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On quasi Einstein manifolds

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Introduction

A non-flat Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0,2)$ is not identically zero and satisfies the condition

(1) $$S(X,Y) = ag = (X,Y) + bA(X)A(Y)$$

where a, b are scalars of which $b \neq 0$ and A is a non-zero 1-form such that

(2) $$g(X,U) = A(X)$$

for all vector fields X, U being a unit vector field.

In such a case a, b are called associated scalars. A is called the associated 1-form and U is called the generator of the manifold. An n-dimensional manifold of this kind shall be denoted by the symbol $(QE)_n$.

This paper deals with $(QE)_n$ $(n > 3)$ which are not conformally flat. The significance of the associated scalars a, b is pointed out by showing that in a $(QE)_n$ the Ricci tensor S has only two distinct eigenvalues $a + b$ and a of which the former is simple and the latter is of multiplicity $n-1$, the generator U being an eigenvector corresponding to the eigenvalue $a+b$. In a $(QE)_n$ the relation $R(X,Y)\cdot S = 0$ does not in general hold. A necessary and sufficient condition is obtained in order that this relation may hold.

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A Riemannian manifold (M^n, g) is said to be conformally conservative [1] if the divergence of its conformal curvature tensor is zero. Thus every conformally flat Riemannian manifold is conformally conservative, but the converse is not, in general, true. Since this paper deals with $(QE)_n$ $(n > 3)$ which are not conformally flat, it is meaningful to look for a sufficient condition in order that a $(QE)_n$ $(n > 3)$ may be conformally conservative. Two such sufficient conditions are considered, corresponding to the two cases when the associated scalars a and b are constants and when they are not so but their sum is zero.

1. Preliminaries

We consider a $(QE)_n$ with associated scalars a, b, associated 1-form A and generator U. Since U is a unit vector field,

$$g(U, U) = 1. \tag{1.1}$$

Again

$$S(X, Y) = ag(X, Y) + bA(X)A(Y). \tag{1.2}$$

Contracting (1.2) over X and Y we get

$$\gamma = na + b \tag{1.3}$$

where γ denotes the scalar curvature of the manifold. Putting $Y = U$ in (1.2) we obtain

$$S(X, U) = (a + b)A(X). \tag{1.4}$$

Let L be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S. Then

$$g(LX, Y) = S(X, Y) \quad \forall X, Y. \tag{1.5}$$

Let R be the curvature tensor of $(QE)_n$. Then $R(X, Y)$ may be regarded as a derivation of the tensor algebra at each point of the tangent space. Hence

$$[R(X, Y) \circ L](Z) = R(X, Y)LZ - L(R(X, Y)Z). \tag{1.6}$$

The transformations $R(X,Y)$ and L are called the curvature transformation and the Ricci transformation respectively.

The commutativity of these two transformations is a consequence of the relation

$$R(X,Y)\circ L = 0. \tag{1.7}$$

This can be proved as follows: If (1.7) holds, then from (1.6) we get

$$R(X,Y)\circ LZ = LR(X,Y)Z \quad \forall Z. \tag{1.8}$$

Hence

$$R(X,Y)\circ L = L\circ R(X,Y). \tag{1.9}$$

This means that $R(X,Y)$ and L commute. These formulas will be used in the sequel.

2. Significance of the associated scalars in a $(QE)_n$ $(n > 3)$

We can express (1.4) as follows:

$$S(X,U) = (a+b)g(X,U). \tag{2.1}$$

From (2.1) we conclude that $a+b$ is an eigenvalue of the Ricci tensor S and U is an eigenvector corresponding to this eigenvalue.

Let V be any other vector orthogonal to U. Then

$$g(U,V) = 0 \quad \text{i.e.} \quad A(V) = 0. \tag{2.2}$$

From (1.2) we obtain

$$S(X,V) = ag(X,V) + bA(X)A(V).$$

Hence in virtue of (2.2) we obtain

$$S(X,V) = ag(X,V). \tag{2.3}$$

From (2.3) we see that a is an eigenvalue of the Ricci tensor and V is an eigenvector corresponding to this eigenvalue. Since the manifold under consideration is n-dimensional and V is any vector orthogonal to U, it follows from a known result in linear algebra [2] that the eigenvalue a is of multiplicity $n-1$. Hence the multiplicity of the eigenvalue $a+b$ must be 1. So there are only two distinct eigenvalues of the Ricci tensor, namely $a+b$ and a, of which the former is simple and the latter is of multiplicity $n-1$.

