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Further remarks on Steinhaus sets

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Abstract. A planar set is said to have the Steinhaus property if however it is placed on $\mathbb{R}^2$, it contains exactly one integer lattice point. The main open question regarding sets having the Steinhaus property (henceforth referred to as Steinhaus sets) is whether such a set exists at all or not. It was proved recently by Mihai Ciucu [3] that a Steinhaus set, if it exists, has empty interior. This was later strengthened [1] to the result that a Steinhaus set can not even contain a circle of positive radius in it and it was further conjectured in [1] that a set having the Steinhaus property cannot contain any homeomorphic image of the unit circle. In the present note, we establish this conjecture.

1. Introduction

A subset of $\mathbb{R}^2$ is said to satisfy the Steinhaus property if under any rigid motion of $\mathbb{R}^2$, it contains exactly one integer lattice point. We shall use the terminology 'Steinhaus sets' for sets satisfying the Steinhaus property. Here the major open problem is whether or not such a set exists.

The earliest result known about Steinhaus sets was obtained by Sierpiński [6] who proved that any Steinhaus set can be neither compact nor bounded open.

Recently, Mihai Ciucu [3] has proved the following theorem.

Theorem 1.1. *Any set S having the Steinhaus property has empty interior.*

As a corollary to the above theorem, it follows (see [3]) that closed sets do not have the Steinhaus property, the boundedness assumption in Sierpiński's result being thus removed.

Regarding other results related to sets having the Steinhaus property, Croft [4] and later J. Beck [2] (apparently unaware of Croft's result and by a different method) had proved the following result.

Theorem 1.2. *There is no bounded and Lebesgue measurable set satisfying Steinhaus' property.*

In [1] it was shown that the ideas in [3] can be pushed to yield finer results. More precisely, it was proved in [1] that if a set $\boldsymbol{S}$ satisfying the Steinhaus property contains a circle of positive radius, then it contains the disk enclosed by the circle. In light of Theorem 1.1 of Mihai Ciucu stated above, as a corollary it then follows that a Steinhaus set cannot contain a circle of positive radius in it.

Proving a conjecture made in [1], we prove the following strengthening of this result in the present note.

Theorem 1.3. *A set satisfying the Steinhaus property can not contain any homeomorphic image of the unit circle.*

We would like to mention that the argument which was employed to tackle the case of the circle in [1], was dependent on special properties of the circle and could not be generalized to yield Theorem 1.3 stated above. To be more precise, in the case of the circle, the proof for the result corresponding to Lemma 2.2 below was different (see the remark following the proof of Lemma 2.2) and at that time the present general proof for this lemma was not seen by its authors. As for the proof of the main theorem of [1], it was observed that if a Steinhaus set contains a circle of positive radius, then it contains the disc enclosed by it. In our Theorem 1.3, we shall show that if a Steinhaus set contains a homeomorphic image of a circle then it will contain some non-empty open subset of the plane.

2. Proof of Theorem 1.3

Let us first fix some notations. We denote a circle centered at a point $\boldsymbol{a}$ and of radius $r > 0$ by $C(\boldsymbol{a}, r)$. Also, for $m, r > 0$, $A(O; m, m+r)$ will denote the closed annulus having its centre at the origin $O = (0,0)$ and of inner radius m and outer radius $m + r$. Assuming the coordinates to increase downward and to the right, for integers m and n, let $A_{m,n}$ denote the unit square of $\mathbb{R}^2$ having its upper left corner at (m, n) and containing the north and west sides.

For any subset $\boldsymbol{M} \subset A_{m,n}$, we denote by $\boldsymbol{M} \pmod 1$ the set $\boldsymbol{M} + (-m, -n)$.

For $\theta \in [0, 2\pi)$, $\boldsymbol{S}(\theta)$ will be the set obtained from S by a rotation through an angle θ around the origin in the anticlockwise direction.

We need two lemmas.

Lemma 2.1 (M. Ciucu [3])**.** *A set* $\boldsymbol{S}$ *satisfies the Steinhaus property if and only if* $\{\boldsymbol{S}(\theta) \cap A_{m,n} \pmod 1\}_{m,n}$ *is a partition of* $A_{0,0}$*, for all* $\theta \in [0, 2\pi)$.

