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Class number problems for dicyclic CM-fields

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Abstract. We prove that the least relative class number of dicyclic CM-fields of degree $4p$ (p any odd prime) is equal to four and we determine all the dicyclic CM-fields of relative class number four. This determination provides us (1) with interesting examples of numerical computations of relative class numbers of non-abelian CM-fields by using evaluations at $s = 1$ of Hecke L-functions over real quadratic fields for which their Artin root numbers may be equal either to $+1$ or to -1, and (2) with interesting illustrations of the use of our theorem on upper bounds of values at $s = 1$ of some abelian Hecke L-functions. We also point out that Shintani's method enables to understand why relative class numbers of various types of CM-fields are always perfect squares.

1. Introduction

Let us fix some of the notation we will be using throughout this paper. We let L denote a real quadratic number field and let A_L, d_L and χ_L denote the ring of algebraic integers, the discriminant and the even primitive Dirichlet character of conductor d_L associated with L, respectively. We let $p \geq 3$ denote an odd prime and N a dicyclic number field of degree $4p$, i.e., N is a number field (considered as a subfield of the field of complex numbers) such that the extension N/Q is a normal extension of degree $4p$ with Galois group the dicyclic group Q_{4p} of order $4p$ defined by the presentation $Q_{4p} = \langle a, b : \ a^{2p} = 1, \ a^p = b^2, \ b^{-1}ab = a^{-1}\rangle$. Note that the centre $Z(Q_{4p}) = \{1, a^p\}$ of Q_{4p} has order 2. We let N^+ denote the subfield of N fixed by the cyclic subgroup generated by a^p and M denote

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the subfield of N fixed by the cyclic subgroup generated by a^2. We have the following lattice of subfields:

$$\begin{tikzcd} & & N \\ & N^+ \arrow[ur, dash, "2"] & \\ & & M \arrow[uu, dash, "p"'] \\ & L \arrow[uu, dash, "p"] \arrow[ur, dash, "2"] & \\ Q \arrow[ur, dash, "2"] & & \end{tikzcd}$$

N/Q dicyclic

N/L cyclic

M/Q cyclic quartic

N^+/Q dihedral

The conductor $\mathcal{F}_{N^+/L}$ of the cyclic extension N^+/L is given by $\mathcal{F}_{N^+/L} = (f_+)$ for some positive rational integer $f_+ \geq 1$ of the form

$$(1)\quad f_+ = p^a \prod_{i=1}^{r} q_i \text{ where } a = 0 \text{ or } a = \begin{cases} 2 & \text{if } p \text{ does not divide } d_L \\ 1 & \text{if } p \geq 5 \text{ divides } d_L, \\ 1 \text{ or } 2 & \text{if } p = 3 \text{ divides } d_L, \end{cases}$$

where the q_i's are primes not equal to p satisfying $q_i \equiv \chi_L(q_i) \pmod p$ (see [Mar] and [LPL]). Notice that since M is cyclic then any prime which divides d_L is not equal to 3 modulo 4 and the latter occurence will never happen. Recall that a number field E is called a CM-field if it is a quadratic extension of its maximal totally real subfield E^+. In that situation, the class number h_{E^+} of E^+ divides the class number h_E of E and $h_E^- = h_E/h_{E^+}$ is called the relative class number of E. If E has degree $2n$ then we have

$$(2)\qquad h_E^- = \frac{Q_E w_E}{(2\pi)^n}\sqrt{\frac{d_E}{d_{E^+}}}\,\frac{\mathrm{Res}_{s=1}(\zeta_E)}{\mathrm{Res}_{s=1}(\zeta_{E^+})} = 2^{-n} Q_E w_E (\zeta_E/\zeta_{E^+})(0)$$

where d_E and d_{E^+} denote the absolute values of the discriminants of E and E^+, respectively. If E is a normal CM-field then complex conjugation c is in the center $Z(G)$ of its Galois group G ([LOO, Lemma 2]). Hence, E^+/Q is a normal extension. In particular, if N is a dicyclic CM-field then $c = a^p$ and our present notation is consistent with the previous one we used in our lattice of subfields. The motivation of this paper is to prove the following result related to [Lou5, Theorem 7]:

Theorem 1. *Let N be a dicyclic CM-field of degree $4p$, p any odd prime. Then $h_N^- \geq 4$. Moreover, $h_N^- = 4$ if and only if $N = KM$ is one of the following four dicyclic CM-fields of degree 12 where K is a non-normal totally real cubic field of discriminant d_K and M is an imaginary cyclic quartic field of conductor f_M:*

$P_K(X)$	d_K	f_M	f_+	$M = Q(\sqrt{-\alpha_M})$ with	h_{N^+}
$X^3 + X^2 - 3X - 1$	$148 = 37 \cdot 2^2$	37	2	$\alpha_M = 37 + 6\sqrt{37}$	1
$X^3 - 10X - 10$	$1300 = 13 \cdot 10^2$	13	10	$\alpha_M = 13 + 2\sqrt{13}$	1
$X^3 + X^2 - 7X - 2$	$1573 = 13 \cdot 11^2$	13	11	$\alpha_M = 13 + 2\sqrt{13}$	1
$X^3 - 12X - 14$	$1620 = 5 \cdot 18^2$	5	18	$\alpha_M = 5 + 2\sqrt{5}$	1

In that case, $N^+ = KL$ is a dihedral sextic field where L is the real quadratic subfield of M.

Remark. For the four CM-fields N of degree 12 which appear in the table above we have $h_N = h_N^- h_{N^+} = 4$. Since the extension N/K is cyclic quartic, then according to [Lou1, Lemma 1] the ideal class group of N cannot be cyclic. Hence, in these four cases the ideal class group of N is of type $(2, 2)$.

2. The least possible relative class number

Let p be an odd prime. *A pure real dihedral number field of degree* $2p$ is a normal field F of degree $2p$ and of Galois group the dihedral group of order $2p$ such that p is totally ramified in F/Q and such that p is the only rational prime which is ramified in F/Q. Note that if there exists a pure real dihedral field F of degree $2p$ then $p \equiv 1 \pmod 4$ and $Q(\sqrt{p})$ is the quadratic subfield of F. We now collect known results we will use to prove that there is no dicyclic CM-field with relative class number less than 4:

Proposition 3.

1. *(See* [LOO, Theorem 5]*.) Let $k \subseteq K$ be two CM-fields. If the degree $[K : k]$ of the extension K/k is odd, then h_k^- divides h_K^-.*
2. *(See* [LO]*.) Let K be a CM-field. If t prime ideals of K are ramified in K/K^+ then 2^{t-1} divides h_K^-.*
3. *(See* [LOO, Proposition 8]*.) Let N/M be a cyclic extension of CM-fields of degree p an odd prime and assume that N^+/M^+ also is a*

cyclic extension of degree p. If T prime ideals of M^+ split in M/M^+ and are ramified in N^+/M^+ then $p^{T-1}h_M^-$ divides h_N^-.

4. *(See* [LOO, Proposition 9]*.) Let $p \equiv 1 \pmod 4$ be a prime and let $\epsilon_p = (u_p + v_p\sqrt{p})/2 > 1$ be the fundamental unit of $Q(\sqrt{p})$. If p does not divides v_p, then there does not exist any pure real dihedral number field of degree $2p$.*

5. *(See* [Mar]*.) Let F be a dihedral field of degree $2p$, let L denote its quadratic subfield, let χ_L denote the primitive quadratic Dirichlet character associated with L and let q denote any rational prime. Then,*
   (a) *q is not inert in F/Q.*
   (b) *If q is ramified in L/Q, say $(q) = \mathcal{Q}^2$ in L, then either $\mathcal{Q}$ splits completely in F/L or $\mathcal{Q}$ is totally ramified in F/L. In the latter case, $q = p$.*
   (c) *If q is different from*[1] *p and if the prime ideals of L above q are ramified in F/L then $q \equiv \chi_L(q) \pmod p$.*

6. *(See* [Lou1]*.) Let M be an imaginary cyclic quartic field of conductor f_M. Then h_M^- is odd if and only if $f_M = 16$ or $f_M = q \equiv 5 \pmod 8$ is prime. Moreover, if h_M^- is odd then $h_M^- \equiv 1 \pmod 4$, hence we cannot have $h_M^- = 3$. Finally, $h_M^- = 1$ if and only if $f_M \in \{16,\ 5,\ 13,\ 29,\ 37,\ 53,\ 61\}$.*