Hence we can state the following

Theorem 1. *In a $(QE)_n$ $(n>3)$, the Ricci tensor S has only two distinct eigenvalues $a+b$ and a of which the former is simple and the latter is of multiplicity $n-1$.*

It is to be noted that if $a+b=0$ then a cannot be zero, because $b\neq 0$.

3. $(QE)_n$ $(n>3)$ satisfying the relation $R(X,Y)\cdot S=0$

We have

$$\begin{aligned}(3.1)\quad [R(X,Y)\cdot S](Z,W) &= -S[R(X,Y)Z,W]-S[Z,R(X,Y)W]\\ &= -[ag(R(X,YZ),W)+bA(R(X,Y,Z)A(W)]\\ &\quad -[ag(R(X,YW),Z)+bA(R(XYW)A(Z)]\\ &\qquad\qquad\qquad [\text{by } (1.2)]\\ &= -b[A(R(X,Y,Z)A(W)+A(R(X,Y,W)A(Z)].\end{aligned}$$

Since $b\neq 0$, it follows from (3.1) that in a $(QE)_n$ $(n>3)$ the relation $R(X,Y)\cdot S=0$ does not, in general, hold. From (3.1) we see that if $A(R(X,Y,Z))=0$ then $[R(X,Y)\cdot S](Z,W)=0$ $\forall Z,W$.

Hence $R(X,Y)\cdot S=0$. Let us now suppose that $R(X,Y)\cdot S=0$. Then from (3.1) we get

$$(3.2)\qquad A(R(X,Y,Z))A(W)+A(R(X,Y,W))A(Z)=0.$$

Putting $W=U$ in (3.2) we have

$$A(R(X,Y,Z))A(U)+A(R(X,Y,U))A(Z)=0$$

or

$$A(R(X,Y,Z))+g(R(X,Y,U),U)A(Z)=0.$$

Hence $A(R(X,Y,Z))=0$ $[\because g(R(X,Y,U),U)=0]$.

Thus we can state the following

Theorem 2. *In a* $(QE)_n$ $(n > 3)$*, the relation* $R(X,Y)\cdot S = 0$ *holds if and only if* $A(R(X,Y,Z)) = 0$*. If* $R(X,Y)\cdot S = 0$*, then*

$$S(R(X,Y,Z),W) + S(R(X,Y,W),Z) = 0$$

or

$$g[LR(X,Y,Z),W] + g[R(X,Y,W),LZ] = 0$$

or

$$g[LR(X,Y,Z),W] - g[R(X,Y,LZ),W = 0$$

or

$$g[LR(X,Y,Z) - R(X,Y,LZ),W] = 0. \tag{3.3}$$

From (3.3) *we get* $LR(X,Y,Z) - R(X,Y,LZ) = 0$*, i.e.* $[R(X,Y)\circ L](Z) = 0$*,* $\forall Z$*. Hence*

$$R(X,Y)\circ L = 0. \tag{3.4}$$

Again, if (3.4) *holds, then* $R(X,Y)\cdot S = 0$*. In virtue of* (3.4) *it follows that the curvature and the Ricci transformations commute [by (1.7)].*

The converse is also true. That is, if the curvature and the Ricci transformations commute, then (3.4) holds and therefore $R(X,Y)\cdot S = 0$. This leads to the following result:

Theorem 3. *In a* $(QE)_n$ $(n > 3)$*, the curvature and the Ricci transformations commute if and only if the relation* $A(R(X,Y,Z)) = 0$ *holds.*

4. $(QE)_n$ $(n > 3)$ with divergence-free conformal curvature tensor i.e. $\operatorname{div}\mathcal{C} = 0$

The conformal curvature tensor $\mathcal{C}$ of a Riemannian manifold (M^n, g) is said to be conservative [1] if the divergence of $\mathcal{C}$ is zero. In such a case the manifold is said to be conformally conservative.

In this section we shall obtain two sufficient conditions for a $(QE)_n$ $(n > 3)$ to be conformally conservative. Let

$$\begin{aligned} H(X,Y,Z) &= (\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) \\ &\quad - \frac{1}{2(n-1)}[d\gamma(X)g(Y,Z) - d\gamma(Z)g(Y,X)]. \end{aligned} \tag{4.1}$$

Then it is known [3] that $\operatorname{div}\mathcal{C} = 0$ if and only if $H(X, Y, Z) = 0$. We shall consider two types of $(QE)_n$ $(n > 3)$:

Type I: the associated scalars a and b are constants and therefore γ is constant.

