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Remark on the characterization of continuous functions

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Abstract. W. Ring, P. Schöpf and J. Schwaiger showed in [RSS] that if E is a finite dimensional normed space then a function $f : E \to \mathbb{R}$ is continuous iff $f \circ \gamma$ is continuous for every regular curve $\gamma : [0,1] \to E$.

In the case E is infinite-dimensional we construct a function $f : E \to \mathbb{R}$ not continuous at zero such that an analogue of the above result fails to hold.

By $C^1(I, E)$ we denote the space of all regular curves from an interval I to a Banach space E. To explain our results we first quote Theorem 4 from [RSS]:

Theorem RSS. *Suppose $D \subset \mathbb{R}^2$ is open and $f : D \to \mathbb{R}$. Moreover, assume that $f \circ \gamma : I \to \mathbb{R}$ is continuous for every regular curve $\gamma : I \to D$, $\gamma \in C^1\ (I, \mathbb{R}^2)$ and $I \subset \mathbb{R}$ compact. Then f is continuous.*

W. Ring, P. Schöpf and J. Schwaiger mention in Remark 4.3 in [RSS] that the restriction to the two-dimensional case $D \subset \mathbb{R}^2$ is not essential, and that with some minor adjustments their proof works for $D \subset \mathbb{R}^n$. They also state that it is not clear what happens in the infinite dimensional case, due to the noncompactness of unit spheres.

What may seem surprising, we show that Theorem RSS is really specific to finite dimensional spaces.

Theorem 1. *Let E be an arbitrary infinite-dimensional Banach space. Then there exists a function $f : E \to \mathbb{R}$ which is not continuous at zero such that $f \circ \gamma : I \to \mathbb{R}$ is continuous for every regular curve $\gamma : I \to E$, $\gamma \in C^1(I, E)$, where I is an arbitrary interval.*

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Moreover, if $\gamma(t_0) = 0$ *for some* $t_0 \in I$*, then* $f \circ \gamma$ *is zero on some neighbourhood of* t_0*.*

The proof of the above theorem is divided into a few technical lemmas. We will use the following well-known result from functional analysis:

Lemma (Lemma 1.12 from Chapter III [To]). *Let* E *be an arbitrary infinite dimensional normed space. Then there exists a sequence* $\{e_n\}_{n\in\mathbb{N}} \subset E$ *such that*

$$\|e_n\| = 1 \quad \text{for } n \in \mathbb{N} \tag{1}$$

and

$$\|e_n - e_k\| \geq 1 \quad \text{for } n, k \in \mathbb{N},\ n \neq k. \tag{2}$$

By $B(x, r)$ we denote the closed ball with center at x and radius r.

Lemma 1. *Let the sequence* $\{e_n\}$ *be as in the Lemma. Then*

$$\frac{1}{n}B\left(e_n, \frac{1}{5}\right) \cap \frac{1}{k}B\left(e_k, \frac{1}{5}\right) = \emptyset \quad \text{for } n, k \in \mathbb{N},\ n \neq k. \tag{3}$$

Proof. Let $k, n \in \mathbb{N}$ and x be such that $x \in \frac{1}{n}B(e_n, \frac{1}{5}) \cap \frac{1}{k}B(e_k, \frac{1}{5})$. We are going to show that $n = k$. We have

$$\|nx - e_n\| \leq \frac{1}{5}, \quad \|kx - e_k\| \leq \frac{1}{5}, \tag{4}$$

which by (1) gives us

$$\|kx\| \in \left[\frac{4}{5}, \frac{6}{5}\right], \quad \|nx\| \in \left[\frac{4}{5}, \frac{6}{5}\right].$$

This and (4) yields

$$\begin{aligned} \|e_n - e_k\| &\leq \|kx - e_k\| + \|nx - e_n\| + \|kx - nx\| \\ &\leq \frac{1}{5} + \frac{1}{5} + |k - n| \cdot \|x\| = \frac{2}{5} + \big|\, \|kx\| - \|nx\| \,\big| \\ &\leq \frac{2}{5} + \frac{2}{5} = \frac{4}{5}, \end{aligned}$$

which by (2) implies that $n = k$. □

By $\operatorname{supp}(f)$ we denote the support of the function f, that is the closure of the set of all points $x \in E$ such that $f(x) \neq 0$.

Lemma 2. *For $n \in \mathbb{N}$ let $f_n : E \to \mathbb{R}$ be an arbitrary continuous function such that* $\operatorname{supp}(f_n) \subset \frac{1}{n}B(e_n, \frac{1}{5})$ *and* $f(\frac{1}{n}e_n) = n$.

We put

$$f(x) = \sum_{i=1}^{\infty} f_n(x) \quad \text{for } x \in E.$$

Then f is a well-defined real valued function on E which is continuous on $E \setminus \{0\}$ and discontinuous at zero.

