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Linked dichotomies and asymptotic theory of nondiagonal differential systems

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Abstract. This paper introduces the notion of linked dichotomies for a linear system of ordinary differential equations. Roughly speaking, this is a family of (h_i, k_i)-dichotomies with nested subspaces of k_i-bounded solutions. This concept is stable under L^1-perturbations. We show that if $B(t) \in L^1$ and the system $x' = A(t)x$, with fundamental matrix $\Phi(t)$, has an exhaustive chain of linked dichotomies, then there exists a fundamental matrix $\widetilde{\Phi}(t)$ of $x' = [A(t) + B(t)]x$, such that $\widetilde{\Phi}(t) = \sum_{i=1}^{r} \left((\Phi(t) + o(h_i(t)))R_i + (\Phi(t) + o(k_i(t)))S_i\right)$, and R_i, S_i are orthogonal projections satisfying $\sum_{i=1}^{r}(R_i + S_i) = I$.

1. Introduction

This paper concerns the problem of asymptotic integration of the linear system

$$y' = [A(t) + B(t)]\, y, \tag{1}$$

where $A(t)$ is continuous and in general a nondiagonal matrix function; the function $B(t)$ is absolutely integrable.

The theory of asymptotic integration of differential systems is an important field of research of applicable analysis, whose fundamentals were given by Levinson in his research papers [11], [12]. This theory has received the contributions of many mathematicians [1–3], [6–9], [11], [12], [18], who have investigated different conditions on the linear system

$$x' = A(t)x \tag{2}$$

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and on the function $B(t)$, in order to characterize the solutions of System (1) assuming knowledge of the solutions of System (2). Concerning this study, we point out the role played by COPPEL's book [5], where the importance of the notion of asymptotic equivalence in the study of System (1) is emphasized. In these bibliographic notes we point out also the monography [7], which expounds the asymptotic integration theory from the point of view of the Levinson asymptotic theorem.

In [11], in order to study the asymptotic integration of System (1), Levinson introduced the notion of asymptotic equivalence between

$$x' = Ax, \quad A = \text{constant}, \tag{3}$$

and its perturbed system

$$y' = [A + B(t)]y, \quad \int_0^\infty |B(t)|dt < \infty. \tag{4}$$

He established that if the real parts of the eigenvalues of the matrix A are nonpositive and the eigenvalues with vanishing real parts are simple in the Jordan sense, then the solutions of Systems (3) and (4) are in bijective correspondence satisfying

$$\lim_{t\to\infty} (x(t) - y(t)) = 0.$$

Later on, the construction of the theory of dichotomies given by MASSERA and SCHÄFFER [10] for the study of the nonautonomous System (2) and the important contributions of COPPEL [5] and PALMER [19] to this subject, gave an important impulse to the investigation of asymptotic equivalence [2], [6], [8], [18], where essentially a correspondence between bounded solutions of Systems (2) and (1) and those of System (2) and the nonlinear system

$$y' = A(t)y + f(t, y) \tag{5}$$

were established [2], [6]. In this historical account we emphasize also the recent paper [14] where by means of the notion of (h, k)-dichotomy [14], [22], the asymptotic equivalence between the bounded (and respectively the unbounded) solutions of Systems (2) and (5) was given.

In [12] Levinson obtained another important result concerning System (1), with nonperturbed System (2) having the diagonal form

$$x' = \Lambda(t)x, \tag{6}$$
$$\Lambda(t) = \operatorname{diag}(\lambda_1(t), \lambda_2(t), \dots, \lambda_n(t)),$$

where the complex valued functions $\lambda_j(t)$ are continuous. If the coefficients of System (6) satisfy the Levinson dichotomic conditions and $B(t) \in L^1$ (for definitions see Section 6 of this paper), then the system

$$y' = [\Lambda(t) + B(t)]y \tag{7}$$

has a fundamental matrix $\Psi(t)$ satisfying the asymptotic formula

$$\Psi(t) = (I + o(1)) \exp\left\{ \int_{t_0}^{t} \Lambda(s)ds \right\}, \tag{8}$$

where we have used the Landau asymptotic symbol $o(h)$ to represent a function with the property $\lim_{t\to\infty} h(t)^{-1} o(h)(t) = 0$. A similar result, using other dichotomic conditions, was obtained by Hartman and Wintner [7] for a System (1) and a function $B(t) \in L^p$, $1 < p \leq 2$.

In summary, the theory of asymptotic integration, started by Levinson was developed in two directions: the theory of asymptotic equivalence and the asymptotic theory of diagonal systems.

