetilde{C}^1 Q$.*

Proof. Since the action of G on Q is transitive, it is sufficient to prove that the action of G^0 on $\widetilde{C}^1_{p_0} Q$, the contact elements of order 1 transversal to D which project onto p_0, is transitive. Now, given $X \in \widetilde{C}^1_{p_0} Q$, we have (see [11]),

$$\begin{aligned} F^0_\ell(X) = &\left(\sum_{\gamma=2}^{n-1} \ell^\gamma_1 \lambda^\gamma_0 + \left(-\ell^0_0 + \ell^1_1 + 2\operatorname{Re}\ell^0_0\right)\lambda^1_0 - i\overline{\ell^0_1}\right)\frac{\partial}{\lambda^1_0}\Big|_X + \cdots \\ &\cdots + \left(\sum_{\gamma\neq\alpha} \ell^\gamma_\alpha \lambda^\gamma_0 + \left(-\ell^0_0 + \ell^\alpha_\alpha + 2\operatorname{Re}\ell^0_0\right)\lambda^\alpha_0 - i\overline{\ell^0_\alpha}\right)\lambda^{n-1}_0 \frac{\partial}{\partial\lambda^\alpha_0}\Big|_X + \cdots \\ &\cdots + \left(\sum_{\gamma=1}^{n-2} \ell^\gamma_{n-1} \lambda^\gamma_0 + \left(-\ell^0_0 + \ell^{n-1}_{n-1} + 2\operatorname{Re}\ell^0_0\right) - i\overline{\ell^0_{n-1}}\right)\frac{\partial}{\lambda^1_{n-1}}\Big|_X . \end{aligned}$$

So we can choose $\ell_1, \ldots, \ell_{n-1} \in \mathcal{G}^0$ such that

$$\left\{F^0_{\ell_1}(X), \ldots, F^0_{\ell_{n-1}}(X)\right\}$$

generate $T_X\mathcal{H}^1$, hence G^0 acts transitively on $\mathcal{H}^1$. □

Corollary 1. *Let $X^1 \in \widetilde{C}^1_{p_0} Q$, and $\mathcal{G}^1$ the Lie algebra of the isotropy group $G^1 \subset G^0$ of X^1; then*

$$\widetilde{C}^1 Q \simeq G/G^1.$$

Proof. The orbit

$$G \cdot X^1 = \mathcal{O}^1 \simeq \widetilde{C}^1 Q,$$

is diffeomorphic to G/G^1. □

Proposition 3. *Let $X^1_0 \in \widetilde{C}^1_{p_0} Q$ be defined in coordinates as follows:*

$$\omega^n_0 | X^1_0 \neq 0, \quad \omega^\alpha_0 | X^1_0 = 0, \qquad 1 \leq \alpha \leq n-1;$$

then the Lie algebra $\mathcal{G}^1$ of the isotropy group $G^1 \subset G^0$ of X^1_0 is given by

$$\mathcal{G}^1 = \left\{\ell \in \mathcal{G}^0 : \ell^0_1 = \cdots = \ell^0_{n-1} = 0\right\},$$

i.e.

$$\mathcal{G}^1 = \left\{ \ell = \begin{pmatrix} \ell_0^0 & 0 & \ell_n^0 \\ 0 & (\ell_\beta^\alpha) & 0 \\ 0 & 0 & -\overline{\ell_0^0} \end{pmatrix}; \quad \ell_\beta^\alpha + \overline{\ell_\alpha^\beta} = 0,\ \operatorname{tr}\ell = 0 \right\}.$$

Proof. We have

$$F_\ell^0(X_0^1) = 0 \ \Leftrightarrow\ \ell_1^0 = \cdots = \ell_\alpha^0 = \cdots = \ell_{n-1}^0 = 0. \qquad \square$$

The forms

$$\left\{\omega_0^\alpha, \omega_0^n, \omega_\alpha^0\right\},$$

vanishing on G^1, define $4n-3$ linearly independent real forms. These forms can be projected onto $T_{X_0^1}\mathcal{O}^1$, and the projected forms, denoted by

$$\left\{\widetilde{\omega_0^\alpha}, \widetilde{\omega_\alpha^0}, \widetilde{\omega_0^n}\right\},$$

define a basis of $T^*_{X_0^1}\mathcal{O}^1 = T^*_{X_0^1}\widetilde{C}^1 Q$.