Proof. Only the last point needs a proof. If h_M^- is odd then according to point 2 at most one prime ideal of M^+ is ramified in M/M^+, and since M/Q is cyclic quartic, then at most one rational prime q is ramified in M/Q. Conversely, if only one rational prime is ramified in M/Q then h_M is odd (see [Wa, Theorem 10.4(b)]), hence h_M^- is odd. Now, if $f_M = 16$ or if $f_M = 5$ then $h_M^- = 1$. If $f_M = q \equiv 5 \pmod 8$ is not equal to 5 and χ_M denotes any one of the two quartic Dirichlet characters modulo q associated with M, then $2q^2h_M^- = a^2 + b^2 = N_{Q(i)/Q}(a + bi)$ where $a + bi = \sum_{x=1}^{q-1} x\chi_M(x) \in Z[i]$ (use [Wa, Theorem 4.17]): Since h_M^- is odd we get $h_M^- \equiv 1 \pmod 4$. □

[1]Note that we forgot to mention this restriction in [LOO, Lemma 4(ii)].

Theorem 4. *Let N be a dicyclic CM-field of degree $4p$, $p \geq 3$ a prime.*

1. *Assume that $p \equiv 1 \pmod 4$ and $L = Q(\sqrt{p})$. Any rational prime $q \neq p$ which is ramified in N^+/L satisfies $q \equiv 1 \pmod p$ and splits in L/Q. Therefore, if $f_M = p$ and N^+ is not pure, then p divides h_N^-.*
2. *Assume that 2^p does not divide h_N^-. Then we are in one of following two situations:*
   (a) *$p \equiv 1 \pmod 4$, $L = Q(\sqrt{p})$, p is totally ramified in N^+/Q and if a prime $q \neq p$ is ramified in M/Q then q splits completely in L/Q. Therefore, either*
   4 divides h_M^-,
   or $f_M = p \equiv 5 \pmod 8$, h_M^- is odd, and if N^+ is not pure then $p \geq 5$ divides h_N^-.
   (b) *$f_M = 16$ or $f_M = q \equiv 5 \pmod 8$ is prime, and if $q = p$ then p is not totally ramified in N^+/Q. In that situation 2^{p-1} divides h_N^- and h_M^- is odd.*
3. *If $h_N^- < 4$ then $h_M^- = 1$ and N^+ is a pure real dihedral field of degree $2p$. Hence, we must have $p \in \{5,\ 13,\ 29,\ 37,\ 53,\ 61\}$.[2] Therefore, we always have $h_N^- \geq 4$.*
4. *Assume that $h_N^- = 4$. Then, either*
   (a) *$p \equiv 1 \pmod 4$, $L = Q(\sqrt{p})$, p is totally ramified in N^+, $h_M^- = 4$ and $f_M = p \cdot q^a \in \{5\cdot 29,\ 13\cdot 17,\ 17\cdot 4,\ 17\cdot 8,\ 29\cdot 5,\ 41\cdot 4,\ 73\cdot 3\}$,*
   (b) *or $p = 3$, $h_M^- = 1$ and $f_M = q \in \{16,\ 5,\ 13,\ 29,\ 37,\ 53,\ 61\}$.*

Proof.

1. Since $q \equiv \chi_L(q) \pmod p$ (Proposition 3, point 5) we do get $\chi_L(q) = (\frac{q}{p}) = (\frac{\pm 1}{p}) = +1$ and $q \equiv \chi_L(q) = 1 \pmod p$. If $f_M = p$ then M is a subfield of the cyclotomic field $Q(\zeta_p)$ and since $q \equiv 1 \pmod p$ implies that q splits completely in $Q(\zeta_p)/Q$, we get that q splits completely in M/Q and if N^+ is not pure then p^{T-1} with $T \geq [M^+ : Q] = 2$ divides h_N^- (Proposition 3, point 3).
2. Assume that at least two rational primes q_1 and q_2 are ramified in L/Q. We may assume that $q_2 \neq p$. Then the prime ideal of L lying above q_2 splits in N^+/L (Proposition 3, point 5) and at least $p+1$ ideals of N^+ are ramified in N/N^+ (the p ones above q_2 and the

[2] Note that the possibility $p = 61$ was not taken care of in [LOO, Theorem 6(iii)].

ones above q_1) and 2^p divides h_N^- (Proposition 3, point 2). Therefore, $L = Q(\sqrt{q})$ for some prime $q \not\equiv 3 \pmod 4$.