Type II: a and b are not constants but $a + b = 0$.

Type I: For this type $da(X) = 0$ and $db(X) = 0$ and therefore $d\gamma(X) = 0$. Now from (1.2) we get

$$\begin{aligned}(\nabla_X S)(Y, Z) &= da(X)g(Y, Z) + db(X)A(Y)A(Z)\\ &\quad + b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(\gamma)]\\ &= b[(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(\gamma)].\end{aligned}$$

Hence

$$\begin{aligned}(\nabla_X S)(X, Z) - (\nabla_Z S)(Y, X) &= b[(\nabla_X A)(Y)A(Z)\\ +(\nabla_X A)(Z)A(Y) - (\nabla_Z A)(Y)A(X) &- (\nabla_Z A)(X)A(Y)].\end{aligned} \tag{4.2}$$

Therefore (4.1) takes the following form:

$$\begin{aligned}H(X, Y, Z) = b[&(\nabla_X A)(Y)A(Z) + (\nabla_X A)(Z)A(Y)\\ &-(\nabla_Z)(Y)A(X) - (\nabla_Z A)(X)A(Y)]\\ &[\because d\gamma(X) = 0].\end{aligned} \tag{4.3}$$

From (4.3) we see that in this case $H(X, Y, Z)$ is not, in general, equal to zero.

We now impose the condition that the generator U of the manifold is a recurrent vector field [4] with the associated 1-form A not being the 1-form of recurrence. Then $\nabla_X U = B(X)U$, where B is the 1-form of recurrence. Hence

$$g(\nabla_X U, Y) = g(B(X)U, Y)$$

or

$$(\nabla_X A)(Y) = B(X)A(Y). \tag{4.4}$$

In virtue of (4.4) we can express (4.3) as follows:

$$\begin{aligned}H(X, Y, Z) = b[&B(X)A(Y)A(Z) + B(X)A(Z)A(Y)\\ &- B(Z)A(Y)A(X) - B(Z)A(X)A(Y)].\end{aligned} \tag{4.5}$$

Since $(\nabla_X A)(U)=0$, it follows from (4.4) that $B(X)=0$. So $H(X,Y,Z)=0$.

Hence we can state the following

Theorem 4. *If in a $(QE)_n$ $(n>3)$ the associated scalars are constants and the generator U of the manifold is a recurrent vector field with the associated 1-form A not being the 1-form of recurrence, then the manifold is conformally conservative.*

Next we consider Type II. For this type, (4.6) $\gamma=(n-1)a$ [by (1.3)]. Hence γ is neither zero nor a non-zero constant. From (4.6) we get

$$d\gamma(X)=(n-1)da(X). \tag{4.7}$$

Now

$$\begin{aligned}(\nabla_X S)(Y,Z) &= da(X)g(Y,Z)-da(X)A(Y)A(Z)\\ &\quad - a[(\nabla_X A)(Y)A(Z)+(\nabla_X A)(Z)A(Y)] \quad [\because a+b=0].\end{aligned}$$

Hence

$$\begin{aligned}(\nabla_X S)(Y,Z)-(\nabla_Z S)(Y,X) &= da(X)g(Y,Z)-da(X)A(Y)A(Z)\\ &\quad - da(Z)g(Y,X)+da(Z)A(Y)A(X)\\ &\quad - a[(\nabla_X A)(Y)A(Z)+(\nabla_X A)(Z)A(Y)]\\ &\quad + a[(\nabla_Z A)(Y)A(X)+(\nabla_Z A)(X)A(Y)].\end{aligned} \tag{4.8}$$

Again

$$\begin{aligned}&\frac{1}{2(n-1)}[d\gamma(X)g(Y,Z)-d\gamma(Z)g(Y,X)]\\ &= \frac{1}{2}[da(X)g(Y,Z)-da(Z)g(Y,X)] \qquad [\text{by } (4.7)].\end{aligned} \tag{4.9}$$