Given $\ell \in \mathcal{G}^1$, in [11] we proved that

$$\ell.\widetilde{\omega}_\alpha^0 = -\sum_{\gamma\neq\alpha} \ell_\alpha^\gamma \widetilde{\omega}_\alpha^0 + \left(\ell_\alpha^0 - \ell_\alpha^\alpha\right)\widetilde{\omega}_\alpha^0.$$

Now, the orbit $\mathcal{O}^1 = G \cdot X_0^1$ is principal ([1], [11]), and diffeomorphic to G/G^1.

Then, the forms

$$\omega_0^\alpha,\ \omega_\alpha^0,\ \omega_0^n,$$

which vanish on G^1, define $4n-3$ real forms, which are linearly independent.

These forms can be projected on $T_{X_0^1}\mathcal{O}^1$, using an argument similar to (1.2). The projected forms give a basis of $T^*_{X_0^1}\mathcal{O}^1 = T^*_{X_0^1}\widetilde{C}^1 Q$.

Let $\mathcal{H}^2$ be the fiber of the contact elements of order 2 which project onto X_0^1. Let $i : C^2Q \to C^1(C^1Q)$ be the canonical immersion, and

$$\pi_0^1 : C^1Q \to Q, \qquad \pi_0^{1,1} : C^1\left(C^1Q\right) \to C^1Q,$$

the canonical projections. Then

$$i(\mathcal{H}^2) = \left\{X^2 \in C^2_{p_0}Q : \ \widetilde{\omega}^\alpha_0|X^2 = 0, \ 2 \leq \alpha \leq n-1\right\}.$$

Indeed, given $X^2_p \in C^2_p Q$ the image

$$i(X^2_p) \subset C^1_{X^1_p} C^1 Q$$

can be identified with a 1-dimensional subspace in $T_{\pi_0^{1,1}(X^2_p)} C^1 Q$.

Then $(\pi^1_0)_* i(X^2_p)$ is identified with a subspace in T_pQ.

The following can be verified using coordinates, see [13]:

If $X^{1,1} \in C^1(C^1Q)$, then there exists $X^2 \in C^2Q$, $i(X^2) = X^{1,1}$

$$\Leftrightarrow T\left(\pi^1_0\right)(X^{1,1}\big) = \pi_0^{1,1}\left(X^{1,1}\right).$$

Consider coordinates in $\mathcal{H}^2$ defined as

$$X^2 = \left(\lambda^0_1, \dots, \lambda^0_{n-1}\right), \quad \text{where } \widetilde{\omega}^0_\alpha|X^2 = \lambda^0_\alpha \widetilde{\omega}^n_0|X^2.$$

Let $\widetilde{C}^2Q$ be the contact elements of order 2, tranversal to D, which project onto $\widetilde{C}^1Q$.

4. Singular orbits on second order contact elements of Q

The space C^2Q, together with the action α^2 of G, is a G-space. The subspace

$$\mathcal{O} = G(X) = \{g(X) \in C^2Q; \ g \in G\}$$

is the orbit of X (under G).

If $\mathcal{O}$ is a G-orbit, then we let type $(\mathcal{O})$ denote its type, that is, its equivalence class under equivariant homeomorphisms.

Type $(\mathcal{O})$ contains a coset space G/H, with H the isotropy group of G at X. Moreover, type (G/H) = type(G/K) iff H and K are conjugate in G.

The maximum orbit type of orbits in C^2Q, guaranteed in [1, 3.1], is called the principal orbit type, and orbits of this type are called principal orbits. If $\mathcal{O}_1$ is a principal orbit and $\mathcal{O}_2$ is any orbit, then there is an equivariant map $\mathcal{O}_1 \to \mathcal{O}_2$.

If $\mathcal{O}_1 \simeq G/H$ and $\mathcal{O}_2 \simeq G/K$, then H is conjugate to a subgroup of K and without loss of generality we may assume that $H \subset K$. Then the equivariant maps

$$\mathcal{O}_2 \to \mathcal{O}_1; \qquad G/H \to G/K,$$

are fiber bundle projections with fiber K/H.

If

$$\dim \mathcal{O}_2 > \dim \mathcal{O}_1, \text{ i.e. } \dim K/H > 0,$$

then $\mathcal{O}_2$ is called a singular orbit.

Proposition 4. *The orbit $\widehat{\mathcal{O}}^2$ by the action of G on $\widetilde{C}^2Q$, defined as*

$$\widehat{\mathcal{O}}^2 = G.\widehat{X}_0^2, \quad \textit{with } \pi_0^2\left(\widehat{X}_0^2\right) = X_0^1,\ \lambda_\alpha^0(\widehat{X}_0^2) = 0,\ 1 \le \alpha \le n-1,$$

is a singular orbit.