First, assume that we are in case (a) and let $q \neq p$ be ramified in M/Q. If q were inert in L/Q then q would not be ramified in N^+/L (point 1) and would split completely in N^+/L (Proposition 3, point 5). Hence, at least $p+1$ ideals of N^+ would be ramified in N/N^+ (the p ones above q and the one above p) and 2^p would divides h_N^- (Proposition 3, point 2), a contradiction. Therefore, q splits in L/Q, hence at least three prime ideals of $L = M^+$ are ramified in M/M^+ (those above q and the one above p) and 4 divides h_M^- (Proposition 3, point 2). We then use Proposition 3, point 6, to complete the proof of this case (a).

Second, assume that we are not in case (a). Then either $q \neq p$, or $q = p \equiv 1 \pmod 4$ and p is not totally ramified in N^+/L. In both cases the prime ideals of L above q split in N^+/L and ramify in M/L, hence at least p prime ideals of N^+ ramify in N/N^+ and 2^{p-1} divides h_N^-. Since 2^p does not divide h_N^- then q is the only prime ramified in M/L and since M/Q is cyclic quartic q is the only prime ramified in M/Q and according to Proposition 3, point 6, we do are in case (b).

3. Follows from point 2.
4. Use point 2 and the determination in [PK] of all the imaginary cyclic quartic fields with relative class number 4. □

3. Lower bounds on relative class numbers

In the dicyclic case, we set $\zeta_{N^+/L} = \zeta_{N^+}/\zeta_L$, $\zeta_{N/L} = \zeta_N/\zeta_L$ (both of them being entire functions), rewrite (2) in the following form

$$h_N^- = \frac{Q_N w_N}{(2\pi)^{2p}} \sqrt{\frac{d_N}{d_{N^+}}} \frac{\zeta_{N/L}(1)}{\zeta_{N^+/L}(1)}. \tag{3}$$

To get lower bounds on h_N^- we need upper bounds on $\zeta_{N^+/L}(1)$ and lower bounds on $\zeta_{N/L}(1)$. For a CM-field N of degree $2n$ we set

$$\epsilon_N = \max(\epsilon'_N, \epsilon''_N) \text{ where } \epsilon'_N = \frac{2}{5} \exp\left(-\frac{2\pi n}{d_N^{1/2n}}\right) \text{ and } \epsilon''_N = 1 - \frac{2\pi n}{d_N^{1/2n}}. \tag{4}$$

We have:

Proposition 5. *Let N be a dicyclic CM-field of degree $4p$, let L be its real quadratic subfield and N^+ is maximal totally real subfield. Let χ_{N^+} be a character of order p generating the group of characters of order p associated with the abelian extension N^+/L and let f_+ denote its conductor.*

1. *(See* [Lou2], [Lou3]*.) Set $c = 2 + \gamma - \log(4\pi) = 0.046\cdots$ where $\gamma = 0.577\cdots$ denotes Euler's constant. Then,*
   $\mathrm{Res}_{s=1}(\zeta_L) \leq (\log d_L + c)/2,$
   $$\lambda_L \stackrel{\text{def}}{=} \left(1 + \log\left(\sqrt{d_L}/4\pi\right)\right) L(1, \chi_L) \ + L'(1, \chi_L) \leq \frac{1}{8}\log^2 d_L$$
   and
   $$\zeta_{N^+/L}(1) = \prod_{i=1}^{p-1} |L(1, \chi_{N^+}^i)| \leq (\mathrm{Res}_{s=1}(\zeta_L)\log f_+ \ + 2\lambda_L)^{p-1}. \tag{5}$$
2. *We have $\zeta_{N/L}(s) \geq 0$ for $0 < s < 1$.*
3. *(See* [Lou1, Section 3.1]*.) Let L be a real quadratic field, N be a totally imaginary number field and assume that the extension N/L is normal and such that $\zeta_{N/L}(s) \geq 0$ for $0 < s < 1$. Then*
   $$\zeta_{N/L}(1) \geq \epsilon_N \frac{4}{e(\log d_L + c)\log d_N}. \tag{6}$$

