Publ. Math. Debrecen
57 / 3-4 (2000), 269–275

On a class of modules

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Abstract. Let R be a ring with identity and M a unital right R-module. Let $\mathrm{Z}^*(M) = \{m \in M : mR \ll \mathrm{E}(mR)\}$. In this study we consider the property (T): For every right R-module M with $\mathrm{Z}^*(M) = \operatorname{Rad} M$, M is injective. We give a characterization of the property (T) when R is a prime PI-ring. Also, over a right Noetherian ring R we prove that if R satisfies (T) then every right R-module is the direct sum of an injective module and a Max-module.

1. Introduction and notations

All rings have identity and all modules are unital right modules.

Let R be a ring and M a right R-module. We write $\mathrm{E}(M)$, $\operatorname{Rad} M$ and $\operatorname{Soc}(M)$ for the injective envelope, the radical and the socle of M, respectively. For the right annihilator of M in R we write $\operatorname{ann}(M)$. As usual, $\mathbb{N}$, $\mathbb{C}$ represent the sets of natural numbers and complex numbers. A submodule N of M is indicated by writing $N \leq M$. The notation $N \leq_e M$ is reserved for essential submodules.

Let N be a submodule of M. N is called a *small submodule* if whenever $N + L = M$ for some submodule L of M we have $M = L$, and in this case we write $N \ll M$. In [7] LEONARD defined a module M to be *small* if it is a small submodule of some R-module. He showed that M is small if and only if M is small in its injective hull. We put

$$\mathrm{Z}^*(M) = \{m \in M : mR \text{ is small}\} \quad [5].$$

Mathematics Subject Classification: 16A12, 16A34, 16A38, 16A52.
Key words and phrases: injective modules, small modules, prime PI-rings, hereditary rings, Noetherian rings.

Since $\mathrm{Rad}(M)$ is the union of all small submodules in M, $\mathrm{Z}^*(E) = \mathrm{Rad}(E)$ for any injective module E and

$$\mathrm{Z}^*(M) = M \cap \mathrm{Rad}\,\mathrm{E}(M) = M \cap \mathrm{Rad}\,E'$$

for an injective $E' \supseteq M$.

In this note we consider the following property:

(T) For every right R-module M with $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$, M is injective.

Clearly, semisimple rings satisfy (T). We will prove that the following are equivalent for a prime PI-ring R:

i) R satisfies (T),

ii) For every left R-module with $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$, M is injective,

iii) R is a hereditary Noetherian ring.

After that we show that over a right Noetherian ring R, if R satisfies (T) then every right R-module is the direct sum of an injective module and a Max-module. Also, if R is a prime right Goldie ring which is not primitive then the converse of the above result holds.

2. Results

We start with the following

Lemma 1. *For any module M, $\mathrm{Z}^*(M)$ is a submodule of M and $\mathrm{Rad}(M) \leq \mathrm{Z}^*(M)$.*

Proof. Elementary. □

Let R be a ring with identity and M be a unital right R-module. An ideal P of R is called *right primitive* if there exists a simple right R-module U such that P is the annihilator of U in R.

Lemma 2. *Suppose that $M = MP$ for every right primitive ideal P. Then $M = \mathrm{Rad}\,M$.*

Proof. Suppose that M contains a maximal submodule N and let $P = \mathrm{ann}(M/N)$. Then $M = MP \leq N$, a contradiction. □

The ring is called *right bounded* if every essential right ideal contains a two-sided ideal which is essential as a right ideal. Moreover, R is *fully right bounded* if R/P is a right bounded ring for every prime ideal P of R. The abbreviation right FBN or FBN is commonly used for a right Noetherian right fully bounded or a Noetherian fully bounded ring, respectively. A ring R is a *PI-ring* if R satisfies a polynomial identity.

Lemma 3. *Suppose that R is right FBN or a PI-ring. Then the ring R/P is (right) Artinian for every right primitive ideal P of R.*

Proof. (See for example [4, Proposition 8.4].) □

Remark. Certain group rings and certain universal enveloping algebras R have the property that the ring R/P is Artinian (because R/P is prime, R/P is right Artinian implies R/P is also left Artinian) for every right primitive ideal P (see [8]). Of course, simple right Noetherian rings which are not (right) Artinian, for example the Weyl algebras $\mathrm{A}_n(\mathbb{C})$ ($n \in \mathbb{N}$), do not have this property.

Lemma 4. *Let R be a ring such that R/P is an Artinian ring for every right primitive ideal P of R. Then $M = \operatorname{Rad} M$ if and only if $M = MP$ for every right primitive ideal P of R.*

Proof. The sufficiency follows by Lemma 2. Conversely, suppose that $M = \operatorname{Rad} M$, i.e. M has no maximal submodule. Let P be any right primitive ideal of R. Then M/MP is a right module over the simple Artinian ring R/P so that M/MP is semisimple. Because M, and hence M/MP, does not have a maximal submodule, it follows that $M = MP$. □

Let M be an injective module. Then $M = Mc$ for every regular (i.e. non-zero divisor) element c in R. A right R-module N is called *divisible* if $N = Nc$ for every regular c. Thus injective modules are divisible [9, Proposition 2.6].