Moreover, if g^2 is the Lie algebra of the isotropy group $G^2 \subset G^1$ of $\widehat{X}_0^2$, then

$$g^2 = g^1.$$

Proof. Since G^0 acts transitively on $\widetilde{C}^1Q$, it is sufficient to prove that

$$\widehat{\mathcal{O}}_2 = G^1(\widehat{X}_0^2)$$

is a singular orbit by the action of G^1 on $\mathcal{H}^2$.

Let $F^1 : \mathcal{G}^1 \to \chi(\mathcal{H}^2)$ be defined as

$$F_\ell^1(X^2) = \frac{d}{dt}\ |_{t=0}\ (\exp{(t\ell)}.X^2)\ .$$

Using coordinates we have

$$F_\ell^1(X^2) = \sum_1^{n-1} B_\alpha^0 \frac{\partial}{\partial \lambda_\alpha^0}\ |_{X^2}\ .$$

Now, given

$$X^2 = \left(\lambda_1^0, \ldots, \lambda_{n-1}^0\right) \quad \text{and} \quad \ell \in \mathcal{G}^1,$$

let $r(t) = \exp t\ell X^2$ be expressed in coordinates as

$$r(t) = \left(\lambda_1^0(t), \ldots, \lambda_{n-1}^0(t)\right), \quad \text{where} \quad \omega_\alpha^0 | r(t) = \lambda_\alpha^0(t)\omega_0^n | r(t).$$

Deriving the last expression with respect to t at 0, we have

$$F^1_\ell(X^2) = \Big(-\sum_{\alpha\neq 1} \ell^\alpha_1 \lambda^0_\alpha + \left(-\ell^0_0 + \ell^1_1 - 2\operatorname{Re}\ell^0_0\right)\lambda^0_1, \ldots,$$
$$-\sum_{\alpha\neq n-1} \ell^\alpha_{n-1}\lambda^0_\alpha + \left(-\ell^0_0 + \ell^{n-1}_{n-1} - 2\operatorname{Re}\ell^0_0\right)\lambda^0_{n-1}\Big).$$

Now, if $\lambda^0_\alpha \neq 0$ for some α, then we can find $2n-2$ vector fields

$$\ell_1, \ldots, \ell_{2n-2} \in \mathcal{G}^1,$$

such that $\{F^1_{\ell_j}(X^2)\}$ generates $T_{X^2}\mathcal{H}^2$, and we conclude that G^1 acts transitively on the contact elements of order 2 with $\lambda^0_\alpha \neq 0$ for some α.

These elements can be represented as

$$\widetilde{\mathcal{O}}^2 = G^1(X^2_0), \quad \text{with } \lambda^0_1(X^2_0) = 1, \quad \lambda^0_\alpha(X^2_0) = 0,$$

and so we have

$$F^1_\ell(X^2_0) = 0 \Leftrightarrow -\ell^0_0 + \ell^1_1 - \operatorname{Re}\ell^0_0 - \ell^1_2 = \cdots = \ell^1_{n-1} = 0, \quad \ell \in \mathcal{G}^1.$$

Therefore $\mathcal{G}_{X^2_0} \neq \mathcal{G}^1$.

If $\lambda^0_\alpha = 0$, then for all α we have the element $\widehat{X}^2_0$, defined by $\pi^2_0(\widehat{X}^2_0) = X^1_0$ and $\lambda^0_\alpha(\widehat{X}^2_0) = 0$.

We observe that

$$F^1_\ell\left(\widehat{X}^2\right) = 0, \quad \ell \in \mathcal{G}^1.$$

Now $\mathcal{G}_{X^2_0} = \mathcal{G}^1$. □

Corollary 2. *The map $\pi^2_1 : \widehat{\mathcal{O}}^2 \to \widetilde{C}^1 Q$ is a local inmersion.*

Proof. Since $g^1 = g^2$, we see that $G^1 \subset G^2$ is open, moreover G^1 is connected and the group G acts transitively on $\widetilde{C}^1 Q$, hence

$$\pi^2_1 : \widehat{\mathcal{O}}^2 \cong G/G^2 \to \widetilde{C}^1 Q \cong G/G^1,$$

is a local immersion. □

5. The Cartan chains

Elie Cartan [2] introduced the notion of chains, which are curves defined by second order ordinary differential equations. These curves play a role similar to that of lines in Euclidean space.

The Cartan chains are obtained as the intersection of complex projective lines with the hyperquadric.

In this section we shall prove that the orbit $\widehat{\mathcal{O}}^2$ defines a completely integrable differential system on Q, and that the Cartan chains transversal to D are solutions of this system.

Definition 1. By a differential system of order 2 and dimension 1 in M we mean an imbedded submanifold $W \subset C^2Q$ [11].