In this paper we point out the remarkable fact used by Levinson in his asymptotic theory, that the dichotomic character of System (6) is described by a family of h-dichotomies. This idea has been emphasized in the recent papers [13–17], [20–22]. Apparently, this situation has not been exploited for a general System (2). The aim of this paper is the construction of an asymptotic theory for (1), assuming that the nondiagonal System (2) has a family of (h, k)-dichotomies [14], [15]. The basic problem to solve here is the following: How to use a family of (h, k)-dichotomies of System (2) in order to obtain maximal information on System (1)? To solve this problem we introduce the notion of an ordered chain of dichotomies, linked by the subspaces of solutions of (2) with different asymptotic growths. By a step by step procedure of asymptotic equivalence, involving solutions of (1) and (2) of the same growth, we construct an asymptotic theory for the general System (1). We prove an asymptotic formula for $\widetilde{\Phi}$, a fundamental matrix of the nonautonomous System (1), for a general System (2). This formula has the form

$$\Psi(t) = \sum_{i=1}^{r} \left((\Phi(t) + o(h_i(t)))R_i + (\Phi(t) + o(k_i(t)))S_i \right),$$

where $\{R_i, S_i\}_{i=1}^{r}$ is a family of orthogonal projections satisfying $\sum_{i=1}^{r}(R_i + S_i) = I$ and Φ denotes a fundamental matrix of System (2). This formula makes explicit the correspondence between h_i-bounded solutions (and also k_i-bounded ones) of Systems (1) and (2).

We will show that the dichotomy of Levinson for the diagonal System (6) can certainly be treated as a particular case of linked dichotomies. We obtain an asymptotic formula for System (7) that contains formula (8). These results unify the asymptotic theory of diagonal systems and the theory of asymptotic integration developed under the notion of asymptotic equivalence.

2. Preliminaries

Let $J = [t_0, \infty)$. V will denote the space R^n or C^n. For a function $h(t)$ we will denote $h(t)^{-1} = 1/h(t)$. We will say that a function $x : J \to V$ is h-bounded iff the function $h(t)^{-1}x(t)$ is bounded. Let (h, k) denote a pair of continuous and positive functions. We will say that this pair is compensated [14–16], iff there exists a positive constant C such that

$$h(t)h(s)^{-1} \le Ck(t)k(s)^{-1}, \quad t \ge s.$$

Definition 1. We will say that the System (2) has an (h, k)-dichotomy iff (h, k) is compensated and there exists a projection matrix P and a constant K such that

$$\begin{aligned} |\Phi(t)P\Phi^{-1}(s)| &\le Kh(t)h(s)^{-1}, \quad t \ge s, \\ |\Phi(t)(I - P)\Phi^{-1}(s)| &\le Kk(t)k(s)^{-1}, \quad s \ge t. \end{aligned} \tag{9}$$

The (h, h)-dichotomy will shortly be called an h-dichotomy. Since we will use different (h, k)-dichotomies, sometimes we will be identify this concept with the triad (h, k, P) and consequently an h-dichotomy will be identified with (h, P). The notion of (h, k)-dichotomy was introduced in [20].

By $x(t,t_0,\xi)$ we denote the solution of System (2) with initial condition ξ at initial time t_0. In the following we use the notations

$$V_h = \Big\{\xi \in V : \sup_{t\geq t_0} |h(t)^{-1}x(t,t_0,\xi)| < \infty\Big\},$$

$$V_{h,0} = \Big\{\xi \in V_h : \lim_{t\to\infty} h(t)^{-1}x(t,t_0,\xi) = 0\Big\}.$$

Q_h and $Q_{h,0}$ will denote projections such that $Q_h[V] = V_h$, $Q_{h,0}[V] = V_{h,0}$. Similar subspaces and projections are defined for the System (1) and they will be distinguished by a tilde: $\widetilde{V}_h$, $\widetilde{V}_{h,0}$, $\widetilde{Q}_h$, etc.

The following results are basic properties of these dichotomies.

Theorem A [13]. *If the System* (2) *has an* (h,k)*-dichotomy with projection matrix* P*, then it has an* (h,k)*-dichotomy with projection matrix* Q *iff*

$$V_{h,0} \subset V_{k,0} \subset Q[V] \subset V_h \subset V_k.$$

The projection P *of the dichotomy* (9) *can be chosen with the property*

$$\lim_{t\to\infty} h(t)^{-1}\Phi(t)P = 0,$$

iff $V_{h,0} = V_{k,0}$.

Theorem A implies the following assertion: If System (2) has an h-dichotomy with projection P, then it has an h-dichotomy with projection Q, iff

$$V_{h,0} \subset Q[V] \subset V_h.$$

In this case, the definition of an h-dichotomy can be accomplished with the projection $Q_{h,0}$ satisfying

$$\lim_{t\to\infty} h(t)^{-1}\Phi(t)Q_{h,0} = 0. \tag{10}$$

The (h,k)-dichotomies have the following roughness property:

Theorem B [15]. *Let us suppose System* (2) *has an* (h,k)*-dichotomy with projection* P*. If* $B(t) \in L^1$*, then System* (1) *has an* (h,k)*-dichotomy*

$$\begin{aligned} |\widetilde{\Phi}(t)\widetilde{P}\widetilde{\Phi}^{-1}(s)| &\leq \widetilde{K}h(t)h(s)^{-1}, \quad t \geq s,\\ |\widetilde{\Phi}(t)(I-\widetilde{P})\widetilde{\Phi}^{-1}(s)| &\leq \widetilde{K}k(t)k(s)^{-1}, \quad s \geq t, \end{aligned} \tag{11}$$

where $\widetilde{P}$ *is a projection similar to* P.