Hence

$$\begin{aligned}H(X,Y,Z) &= \frac{1}{2}da(X)[g(Y,Z)-2A(Y)A(Z)]\\ &\quad -\frac{1}{2}da(Z)[g(Y,X)-ZA(Y)A(X)]\\ &\quad + a[(\nabla_Z A)(Y)A(X)-(\nabla_X A)(Y)A(Z)\\ &\quad + A(Y)\{(\nabla_Z A)(X)-(\nabla_X A)(Z)\}].\end{aligned} \tag{4.10}$$

From (4.10) we see that $H(X,Y,Z)$ is not, in general, equal to zero. Hence we impose the conditions

$$\text{(i)} \qquad U = \frac{1}{2a}\,\mathrm{grad}\,a$$

and

$$\text{(ii)} \qquad \nabla_X U = -X + A(X)U.$$

From (i) we get $g(X,U) = g\left(\frac{1}{2a}\,\mathrm{grad}\,a,\ X\right)$ or

$$A(X) = \frac{1}{2a} da(X)$$

or

$$\frac{1}{2} da(X) = a(A(X). \tag{4.11}$$

Again from (ii) we get

$$(\nabla_X A)(Y) = -g(X,Y) + A(X)A(Y). \tag{4.12}$$

In virtue of (4.11) and (4.12) we can express (4.10) as follows:

$$\begin{aligned} H(X,Y,Z) = {} & aA(X)g(Y,Z) - 2aA(X)A(Y)A(Z) \\ & - aA(Z)g(Y,X) - 2aA(X)A(Y)A(Z) \\ & + a[A(X)\{-g(Y,Z) + A(Y)A(Z)\} \\ & - A(Z)\{-g(X,X) + A(X)A(Y)\} \\ & + A(Y)\{-g(X,Z) + A(X)A(Z) \\ & + g(X,Z) - A(X)A(Z)\}]. \end{aligned} \tag{4.13}$$

Hence $H(X,Y,Z) = 0$.

Therefore we can state the following

Theorem 5. *If in a* $(QE)_n$ $(n > 3)$ *the associated scalars are not constants but their sum is zero and the generator satisfies the conditions* (i) *and* (ii), *then the manifold is conformally conservative.*

We shall next point out the geometric significance of the condition (ii), namely

$$\nabla_X U = -X + A(X)U. \tag{4.14}$$

Let $U^\perp$ denote the $(n-1)$-dimensional distribution in $(QE)_n$ orthogonal to U. If X and Y belong to $U^\perp$ where $Y \neq \lambda X$, then

$$g(X,U) = 0 \tag{4.15}$$

and

$$g(Y,U) = 0. \tag{4.16}$$

Since $(\nabla_X g)(Y,U) = 0$, if follows from (4.16) that by (4.14)

$$g(\nabla_X Y, U) = g(\nabla_X U, Y) = g(X,Y) - A(X)A(Y). \tag{4.17}$$

Similarly from (4.15) we get

$$g(\nabla_Y X, U) = g(X,Y) - A(X)A(Y). \tag{4.18}$$

Hence

$$g(\nabla_X Y, U) = g(\nabla_Y X, U). \tag{4.19}$$

Now, $[X,Y] = \nabla_X Y - \nabla_Y X$. Therefore

$$g([X,Y],U) = g(\nabla_X Y - \nabla_Y X, U) = g(\nabla_X Y, U) - g(\nabla_Y X, U) = 0$$

by (4.19). Hence $[X,Y]$ is orthogonal to U, i.e. $[X,Y] \in U^\perp$.

Thus the distribution $U^\perp$ is involutive [5]. Hence from Frobenius' theorem [5] it follows that $U^\perp$ is integrable. This implies that the $(QE)_n$ is a product manifold.

We can therefore state the following

Theorem 6. *If in a* $(QE)_n$ $(n > 3)$ *the associated scalars are not constants but their sum is zero and the generator of the manifold satisfies the conditions* (i) *and* (ii) *then this* $(QE)_n$ *is a product manifold.*

Before concluding we would like to mention that the notion of a quasi-Einstein metric was introduced in 1996 by Chave and Valent [6]. They defined such a metric by the following constraint:

$$S(X,Y) = ag(X,Y) + \frac{1}{2}[(\nabla_X A)(Y) + (\nabla_Y A)(X)], \tag{4.20}$$

where S is the Ricci tensor of type $(0,2)$, a is a scalar and A is a non-zero 1-form. Thus the notion of a quasi-Einstein manifold defined by us is different from that of a space of quasi-Einstein metric defined by Chave and Valent and our definition is *new*. This new notion was suggested by an unknown referee.

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