Lemma 5 [6, Proposition 3.5]**.** *Suppose that R is a ring such that every divisible right R-module is injective. Then R is right hereditary.*

Remark. Let R be a semiprime right Goldie ring. Then any torsion free (i.e. non-singular) divisible right R-module is injective.

Lemma 6. *Let R be a prime right or left Goldie ring. Let M be a divisible right R-module. Then $M = MI$ for every non-zero ideal I of R.*

Proof. For any non-zero ideal I of R there exists a regular element c of R such that $c \in I$. Hence $M = Mc \leq MI \leq M$, i.e. $M = MI$. □

Lemma 7. *Let R be a prime right Noetherian ring. Then $M = MI$ for every non-zero ideal I of R if and only if $M = MP$ for every non-zero prime ideal P of R.*

Proof. The necessity is clear. Conversely, suppose that $M = MP$ for every non-zero prime ideal P of R. Let I be any non-zero ideal of R. Then there exists a positive integer n and prime ideals P_i $(1 \leq \mathrm{i} \leq n)$ such that $P_1 \dots P_n \leq I \leq P_1 \cap \dots \cap P_n$. Then $M = MP_n = MP_{n-1}P_n = \dots = MP_1 \dots P_n \leq MI \leq M$, i.e. $M = MI$. □

Lemma 8. *Let R be a left bounded left Goldie prime ring. Then the right R-module M is divisible if and only if $M = MI$ for every non-zero ideal I of R.*

Proof. The necessity follows by Lemma 6. Conversely, suppose that $M = MI$ for every non-zero ideal I of R. Let c be any regular element of R. Then there exists a non-zero ideal J such that $J \leq Rc$. Now $M = MJ \leq MRc = Mc \leq M$, i.e. $M = Mc$. □

Prime PI-rings are right and left bounded and right and left Goldie, so Lemma 7 and Lemma 8 give at once:

Corollary 9. *Let R be a prime PI-ring. Then M is divisible if and only if $M = MI$ for every non-zero ideal I of R. If, in addition, R is right Noetherian, then M is divisible if and only if $M = MP$ for every non-zero prime ideal P of R.*

Lemma 10. *Let R be a prime (right and left) FBN-ring which is not Artinian and for which every non-zero prime ideal is right primitive (maximal in this case). Then the right R-module M satisfies $M = \operatorname{Rad} M$ if and only if M is divisible.*

Proof. By Lemmas 4, 7 and 8. □

We refer to [2, Chapter 6] for the definition of *Krull Dimension*.

Proposition 11. *Let R be a prime PI-ring of right Krull dimension 1. Then the right R-module M satisfies $M = \operatorname{Rad} M$ if and only if M is divisible.*

Proof. Suppose that $S = \operatorname{Soc} R_R \neq 0$. Then S contains a regular element c, because R is prime right Goldie, and $R \cong cR \leq S$ gives that R

is right Artinian, contradicting the fact that R has right Krull dimension 1. Thus $S = 0$.

Let E be any essential right ideal of R. There exists a non-zero ideal I of R such that $I \leq E$. There exists a regular element d such that $d \in I$. Now R/dR is Artinian and hence the right R-module R/I is Artinian (this is because the (right) Krull dimension of R/dR is 0). By the Hopkins–Levitzki Theorem, the right Artinian ring R/I is right Noetherian. Thus R/E is a Noetherian right R-module. It follows that R satisfies the ascending chain condition on essential right ideals and hence the ring R/S is a right Noetherian ring by [2, 5.15]. Thus R is a right Noetherian ring. By [8, 13.6.15 Theorem] R is also left Noetherian.

It is now clear that Lemma 10 can be applied to give the result. □

Proposition 12. *Let R be a prime right Goldie ring which is not primitive. Then $Z^*(M) = M$ for every right R-module M. In addition if R satisfies* (T)*, then every divisible right R-module is injective.*

Proof. Let M be a right R-module, $x \in M$ and $E = \mathrm{E}(xR)$. Suppose that $E = xR + L$ for some $L \leq E$. If x is not in L, then E/L is non-zero and a cyclic module so that there exists a maximal submodule P of E with L contained in P. The module $U = E/P$ is simple, and if I is its annihilator in R we know that I is a non-zero ideal of R by our hypothesis. But in this case I contains a non-zero divisior by Goldie's Theorem [4, Proposition 5.9] and then $E = EI$ by [9, Proposition 2.6] so that $E = P$, a contradiction. Hence $x \in L$ and so $E = L$ and xR is small. Thus $\mathrm{Z}^*(M) = M$.