A solution of a differential system W at $X \in W$ is a 1-dimensional imbedded submanifold $\Gamma \subset Q$ with $x = \pi_0^2(X) \in \Gamma$, such that $C^2\Gamma \subset W$ and $C_x^2\Gamma = X$.

An imbedded submanifold $W \subset C^2Q$ such that the following conditions are satisfied:

1. $\pi_1^2 : W \to C^1Q$; is a local inmersion in a neighborhood of $X \in W$;
2. $\pi_2^3 : C^1W \cap C^2Q \to W$ is a local submersion in a neighborhood of X; defines a differential system with solution $\Gamma \subset M$ passing through X [10].

Let X_0^2 be the contact element of order 2, defined in coordinates as

$$\widetilde{\omega}_0^2|X_0 0^2 \neq 0; \quad \widetilde{\omega}_0^\alpha|X_0^2 = \widetilde{\omega}_\alpha^2|X_0^2 = 0, \quad 1 \leq \alpha \leq n-1.$$

Proposition 5. *Let $\Gamma(t)$ be the curve in Q defined in non-homogeneous coordinates as*

$$\Gamma(t) = (1, 0, \ldots, 0, t),$$

and $g(t)$ the curve in G given by

$$g(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_{n-1} & 0 \\ t & 0 & 1 \end{pmatrix},$$

then

i) $C^2_{p_0}T = X^2_0$

ii) $C^2_{\Gamma(t)}T = g(t)X^2_0$.

PROOF. Consider the curve $\Gamma(t) = (1, 0, \dots, 0, t) \subset Q$ which we identify with

$$[g(t)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_n & 0 \\ t & 0 & 1 \end{pmatrix} \subset G/G^0 \cong Q.$$

Let $X^2_0 = C^2_{p_0}\Gamma$, then

$$\widetilde{\omega}^u_0|X^2_0 \neq 0; \quad \widetilde{\omega}^\alpha_0|X^2_0 = \widetilde{\omega}^0_\alpha|X^2_0 = 0 \qquad 1 \leq \alpha \leq u-1.$$

Moreover

$$g(t) \cdot X^2_0 = g(t)C^2_{p_0}\Gamma = C^2_{g(t)p_0}g(t) = \Gamma = C^2_{\Gamma(t)}\Gamma. \qquad \square$$

Theorem 1. *Let $\mathcal{O}^2$ be the orbit by the action of G on Q, defined as*

$$\mathcal{O}^2 = G \cdot X^2_0,$$
with $\widetilde{\omega}^n_0|X^2_0 \neq 0$; $\widetilde{\omega}^\alpha_0|X^2_0 = \omega^0_\alpha|X^2_0 = 0 \quad 1 \leq \alpha \leq n-1$,

then $\mathcal{O}^2$ is a completely integrable system.

PROOF. The submanifold $\mathcal{O}^2$ is regularly immersed, because it is an orbit of the G-action on $\widetilde{C}^2Q$.

Now, the map $\pi^2_1 : O^2_0 \to C^1Q$ is a local inmersion; indeed by Proposition 4, we have $g^1 = g^2$, and $G^1 \subset G^2$ is open. Moreover G^1 is connected and the group G acts transitively on $\widetilde{G}^1Q$, therefore

$$\pi^2_1 : \mathcal{O}^2 \cong G/G^2 \to \widetilde{C}^1Q \cong G/G^1$$

is a local inmersion, and Condition 1 of Theorem 2 is satisfied.

Let $X^2_p \in \mathcal{O}^2$ and $g \in G$ such that $X^2_p = g \cdot X^2_0$, then by Proposition 2.1 we have that the curve $\Gamma = (t)g \cdot g(t) \cdot P_0$ satisfies

$$C^2_{\Gamma(t)}\Gamma \subset \mathcal{O}^2,$$

$$C^3_{\Gamma(t)}\Gamma \cap C^1\mathcal{O}^2 \xrightarrow{\pi^3_2} C^2_{\Gamma(t)}\Gamma = X^2_p,$$

hence π_2^3 is a local submersion in a neigborhood of X_0^2, and we obtain the result. $\square$

Theorem 2. *Let $\Gamma : J \subset \mathbb{R} \to Q$ be a curve given by the intersection of a complex projective line transversal to D with Q, then Γ is the solution of the differential system defined by $\mathcal{O}^2$.*

Proof. Let $p_0 = (1, 0, \dots, 0) \in V$; $v_0 = (0, \dots, 0, 1) \in T_{p_0}V$;

$$P_0 = [\mu p_0 + \lambda v_0], \qquad \mu, \lambda \in \mathbb{C}.$$

It is clear that $\omega_0^n(\pi_*(v_0)) = 1$, and

$$\Gamma(t) = Q \cap P_0 = \{[1, 0, \dots, 0, t)],\ t \in \mathbb{R}\},$$

expressed in non homogeneous coordinates as

$$\Gamma(t) = (1, 0, \dots, 0, t),$$

is the solution of the differential system defined by $\mathcal{O}^2$ with initial condition X_0^2.

