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Completeness of Finsler manifolds

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Dedicated to Professor Lajos Tamássy on his 70th birthday

Abstract. This paper analyses some constructions that produce complete Finsler manifolds:

1) Let the Finsler manifolds $(M, g(x,y))$ and $(M, \bar{g}(x,y))$ be given. Then $(M, \bar{g}(x,y))$ is complete if $(M, g(x,y))$ is complete and the tensor field $\bar{g} - g$ is positive semi-definite.
2) If $(M, g(x,y))$ is a Finsler manifold and $f : M \to \mathbb{R}$ is a proper function then the Finsler manifold $(M, g(x,y)+df(x)\otimes df(x))$ is complete. Using this construction we prove that a Finsler manifold which supports a proper function whose differential has bounded relative length is complete.
3) Let the Finsler manifolds $(M_1, g_1(x_1,y_1))$ and $(M_2, g_2(x_2,y_2))$ be given and suppose that $f > 0$ is a differentiable function on M_1. The warped product $(M_1 \times M_2,\ g_1 + fg_2)$ is complete if and only if $(M_1, g_1(x_1,y_1))$ and $(M_2, g_2(x_2,y_2))$ are complete.

§1. Complete Finsler manifolds [2], [4], [5]

Let M be an n–dimensional connected C^3–manifold and TM its tangent bundle. Denote by (x,y) an arbitrary point in TM and by x the corresponding point in M.

Definition 1.1. A Finsler tensor field $g(x,y)$ of type $(0,2)$ which is symmetric, positive definite and whose components $g_{ij}(x,y)$ are homogeneous functions of degree zero with respect to y is called a *Finsler metric* on M. The pair $(M, g(x,y))$ is called a Finsler manifold.

The function

$$L : TM \to \mathbb{R}, \quad L(x,y) = \sqrt{g_{ij}(x,y)y^i y^j}$$

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is called fundamental Finsler function and L^2 is called *absolute Finsler energy.*

For a vector field $V = V^i(x)\frac{\partial}{\partial x^i}$ on M we have two kinds of lengths: the absolute length

$$L(x, V(x)) = \sqrt{g_{ij}(x, V(x))V^i(x)V^j(x)}$$

and the relative length

$$\|V(x)\|_y = \sqrt{g_{ij}(x, y)V^i(x)V^j(x)}\,.$$

Remark. If $g_{ij}(x, V(x)) - g_{ij}(x, y)$ is negative semidefinite, then the absolute length of $V(x)$ is the minimum of the relative length of $V(x)$.

In a Finsler manifold $(M, g(x, y))$ the length ℓ of a curve arc $\gamma : [0, 1] \to M$ is given by

$$\ell(\gamma) = \int_0^1 L(\gamma(t), \dot{\gamma}(t))dt\,.$$

Definition 1.2. let $(M, g(x, y))$ be a Finsler manifold. The function

$$\exp_x : 0_x \subset T_xM \to M, \quad X \to \exp_x X\,,$$

where $\exp_x X$ is the terminal point $\gamma(1)$ of the geodesic $\gamma : [0, 1] \to M,\ \gamma(0) = x,\ \dot{\gamma}(0) = X$ is called the *exponential map.*

The curve $\gamma : [0, 1] \to M,\ \gamma(t) = \exp_x(tX),\ X \in T_xM$ is a geodesic which joins the points x and $\exp_x X$. The length of this geodesic is $L(x, X)$.

Definition 1.3. The *distance* $d(x, x')$ between the points $x, x' \in M$ is the infimum of the lengths of all curves from x to x'.

This definition is correctly in sense that the properties:

1) $d(x, x') \geq 0, \quad \forall x, x' \in M$
2) $d(x, x') = 0,$ if and only if $x = x'$
3) $d(x, x') = d(x', x), \quad \forall x, x' \in M$
4) $d(x, x') \leq d(x, x'') + d(x'', x'), \quad \forall x, x', x'' \in M$

are satisfied. Also, the topology of M induced by the distance d coincides with the manifold topology of M.