Proof. Only point 2 needs a proof. Since the extension N/L is cyclic of degree $2p$ then ζ_N/ζ_M is a product of $2p-2$ abelian L-functions $L(s,\chi)$ over non quadratic characters χ which come in conjugate pairs. Hence, $(\zeta_N/\zeta_M)(s) \geq 0$ for $0 < s < 1$. In the same way since M/Q is cyclic quartic then ζ_M/ζ_L is a product of two Dirichlet L-functions associated with two conjugated quartic characters. Hence, $(\zeta_M/\zeta_L)(s) \geq 0$ for $0 < s < 1$ (note that we cannot prove this last assertion if N is a dihedral field). Since $\zeta_M/\zeta_L = (\zeta_N/\zeta_M)(\zeta_M/\zeta_L)$, we get the desired result. □

We developed in [Lou3] an efficient technique for computing numerical approximations of λ_L, technique we have used to fill in Table 1 below.

case	d_L	$\mathrm{Res}_{s=1}(\zeta_L)$	λ_L	f_M	p	$f_+ \leq$
1	8	$0.623\cdots$	$0.0877\cdots$	16	3	2800
2	5	$0.430\cdots$	$0.0436\cdots$	5	3	3900
3	13	$0.662\cdots$	$0.146\cdots$	13	3	3000
4	29	$0.611\cdots$	$0.283\cdots$	29	3	1400
5	37	$0.819\cdots$	$0.352\cdots$	37	3	1500
6	53	$0.540\cdots$	$0.407\cdots$	53	3	700
7	61	$0.938\cdots$	$0.487\cdots$	61	3	1100
8	5	$0.430\cdots$	$0.0436\cdots$	145	5	160
9	13	$0.662\cdots$	$0.146\cdots$	221	13	60
10	17	$1.016\cdots$	$0.220\cdots$	68	17	80
11	17	$1.016\cdots$	$0.220\cdots$	136	17	80
12	29	$0.611\cdots$	$0.283\cdots$	145	29	30
13	41	$1.299\cdots$	$0.473\cdots$	164	41	50
14	73	$1.794\cdots$	$0.756\cdots$	219	73	50

Table 1: bounds on f_+ if $h_N^- = 4$.

Theorem 6. *Let N be a dicyclic CM-field of degree $4p$ where p is an odd prime. Let f_+ and f_M denote the conductors of the extensions N^+/L and M/Q, respectively, and set $f = \mathrm{lcm}(f_+^2, d_M/d_L^2) = \mathrm{lcm}(f_+^2, f_M^2/d_L)$. Then f_+, f_M and f are positive integers and we have $d_N/d_{N^+} = (d_L f)^{p-1} f_M^2$, $d_M/d_{M^+} = d_M/d_L = f_M^2$, $d_{N^+} = d_L^p f_+^{2(p-1)}$, $d_N = d_M(d_L^2 f_+^2 f)^{p-1}$ and*

$$(7) \qquad h_N^- \geq \epsilon_N \frac{2f_M}{e\pi^2(\log d_L + c)\log d_N}\left(\frac{\sqrt{d_L f/16\pi^4}}{\mathrm{Res}_{s=1}(\zeta_L)\log f_+ + 2\lambda_L}\right)^{p-1}.$$

Hence, according to Proposition 5 and Theorem 6, we have

$$(8) \qquad h_N^- \geq \epsilon_N \frac{2f_M}{e\pi^2(\log d_L + c)\log d_N}\left(\frac{\sqrt{d_L f/\pi^4}}{(\log d_L + c)\log(d_L f)}\right)^{p-1}.$$

Proof. Let $\mathcal{F}_+$, $\mathcal{F}_-$ and $\mathcal{F}$ denote the conductors of the extensions N^+/L, M/L and N/L, respectively. Then $\mathcal{F} = \mathrm{lcm}(\mathcal{F}_+, \mathcal{F}_-)$. Since $\mathcal{F}_+$ and $\mathcal{F}_-$ are clearly invariant under the action of the Galois group of L/Q we get $N_{L/Q}(\mathcal{F}) = \mathrm{lcm}(N_{L/Q}(\mathcal{F}_+), N_{L/Q}(\mathcal{F}_-)) = \mathrm{lcm}(f_+^2, d_M/d_L^2) = f$.