Proof. Let $x \in E \setminus \{0\}$ be arbitrary and let U_x be an arbitrary neighbourhood of x which does not contain zero. By the definition of the sequence f_n one can easily notice that only a finite number of elements of f_n is nonzero on U_x. By the definition of f this implies that f is continuous on U_x. As x was chosen arbitrarily this implies that f is continuous on $E \setminus \{0\}$.

We show that f is discontinuous at zero. Obviously, by the definition $f(\frac{1}{k}e_k) = k$. However, the sequence $\{\frac{1}{k}e_k\}$ converges to zero as $k \to \infty$, which implies that f is not bounded on any neighbourhood of zero, and therefore not continuous. $\square$

Before proceeding to the next lemma we would like to remark that functions f_n satisfying the assumptions of Lemma 2 exist in an arbitrary normed space, for example we may put

$$f_n(x) := n \cdot \max\{0, 1 - 5\|nx - e_n\|\} \quad \text{for } x \in E.$$

If E is a unitary space then these functions can be chosen to be of the class C^∞.

From now on, by f_n and f we denote functions chosen as in the previous lemma.

Lemma 3. *Let $a > 0$ and let $\gamma : [0, a) \to E$ be differentiable at zero and such that $\gamma(0) = 0$, $\gamma'(0) \neq 0$.*

Then there exists $\varepsilon > 0$ such that if for certain $k \in \mathbb{N}$, $t \in (0, \varepsilon)$

$$\gamma(t) \in \operatorname{supp}(f_k),$$

then

$$\left\| e_k - \frac{1}{\|\gamma'(0)\|}\gamma'(0) \right\| < \frac{1}{2}.$$

Proof. Applying, if necessary, rescaling by $\frac{1}{\|\gamma'(0)\|}$ we may assume that $\|\gamma'(0)\| = 1$.

Let $\alpha \in (0, a)$ be chosen so that

$$\max\left\{1 - \frac{4}{5} \cdot \frac{1}{1+\alpha},\ \frac{6}{5} \cdot \frac{1}{1-\alpha} - 1\right\} + \frac{6}{5} \cdot \frac{\alpha}{1-\alpha} + \frac{1}{5} < \frac{1}{2} \tag{5}$$

(such an α exists as the left side of the above inequality is a continuous function on the interval $(0, a)$ which for $\alpha = 0$ takes the value $\frac{2}{5}$).

As γ is differentiable at zero, we can find $\varepsilon > 0$ such that

$$\|\gamma(t) - t\gamma'(0)\| \le \alpha t \quad \text{for } t \in (0, \varepsilon). \tag{6}$$

Now suppose that there exists $t_0 \in (0, \varepsilon)$ and $k \in \mathbb{N}$ be such that

$$\gamma(t_0) \in \operatorname{supp}(f_k).$$

Then

$$\|k\gamma(t_0) - e_k\| \le \frac{1}{5}. \tag{7}$$

By (6) we get

$$\|k\gamma(t_0) - kt_0\gamma'(0)\| \le kt_0\alpha, \tag{8}$$

which by the triangle inequality yields

$$\|k\gamma(t_0)\| - kt_0\|\gamma'(0)\| \le kt_0\alpha, \quad kt_0\|\gamma'(0)\| - \|k\gamma(t_0)\| \le kt_0\alpha.$$

As $\|\gamma'(0)\| = 1$ this means that

$$\|k\gamma(t_0)\| \le kt_0(1+\alpha), \quad \|k\gamma(t_0)\| \ge kt_0(1-\alpha),$$

and consequently

$$kt_0 \in \left[\frac{1}{1+\alpha},\ \frac{1}{1-\alpha}\right] \cdot \|k\gamma(t_0)\|.$$

By (7) we obtain that $\|k\gamma(t_0)\| \in [\frac{4}{5}, \frac{6}{5}]$, so as $t_0 > 0$

$$kt_0 \in \left[\frac{4}{5} \cdot \frac{1}{1+\alpha},\ \frac{6}{5} \cdot \frac{1}{1-\alpha}\right]. \tag{9}$$

Using once more (7) and applying (8) and (9) we get

$$\begin{aligned}\|\gamma'(0)-e_k\| &\le \|\gamma'(0)-k\gamma(t_0)\|+\|k\gamma(t_0)-e_k\|\\ &\le \|\gamma'(0)-kt_0\gamma'(0)\|+\|kt_0\gamma'(0)-k\gamma(t_0)\|+\frac{1}{5}\\ &\le |1-kt_0|\cdot\|\gamma'(0)\|+kt_0\alpha+\frac{1}{5}\\ &\le \max\left\{1-\frac{4}{5}\cdot\frac{1}{1+\alpha},\ \frac{6}{5}\cdot\frac{1}{1-\alpha}-1\right\}+\frac{6}{5}\cdot\frac{\alpha}{1-\alpha}+\frac{1}{5}.\end{aligned}$$

By (5) this yields

$$\|e_k-\gamma'(0)\|<\frac{1}{2}.$$

□

Lemma 4. *Let $\gamma:[0,1)\to E$ be a function which is differentiable at zero and such that $\gamma(0)=0$, $\gamma'(0)\neq 0$.*

Then there exists $\delta>0$ such that

$$f(\gamma(t))=0 \quad \textit{for } t\in[0,\delta).$$

Proof. Let $\varepsilon>0$ be chosen as in the previous lemma.