Definition 1.4. The Finsler manifold $(M, g(x, y))$ is called *geodesically complete* if the exponential map $\exp_x$ is defined on the whole of T_xM for any point of M.

Definition 1.5. The Finsler manifold $(M, g(x, y))$ is called *metrically complete* if the metric space (M, d) is complete.

Theorem 1.1. [2] *For a Finsler manifold $(M, g(x,y))$ the following three conditions are equivalent:*
1) *$(M, g(x,y))$ is geodesically complete.*
2) *$(M, g(x,y))$ is metrically complete.*
3) *Any bounded closed subset of M is compact.*

Remarks. 1) Let (M, g) and $(M, \bar{g})$ be two Finsler manifolds. Then $(M, \bar{g})$ is complete if (M, g) is complete and the tensor field $\bar{g} - g$ is positive semidefinite.

2) Let (M_1, d_1) and (M_2, d_2) be complete metric spaces. The product space $(M_1 \times M_2,\ d_1 + d_2)$ is complete.

3) Let $g_{ij}(x,y) = \gamma_{ij}(x) + c^{-2} y_i y_j$,
where γ_{ij} is a Riemann metric tensor, c is the universal speed-of light constant, $\dot{x}^i$ is the tangent vector supported by a point $x = (x^i)$ and $y_i = \gamma_{ij}(x)\dot{x}^j$. If the Riemann manifold $(M, \gamma_{ij}(x))$ is complete, then the generalized Lagrange manifold $(M, g_{ij}(x,y))$ is complete. This generalized Lagrange manifold is not reducible to a Lagrange manifold, neither to a Finsler manifold nor to a Riemannian manifold [3].

§2. Analytical criterion for completeness

Definition 2.1. A continuous function $f : M \to \mathbb{R}$ is called *proper* if $f^{-1}(K)$ is a compact set whenewer K is compact.

Theorem 2.1. *Let $(M, g(x,y))$ be a Finsler C^3–manifold (not necessarily complete) and $f : M \to \mathbb{R}$ a proper C^3 function.*

The Finsler manifold $(M, \tilde{g}(x,y) = g(x,y) + df(x) \otimes df(x))$ is complete.

Proof. We consider the Finsler manifold $(M \times \mathbb{R}, h)$, where $h_{ij} = g_{ij},\ h_{i\,n+1} = 0,\ i,j = 1,2,\dots,n,\ h_{n+1n+1} = 1$. The graph

$$G(f) = \{(x, f(x)) | x \in M\}$$

is a submanifold of the product manifold $M \times \mathbb{R}$, diffeomorphic to M.

The Finsler metric $h_{\alpha\beta}$ induces on $G(f)$ the Finsler metric $\tilde{g} = g + df \otimes df$.

If $\{(x_n,\ f(x_n))\}$ is a Cauchy sequence of elements in $G(f)$, then $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$ because

$$d_{G(f)}[(x, f(x)),\ (x', f(x'))] \ge d_{M \times R}[(x, f(x)),\ (x', f(x'))] \ge \\ \ge |f(x) - f(x')|\,.$$

Then there exists $z \in \mathbb{R}$ with $z = \lim_{n \to \infty} f(x_n)$ and hence $\{z, f(x_1), \dots, f(x_n), \dots\}$ is a compact set in $\mathbb{R}$. But $\{x_1, \dots,\ x_n, \dots\} \subset f^{-1}(\{z, f(x_1),$

$\dots, f(x_n), \dots\})$ and f is proper. Hence the sequence $\{x_n\}$ contains a convergent subsequence since $\{x_n\}$ is contained in a compact set.

So the sequence $\{(x_n, f(x_n))\}$ is covergent and hence $(M, \tilde{g}(x,y))$ is complete.