We then use the conductor-discriminant formula to compute all these discriminants. Finally, we use formula (3) and Proposition 5 to complete the proof. □

Corollary 7. *Let N be a dicyclic CM-field of degree $4p$, p any odd prime and assume that (p, d_L) is one of the fourteen occurrence which appear in Theorem 4, point 4. Then Table 1 provides us with upper bounds on f_+ the conductor of the extension N^+/L whenever $h_N^- = 4$.*

4. Computation of relative class numbers

Now, Theorem 1 follows from Corollary 7 which reduces the determination of all dicyclic CM-fields with relative class number 4 to the computation of the relative class numbers of finitely many dicyclic fields (excerpts of our computation appear in the tables of the following section). This section is devoted to explaining how we performed these computations. To begin with, we prove a phenomenon which was observed but not explained in [Lef]:

Theorem 8. *Let p be any odd prime. Let N be either a dihedral or a dicyclic CM-field of degree $4p$ and let M denote its imaginary quartic subfield. Let χ_N denote any one of the $p-1$ characters of order $2p$ associated with the cyclic extension N/L of degree $2p$.*

1. *(See* [Loo, Theorem 5]*.) $Q_N = Q_M$, $w_N = w_M$ and h_M^- divides h_N^-.*
2. *$h_N^-/h_M^- = (h_{N/M}^-)^2$ is a perfect square and if we let $\mathcal{F}_{N/L}$ denote the conductor of the extension N/L and set $f = N_{L/Q}(\mathcal{F}_{N/L})$, then*

$$h_{N/M}^- = \prod_{j=0}^{(p-3)/2} \frac{\sqrt{d_L f}}{4\pi^2} L\left(1, \chi_N^{2j+1}\right) = 2^{1-p} \prod_{j=0}^{(p-3)/2} L\left(0, \chi_N^{2j+1}\right). \tag{9}$$

Proof. In using (2) for both $E = M$ and $E = N$, we obtain:

$$h_N^-/h_M^- = 2^{2-2p} \prod_{\substack{j=0 \\ \gcd(j,2p)=1}}^{2p-1} L(0, \chi_N^j).$$

Now, χ_N has order $2p$ and according to Siegel–Klingen's Theorem we have $L(0,\chi_N) \in Q(\zeta_p)$ (see [Hid, Corollary 1, p. 57]). Therefore, the previous formula writes

$$h_N^-/h_M^- = 2^{2-2p} N_{Q(\zeta_p)/Q}(L(0,\chi_N)).$$

Finally, let $Q^+(\zeta_p)$ denote the maximal real subfield of $Q(\zeta_p)$. Since $L(0,\chi_N)$ is real (for the character χ_N^* of Q_{4p} induced by χ_N is real valued), we get that $h_N^-/h_M^- = (h_{N/M}^-)^2$ is the square of the rational number

$$h_{N/M}^- = 2^{1-p} N_{Q^+(\zeta_p)/Q}(L(0,\chi_N)),$$

hence is the square of the rational integer $h_{N/M}^-$. □

We explained in details in [Lou4] how one can compute relative class numbers of CM-fields N which are abelian extensions of real quadratic fields L and gave there several examples of computation of relative class numbers of dihedral CM-fields N of degree $4p$. Such computations include the computation of various Artin root numbers W_χ for characters χ associated with the abelian extension N/L. In this respect, it is worth mentioning that the Artin root numbers W_χ of the $p-1$ characters χ of order $2p$ associated with dihedral CM-fields N of degree $4p$ are all equal to $+1$ (see [FQ]). In contrast, the Artin root numbers W_χ of the $p-1$ characters χ of order $2p$ associated with dicyclic CM-fields N of degree $4p$ can be equal either to $+1$ or to -1 and may not only depend on N but also on χ (however, if p is not totally ramified in N/Q then these $W_\chi \in \{\pm 1\}$ depend on N only and will be denoted by W_N in Tables 2 and 3 below). To avoid their lengthy computation, we used [Lou6] in which explicit formulae for these roots numbers associated with dicyclic CM-fields of degree $4p$ are given.