Now assume that R satisfies (T). Let M be a divisible right R-module and N a maximal submodule of M. Then $0 \neq \mathrm{ann}(M/N) \leq_e R$. There exists a non-zero regular element $d \in \mathrm{ann}(M/N)$. Now $M = Md \leq N$ and so $M = N$. Hence $\mathrm{Rad}\, M = M$. By hypothesis M is injective. □

Remark. Let R be a prime PI-ring. Suppose in addition that R is right hereditary. Because R is right Goldie it follows that R is right Noetherian [1, Corollary 8.25] and hence also left Noetherian [8, 13.6.15 Theorem]. By [8, 6.2.8 Corollary] R has right Krull dimension at most 1. Note also that R is left hereditary because R is right and left Noetherian [1, Corollary 8.18].

Theorem 13. *The following are equivalent for a prime PI-ring R:*

(i) *For every right R-module M with $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$, M is injective,*

(ii) *For every left R-module M with $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$, M is injective,*

(iii) *R is a hereditary Noetherian ring.*

Proof. (i) $\Longrightarrow$ (iii) We claim that every divisible right R-module is injective. Let M be divisible. If $M = \mathrm{Rad}\, M$ then $\mathrm{Z}^*(M) = \mathrm{Rad}(M)$ and hence M is injective. Suppose $M \neq \mathrm{Rad}\, M$ and let N be a maximal submodule of M. If $\mathrm{ann}(M/N) = 0$ then R is primitive. By Kaplansky's Theorem, R is semisimple Artinian. Hence M is injective. If $\mathrm{ann}(M/N) \neq 0$, then, by Proposition 12, M is injective. Thus, by Lemma 5, R is right hereditary. Hence by the above remark R is a hereditary Noetherian ring.

(iii) $\Longrightarrow$ (i) Let M be a right R-module and suppose $\mathrm{Z}^*(M) = \mathrm{Rad}\, M$. By Proposition 12, M has no maximal submodule. By the above remark, R has (right or left) Krull dimension 1. By Proposition 11, M is divisible. Hence M is injective by Theorem 3.37 in [3] and Theorem 3.4 in [6].

(ii) $\Longleftrightarrow$ (iii) Symmetrical. □

We call a module M a *Max-module* if for every non-zero submodule N of M, N has a maximal submodule. For any module M we define the radical series of M to be the chain of submodules

$$M = M_0 \geq M_1 \geq \cdots \geq M_\alpha \geq M_{\alpha+1} \geq \cdots$$

where for any ordinal $\alpha \geq 0$, $\mathrm{Rad}\, M_\alpha = M_{\alpha+1}$ and $M_\alpha = \bigcap_{0 \leq \beta < \alpha} M_\beta$ if α is a limit ordinal. Since M is a set, there exists an ordinal $\rho \geq 0$ such that $M_\rho = M_{\rho+1} = \dots$.

Proposition 14 [10, Proposition 2.2]**.** *A module M is a Max-module if and only if $M_p = 0$.*

Theorem 15. *Let R be a right Noetherian ring. If R satisfies* (T) *then every right R-module is the direct sum of an injective module and a Max-module.*

Proof. Let M be any right R-module. Let $\mathcal{S}$ denote the collection of injective submodules of M (note that $0 \in \mathcal{S}$). Let $\{C_\lambda : \lambda \in \Lambda\}$ be a chain in $\mathcal{S}$ and let $C = \bigcup C_\lambda$ ($\lambda \in \Lambda$). Since R is right Notherian, Baer's Lemma gives that C is injective. Thus $C \in \mathcal{S}$. By Zorn's Lemma $\mathcal{S}$ has a maximal member M_1. Because M_1 is injective, we have $M = M_1 \oplus M_2$ for some submodule M_2 of M. Let N be a non-zero submodule of M_2. By the choice of M_1, $M_1 \oplus N$, hence N is not injective. Thus $\mathrm{Rad}\, N \neq \mathrm{Z}^*(N)$ by hypothesis and it follows that $N \neq \mathrm{Rad}\, N$. Therefore N has a maximal submodule. Thus M_2 is a Max-module. □

Theorem 16. *Let R be a prime right Goldie ring which is not primitive. Assume that every right R-module is the direct sum of an injective module and a Max-module. Then R satisfies* (T).

PROOF. Suppose that every right R-module is the direct sum of an injective module and a Max-module. Let M be any right R-module such that $Z^*(M) = \operatorname{Rad} M$. Then by Proposition 12, $\operatorname{Rad} M = M$. Let $M = X \oplus Y$ where X is injective and Y is a Max-module. Now $X \oplus Y = M = \operatorname{Rad} M = \operatorname{Rad} X \oplus \operatorname{Rad} Y$ so that $Y = \operatorname{Rad} Y$ and Y does not contain a maximal submodule. This implies that $Y = 0$ and $M = X$, i.e. M is injective. □

Acknowledgement. This paper was written while Professor P.F. SMITH (University of Glasgow) was visiting the University of Hacettepe (Ankara, Turkey). The author wishes to thank Professor P. F. SMITH for many useful discussions.

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(Received September 2, 1998; revised September 18, 1999)