Theorem 2.2. *Let $(M, g(x,y))$ be a Finsler C^3–manifold. If there exists a proper C^3 function $f : M \to \mathbb{R}$ such that the Finsler tensor field $g(x,y) - df(x) \otimes df(x)$ is positive definite, then $(M, g(x,y))$ is complete.*

Proof. Put, $\tilde{g} = g - df \otimes df$. If $\tilde{g}$ is positive definite, then $(M, \tilde{g}(x,y))$ is a Finsler C^3–manifold. As $f : M \to \mathbb{R}$ is a proper function, we apply theorem 2.1 which says that $(M, \tilde{g} + df \otimes df)$ is complete. But $\tilde{g} + df \otimes df = g$. Hence $(M, g(x,y))$ is complete.

Theorem 2.3. [1]. *Any C^3–manifold M supports a proper C^3 function $f : M \to \mathbb{R}$.*

Theorem 2.4. *Let $(M, g(x,y))$ be a Finsler C^3–manifold and $f : M \to \mathbb{R}$ a C^3 function. Then the Finsler tensor field*

$$\tilde{g}(x,y) = g(x,y) - df(x) \otimes df(x)$$

is positive definite iff $\|df(x)\|_y < 1$, for any vector y.

Proof. Let $V^i(x,y) = g^{ij}(x,y)\frac{\partial f(x)}{\partial x^j}$ and $V(x,y) = V^i(x,y)\frac{\delta}{\delta x^i}$. Here $\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$ is a local base adapted to the horizontal canonical distribution $N = (N_i^j(x,y))$ of the manifold (M, g).

Suppose that $\tilde{g}(x,y)$ is positive definite and x is not a critical point of f. Hence

$$0 < \tilde{g}_{ij}(x,y)V^i(x,y)V^j(x,y) = g_{ij}(x,y)g^{ik}(x,y)\frac{\partial f(x)}{\partial x^k} g^{j\ell}(x,y)\frac{\partial f(x)}{\partial x^\ell} -$$

$$-\frac{\partial f(x)}{\partial x^i}\frac{\partial f(x)}{\partial x^j} g^{ik}(x,y)\frac{\partial f(x)}{\partial x^k} g^{j\ell}(x,y)\frac{\partial f(x)}{\partial x^\ell} =$$

$$= \delta_j^k g^{j\ell}(x,y)\frac{\partial f(x)}{\partial x^\ell}\frac{\partial f(x)}{\partial x^k} - g^{ik}(x,y)\frac{\partial f(x)}{\partial x^i}\frac{\partial f(x)}{\partial x^k} g^{j\ell}(x,y)\frac{\partial f(x)}{\partial x^j}\frac{\partial f(x)}{\partial x^\ell} =$$

$$= \|df(x)\|_y^2 (1 - \|df(x)\|_y^2)\,.$$

Thus $\|df(x)\|_y < 1$, $\forall y$.

Now suppose that $\|df(x)\|_y < 1$, $\forall y$. Then for any vector field $X(x) = X^i(x)\frac{\partial}{\partial x^i}$ we have

$$\tilde{g}_{ij}(x,y)X^i(x)X^j(x) = g_{ij}(x,y)X^i(x)X^j(x) - \frac{\partial f(x)}{\partial x^i}X^i(x)\frac{\partial f(x)}{\partial x^j}X^j(x) =$$

$$= \|X(x)\|_y^2 - \delta^i_k \frac{\partial f(x)}{\partial x^i}X^k(x)\delta^j_\ell \frac{\partial f(x)}{\partial x^j}X^\ell(x) = \|X(x)\|_y^2 -$$

$$-g_{km}(x,y)g^{im}(x,y)\frac{\partial f(x)}{\partial x^i}X^k(x)g_{\ell s}(x,y)g^{js}(x,y)\frac{\partial f(x)}{\partial x^j}X^\ell(x) =$$

$$= \|X(x)\|_y^2 - (g_{km}(x,y)g^{im}(x,y)\frac{\partial f(x)}{\partial x^i}X^k(x))^2 \geq$$

$$\geq \|X(x)\|_y^2 - (g_{km}(x,y)X^k(x)X^m(x))^2\left(g^{is}(x,y)\frac{\partial f(x)}{\partial x^i}\frac{\partial f(x)}{\partial x^s}\right)^2 =$$

$$= \|X(x)\|_y^2(1 - \|df(x)\|_y^2)\,,$$

and hence $\tilde{g}(x,y)$ is positive definite.