Trivially $f(\gamma(0))=f(0)=0$. If $\gamma(t)\notin \operatorname{supp}(f_k)$ for every $k\in\mathbb{N}$ and $t\in(0,\varepsilon)$ then $f_k(\gamma(t))=0$ for every $k\in\mathbb{N}$, so by the definition of f

$$f(\gamma(t))=\sum_{i=1}^{\infty} f_i(\gamma(t))=0,$$

which yields the assertion of the Lemma in this case.

So we may assume that there exists $k\in\mathbb{N}$ such that for a certain $t_0\in(0,\varepsilon)$

$$\gamma(t_0)\in\operatorname{supp}(f_k).$$

We show that such a $k\in\mathbb{N}$ is unique. Let us assume that there exists $l\in\mathbb{N}$ and $t_1\in(0,\varepsilon)$ such that $\gamma(t_1)\in\operatorname{supp}(f_l)$. Applying Lemma 3 to both k, t_0 and l, t_1 we obtain

$$\left\|e_k-\frac{1}{\|\gamma'(0)\|}\gamma'(0)\right\|<\frac{1}{2},\quad \left\|e_l-\frac{1}{\|\gamma'(0)\|}\gamma'(0)\right\|<\frac{1}{2}.$$

Joining the two above inequalities we get

$$\|e_k - e_l\| < 1,$$

which implies by the Lemma that $l = k$.

Thus we have obtained that there is a unique $k \in \mathbb{N}$ such that there exists $t_0 \in (0, \varepsilon)$ with

$$\gamma(t_0) \in \operatorname{supp}(f_k).$$

In other words this means that for every $l \neq k$

$$\gamma(t) \notin \operatorname{supp}(f_l) \quad \text{for } t \in (0, \varepsilon),$$

or equivalently

$$f_l(\gamma(t)) = 0 \quad \text{for } l \in \mathbb{N},\ l \neq k,\ t \in (0, \varepsilon). \tag{10}$$

As $\operatorname{supp}(f_k)$ is isolated from zero, by the continuity of γ there exists $\delta > 0$, $\delta < \varepsilon$ such that

$$\gamma(t) \notin \operatorname{supp}(f_k) \quad \text{for } t \in (0, \delta).$$

This means that $f_k(\gamma(t)) = 0$ for $t \in (0, \delta)$. Joining this with (10) we have

$$f_n(\gamma(t)) = 0 \quad \text{for } n \in \mathbb{N},\ t \in (0, \delta).$$

By the definition of f this yields the assertion of the Lemma. □

After these preparatory lemmas we are able to prove Theorem 1.

Proof of Theorem 1. Let I be an arbitrary interval and let $\gamma : I \to E$ be a regular C^1 function. We will show that $f \circ \gamma$ is continuous.

Let $t_0 \in I$ be arbitrarily fixed. If $\gamma(t_0) \neq 0$ then $f \circ \gamma$ is continuous at t_0 as f is continuous on $E \setminus \{0\}$.

So suppose that $\gamma(t_0) = 0$. We will show that then $f \circ \gamma$ is zero on some neighbourhood of t_0 (which in particular implies that it is continuous at t_0). Applying Lemma 4 to the function $\gamma_1(t) := \gamma(t + t_0)$ we obtain that there exists $\delta_1 > 0$ such that

$$f(\gamma_1(t)) = 0 \quad \text{for } t \in [0, \delta_1) \cap (I - t_0),$$

which means that

$$f(\gamma(t)) = 0 \quad \text{for } t \in [t_0, t_0 + \delta_1) \cap I. \tag{11}$$

Making use of Lemma 4 for the curve $\gamma_2(t) := \gamma(-t + t_0)$, we get $\delta_2 > 0$ such that

$$f(\gamma_2(t)) = 0 \quad \text{for } t \in [0, \delta_2) \cap (t_0 - I),$$

or in other words

$$f(\gamma(t)) = 0 \quad \text{for } t \in (t_0 - \delta_2, t_0] \cap I. \tag{12}$$

By joining (11) with (12) we finally obtain

$$f(\gamma(t)) = 0 \quad \text{for } t \in (t_0 - \delta_2, t_0 + \delta_1) \cap I. \qquad \square$$

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