Lemma 2.2. *If a Steinhaus set* $\boldsymbol{S}$ *contains a homeomorphic image of the unit circle which in turn encloses the circle* $C(O, r)$ *for some* $r > 0$ *in* $\mathbb{R}^2$ *and* $\boldsymbol{D} = \bigcup_{n=x^2+y^2} A(O, \sqrt{n} - r, \sqrt{n} + r)$*, then* $\boldsymbol{S} \cap \boldsymbol{D} = \phi$*, where in the union defining* $\boldsymbol{D}$*,* x *and* y *vary over integers.*

Proof. Let $\boldsymbol{S}$ be a Steinhaus set satisfying the condition in the statement of the lemma. Let T be a homeomorphic image of the unit circle contained in S (see Fig. 1). To elaborate, considering both the unit circle and T as topological spaces equipped with the subspace topology from $\mathbb{R}^2$, there is a continuous bijection f from the circle onto T such that the inverse of f is also continuous. Then, by the Jordan separation theorem (see [5], Chapter 8 for instance) the complement of T in $\mathbb{R}^2$ will not be connected and hence will enclose a circle of positive radius. Here we have assumed that the circle has the origin as its centre; in the general situation we can apply a rigid motion to S to have our circle with centre at O.

We claim that for any integer n representable in the form $x^2 + y^2$ where x and y are integers, the annulus $A(O; \sqrt{n} - r, \sqrt{n} + r)$ cannot contain points of $\boldsymbol{S}$. If possible, suppose $P \in \boldsymbol{S} \cap A(O; \sqrt{n} - r, \sqrt{n} + r)$. Writing $L = (x, y)$, the length of the line segment OL is $\sqrt{n}$. Now let the point (or one of the points, in case there are more) at which the line segment PO meets T be C. Similarly let D be one of the points where the extended line $\overrightarrow{PO}$ meets T. Clearly, the length of the segment PC is less than or equal to that of OL which is equal to $\sqrt{n}$. Similarly, the length of the segment PD is greater than or equal to $\sqrt{n}$. So if we apply intermediate value theorem to this length function on either of the two parts to which the line joining P and O divides T, there is a point on each of these parts between C and D whose distance from P is exactly $\sqrt{n}$. Since the point P as well as any point on T belong to the Steinhaus set $\boldsymbol{S}$, by some rigid motion in $\mathbb{R}^2$ the set $\boldsymbol{S}$ can be placed on $\mathbb{R}^2$ in such a way that it will contain two lattice points. Hence the lemma. $\square$

Remark. If T had been a circle, we could have taken the enclosed circle to be T itself. In that case, if some point P of the annulus $A(O; \sqrt{n} - r, \sqrt{n} + r)$ belongs to S, then rotating S suitably around O, the point P can be brought to lie on the circle of radius r centred at L and it could be observed without much difficulty that in that position, the distance of P from some point of T will be same as the distance between L and O.

Proof of the theorem. If possible, let $\boldsymbol{S}$ contain a homeomorphic image of the unit circle. This will enclose a circle of positive radius, say r in $\mathbb{R}^2$. Without loss of generality, we may assume that the centre of the circle is O. Let $\boldsymbol{D}$ be as in Lemma 2.2. Denoting the connected component of $A_{m,n} \cap \boldsymbol{D} \pmod 1$ containing the origin by $C_{m,n}$, let $I(r) \stackrel{\text{def}}{=} \bigcap_{(m,n)\neq(0,0)} C_{m,n}$.

If $\boldsymbol{x} \in I(r) \subset A_{0,0}$, then by Lemma 2.1 there exist m, n such that $\boldsymbol{x} \in \boldsymbol{S} \cap A_{m,n} \pmod 1$, that is, $\boldsymbol{x} + (m,n) \in \boldsymbol{S}$. Now, if $(m,n) \neq (0,0)$, then, since $\boldsymbol{x} \in I(r)$, we have $\boldsymbol{x} \in C_{m,n} \subset A_{m,n} \cap \boldsymbol{D} \pmod 1$ and so $\boldsymbol{x} + (m,n) \in \boldsymbol{D}$, which contradicts Lemma 2.2 which says that $\boldsymbol{S} \cap \boldsymbol{D}$ is empty. Hence $(m,n) = (0,0)$ and this implies that $I(r) \subset \boldsymbol{S}$.

Since for each $(m,n) \neq (0,0)$, $C_{m,n}$ contains the lower right quarter Q of the closed disk of radius r centered at the origin, Q is contained in $I(r)$ and therefore by the observation above, $Q \subset \boldsymbol{S}$ and hence $\boldsymbol{S}$ contains a closed disk. Since this contradicts Theorem 1.1, we are through. $\square$

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