From theorems 2.2 and 2.4 follows

Theorem 2.5. *A Finsler C^3–manifold $(M, g(x,y))$ which supports a proper C^3 function $f : M \to \mathbb{R}$ such that $\|df(x)\|_y < 1$, $\forall y$, is complete.*

Theorem 2.6. *Let $(M, g(x,y))$ be a Finsler C^3–manifold and $f : M \to \mathbb{R}$ a proper C^3 function. Then*

$$\big(M,\ \tilde{g}(x,y) = e^{\|df(x)\|_y^2}g(x,y)\big)$$

is a complete Finsler manifold.

Proof. Obviously $\tilde{g}(x,y)$ is symmetric, positive definite and its components $\tilde{g}_{ij}(x,y)$ are homogeneous functions of degree zero with respect to y. On the other hand

$$\|\widetilde{df(x)}\|_y^2 = e^{-\|df(x)\|_y^2}\|df(x)\|_y^2 \leq \frac{1}{e}\,.$$

So the proper function $\varphi : M \to \mathbb{R}$, $\varphi = \sqrt{e}f$ satisfies $\|\widetilde{d\varphi}\|_y^2 < 1$, $\forall y$. Applying theorem 2.5 we obtain the desired result.

§3. Warped products of complete Finsler manifolds

Let $(M_1, g_1(x_1,y_1))$ and $(M_2, g_2(x_2,y_2))$ be Finsler manifolds and $f > 0$ a differentiable function on M_1. Consider the product manifold $M_1 \times M_2$ with its projections

$$\pi : M_1 \times M_2 \to M_1, \quad \eta : M_1 \times M_2 \to M_2\,.$$

The Finsler manifold
$$(M_1 \times M_2,\ g_1 + f g_2)$$
is called the warped product between (M_1, g_1) and (M_2, g_2).

Theorem 3.1. *$(M_1 \times M_2,\ g_1 + f g_2)$ is complete if and only if (M_1, g_1) and (M_2, g_2) are complete.*

Proof. If $(M_1 \times M_2,\ g_1 + f g_2)$ is complete, then a Cauchy sequence in (M_1, g_1) or (M_2, g_2) imbeds in a (horizontal) leaf or a (vertical) fiber as a Cauchy sequence, and hence converges.

If (M_1, g_1) and (M_2, g_2) are complete, let $\{p_i = (p_{1i}, p_{2i})\}$ be a Cauchy sequence in $(M_1 \times M_2,\ g_1 + f g_2)$. Denote by α_{ij} a curve from p_i to p_j in $(M_1 \times M_2,\ g_1 + f g_2)$ having length at most $2d(p_i, p_j)$. We can assume that all projections $\pi \circ \alpha_{ij}$ lie in a compact region in M_1, and on this we have $f \geq c > 0$. Consequently the speed of α_{ij} at each point is at least c times the speed of $\eta \circ \alpha_{ij}$. Thus
$$d(p_{2i}, p_{2j}) \leq \frac{2}{c}\, d(p_i, p_j)$$
showing that $\{p_{2i}\}$ is Cauchy and hence convergent.

Since π is distance-nonincreasing, the sequence $\{p_{1i}\}$ is also Cauchy, hence convergent. Thus $\{p_i\}$ is convergent, and $(M_1 \times M_2,\ g_1 + f g_2)$ is complete.

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