Finally, dicyclic CM-fields N of degree $4p$ are composita $N = MN^+$ of imaginary cyclic quartic fields M and of real dihedral field N^+ of degree $2p$. Since such M's are easy to construct, it would remain to explain the constructions of such N^+'s and we refer the reader to [Lou4], [Lef] and [LPL] for their construction. Let us just mention that here the situation is rather simple for in the fourteen occurences of Theorem 4, point 4, we have $h_L = 1$. Hence, for f_+ as in (1), χ_+ may be viewed as a primitive character on the multiplicative group $(A_L/(f_+))^*$ which is trivial on the image of Z and trivial on ϵ_L (the fundamental unit of L) and there is a bijective correspondance between the cyclic groups of order p generated by such characters and the real dihedral field N^+ of degree $2p$ with quadratic subfield L and conductor $\mathcal{F}_{N/L} = (f_+)$. For example, we obtain:

Corollary 9. *There are only two real dicyclic fields of degree $4p$ with $p \in \{5,\ 13,\ 17,\ 29,\ 41,\ 73\}$ totally ramified in N^+/Q and $L = Q(\sqrt{p})$ (i.e., with N^+ as Theorem 4, point 4(a)), with f_+ less than or equal to the bounds given in part 2 of Table 1: those with $p = 5$ of conductors $f_+ = 55$ and $f_+ = 155$.*

5. Tables of relative class numbers

Table 2 lists the values of the relative class numbers of the eight dicyclic CM-fields containing M with the least possible values for f_+ for each of the seven possibilities for M as in Theorem 4, point 4(b). Table 3 gives examples of computations of relative class numbers of dicyclic CM-fields of degree $4p$ with $p > 3$ for which p is not totally ramified in N/Q.

d_L	f_+	W_N	h_N^-
8	29	-1	64
8	35	-1	144
8	45	-1	256
8	55	$+1$	400
8	59	-1	196
8	63	$+1$	900
8	77	-1	900
8	79	$+1$	400

d_L	f_+	W_N	h_N^-
5	18	-1	**4**
5	34	$+1$	16
5	38	-1	16
5	46	$+1$	16
5	47	-1	16
5	62	-1	36
5	106	$+1$	256
5	107	-1	100

d_L	f_+	W_N	h_N^-
13	10	$+1$	**4**
13	11	-1	**4**
13	18	-1	36
13	41	-1	64
13	45	-1	144
13	79	$+1$	144
13	86	-1	400
13	90	$+1$	900

Table 2.

d_L	f_+	W_N	h_N^-
29	9	$+1$	16
29	14	-1	144
29	22	$+1$	64
29	26	-1	676
29	34	$+1$	676
29	41	-1	676
29	77	-1	2304
29	91	$+1$	900

d_L	f_+	W_N	h_N^-
37	2	-1	**4**
37	35	-1	900
37	45	-1	2500
37	63	$+1$	2304
37	70	$+1$	6084
37	70	$+1$	2304
37	73	$+1$	6400
37	85	$+1$	1444

d_L	f_+	W_N	h_N^-
53	7	$+1$	64
53	10	$+1$	400
53	18	-1	484
53	23	-1	1024
53	26	-1	900
53	43	$+1$	1024
53	45	-1	2704
53	65	-1	2304

Table 2 (continued).

d_L	f_+	W_N	h_N^-
61	13	$+1$	576
61	18	-1	1024
61	22	$+1$	2500
61	23	-1	676
61	34	$+1$	12100
61	38	-1	1936
61	53	-1	10816
61	58	$+1$	1024

Table 2 (continued).

d_L	p	f_+	f_M	W_N	h_N^-/h_M^-
5	5	341	145	$+1$	786764^2
8	5	179	16	-1	4016^2
5	7	307	5	-1	20168^2
13	7	211	13	$+1$	84616^2
5	11	859	5	$+1$	2356954016^2
5	11	967	5	-1	5590999072^2
5	13	911	5	$+1$	208547643200^2
29	13	389	29	-1	73454361249088^2

Table 3.

Finally, we quote the following two examples (related to Theorem 4, point 4(a) and to Corollary 9) of dicyclic CM-fields $N = N^+M$ of degree 20 for which $p = 5$ is totally ramified in N/Q, $L = Q(\sqrt{5})$ and $f_M = 145$ (for which $h_M^- = 4$): $f_+ = 55$ for which $h_N^- = 4 \cdot (4061)^2$ and $f_+ = 155$ for which $h_N^- = 4 \cdot (32161)^2$.

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