From the uniqueness of the solution [10] and the transitivity of the G action on the directions transversal to D, we obtain that a transversal curve to D at p_0 is singular, if and only if it is contained in a Cartan chain transversal to D.

Let $\widetilde{p} \in V$, $v \in \Gamma_{\widetilde{p}}V$ be such that

$$\pi(\widetilde{p}) \in Q \quad \pi_*(\widetilde{v}) = v \notin D_p.$$

Let $P = \{[\mu\widetilde{p} + \lambda\widetilde{v}],\ \mu\lambda \in \mathbb{C}\}$. Since G acts transitively on $\widetilde{C}^1Q$, there is a $g \in G$ such that $g \cdot p_o = p$, and $(L_g)_*(v_0) = v$.

Now $L_{g-1}(P)$ is a projective line transversal to D at p_0. Since Q is G-invariant and L_g is an n diffeomorphism,

$$L_{g-1}(Q \cap P) = Q \cap L_{g-1}P.$$

Therefore $Q \cap P$ is a solution of $\mathcal{O}^2$ at p, whose tangent line at p is generated by v.

From the uniqueness of the solution of $\mathcal{O}^2$, we obtain that $\Gamma = Q \cap P$ is the solution of $\mathcal{O}^2$ at P. $\square$

Corollary 3. *Given a point $p \in Q$ and a line $L \subset T_pQ$ transversal to D, there exists a unique Cartan chain tangent to L at P.*

Let be given a homogeneous manifold $M = G/H$, and a closed subgroup $K \subset G$. Let $K(0)$ be the orbit of $0 = \pi(H)$ under the induced action of K on G/H.

The order of the orbit $K(0)$, defined in [9], can be given in terms of contact elements as follows:

If G^s is the isotropy group of $C_0^s K(0)$ by the action of G on $C^s K(0)$ and g^s its Lie algebra, then the first index s such that $g^s = g^{s+1}$ is the order of the orbit $K(0)$.

Corollary 4. *The Cartan chains are orbits of order 1 induced by the action of a closed subgroup K of G on G/H_0.*

References

[1] G. E. Bredon, Introduction to compact transformations groups, *Academic Press*, *New York*, 1972.

[2] E. Cartan, Sur la géométrie pseudo-conforme des hypersurfaces de deux variables complexes, I, II, Ouvres II, 2, 1231–1304; ibid III, 2, 1217–1238.

[3] E. Cartan, Théorie des groupes finis et la géométrie différentielle traitées par la Méthode du repére mobile, *Gauthier-Villars*, *Paris*, 1937.

[4] S. S. Chern and J. K. Moser, Real hypersurfaces in complex manifolds, *Acta Math.* **133** (1975), 219–271.

[5] C. Ehresmann, Les prolongements d'un espace fibré différenntiable, *C. R. Acad. Sci. Paris* **240** (1955), 1755–1757.

[6] G. R. Jensen, Higher order contact of submanifolds of homogeneous spaces, Lecture Notes in Math, Vol. 610, *Springer*, *Berlin*, 1977.

[7] I. Kolář, Canonical forms on the prolongations of principle fibre bundles, *Rev. Roum. Math. Pures et Appl., Bucarest,* Tome XVI, N° **7** (1971), 1091–1106.

[8] H. Poincaré, Les fonctions analytiques de deux variables et la représentatión conforme, *Rend. Circ. Mat. Palermo* (1907), 185–220.

[9] I. Szenthe, Transformation groups on homogeneous spaces, *Rend. Sem. Mat., Univers. Politecn. Torino* (1982).

[10] Y. Villarroel, On completely integrable systems, *Publ. Math. Debrecen* **47**/3–4 (1995), 237–248.

[11] Y. Villarroel, Higher order contact on real curves in a real hyperquadric, Archivum mathematicum, Brno, Tomus 32, 1996/1, 57–73.

[12] Y. Villarroel, Order of orbits in homogeneous spaces, *Acta Math. Hung.* **69**(4) (1995), 263–272.

[13] Y. Villarroel, On higher order contact manifolds, Differential geometry and applications (Brno, 1995), *Masaryk Univ.*, *Brno*, 1996, 207–214.

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