Finally, for the sake of simplicity, we will assume that $\Phi(t_0) = I$ and the functions (h, k) in definitions (9) and (11) satisfy $(h(t_0), k(t_0)) = (1, 1)$. This can be obtained if instead of (h, k) we use $(\frac{h(t)}{h(t_0)}, \frac{k(t)}{k(t_0)})$. Thus, from (9) and (11) we have the estimates

$$|\Phi(t)P| \le Kh(t), \quad |\widetilde{\Phi}(t)\widetilde{P}| \le \widetilde{K}h(t), \qquad t \ge t_0. \tag{12}$$

3. Asymptotic equivalence

If System (2) has an (h, k) dichotomy, then according to Theorem A the projections Q_h and Q_k have the property

$$Q_k Q_h = Q_h. \tag{13}$$

Definition 2. Whe shall say that the h-bounded solutions of System (1) are in bijective correspondence with the h-bounded solutions of System (2) iff dimension V_h = dimension $\widetilde{V}_h$.

Theorem 1. *Let us assume that System* (2) *has an h-dichotomy. If*

$$K\widetilde{K}\int_{t_0}^{\infty} |B(s)|ds < 1, \tag{14}$$

then the h-bounded solutions of System (2) *and the h-bounded solutions of System* (1) *are in bijective correspondence. This correspondence satisfies*

$$y(t) = x(t) + o(h(t)). \tag{15}$$

Moreover, the fundamental matrix Ψ of System (1), $\Psi(t_0) = I$, *has the asymptotic representation*

$$\Psi(t)\widetilde{Q}_h = (\Phi(t) + o(h(t)))Q_h. \tag{16}$$

Proof. From Theorem A, the h-dichotomy of System (2) can be accomplished with the projection $Q_{h,0}$. Moreover we can assume that this h-dichotomy is accomplished with the fundamental matrix Φ satisfying $\Phi(t_0) = I$. Given $x(t)$, an h-bounded solution of System (2), we consider the integral equation

$$(17) \qquad y(t) = x(t) + \int_{t_0}^{t} \Phi(t)Q_{h,0}\Phi^{-1}(s)B(s)y(s)ds - \int_{t}^{\infty} \Phi(t)(I - Q_{h,0})\Phi^{-1}(s)B(s)y(s)ds.$$

Then, following [14] it is possible to prove that this system has a unique h-bounded solution satisfying System (1) and the property (15). If we put $t = t_0$ in (17), we obtain

$$y(t_0) = x(t_0) - \int_{t_0}^{\infty} (I - Q_{h,0})\Phi^{-1}(s)B(s)\Psi(s)y(t_0)ds.$$

The estimate (12) implies $|\Psi(t)\widetilde{Q}_h| \leq \widetilde{K}h(t)$. Henceforth

$$\left| \int_{t_0}^{\infty} (I - Q_{h,0})\Phi^{-1}(s)B(s)\Psi(s)\widetilde{Q}_h ds \right| \leq K\widetilde{K} \int_{t_0}^{\infty} |B(s)|ds < 1.$$

From this estimate we obtain $y(t_0) = \Theta_h x(t_0)$, where

$$(18) \qquad \Theta_h = \left(I + \int_{t_0}^{\infty} (I - Q_{h,0})\Phi^{-1}(s)B(s)\Psi(s)\widetilde{Q}_h ds \right)^{-1}.$$

Thus $\Theta_h : V_h \to \widetilde{V}_h$ is bijective, implying bijective correspondence between the solutions of Systems (1) and (2). The integral equation (17) implies

$$\Psi(t)\widetilde{Q}_h y(t_0) = \Phi(t)Q_h x(t_0) + \int_{t_0}^{t} \Phi(t)Q_{h,0}\Phi^{-1}(s)B(s)\Psi(t)\Theta_h Q_h x(t_0)ds - \int_{t}^{\infty} \Phi(t)(I - Q_{h,0})\Phi^{-1}(s)B(s)\Psi(t)\Theta_h Q_h x(t_0)ds.$$

Since $|\Psi(t)\Theta_h Q_h x(t_0)| \leq \widetilde{K}h(t)$ and $\lim_{t\to\infty} h(t)^{-1}\Phi(t)Q_{h,0} = 0$, the Lebesgue theorem on dominated convergence implies

$$\int_{t_0}^{t} \Phi(t)Q_{h,0}\Phi^{-1}(s)B(s)\Psi(t)\Theta_h Q_h x(t_0)ds = o(h).$$

Since $B \in L^1$, we have

$$\int_{t}^{\infty} \Phi(t)(I - Q_{h,0})\Phi^{-1}(s)B(s)\Psi(t)\Theta_h Q_h x(t_0)ds = o(h).$$

Formula (16) is a consequence of these asymptotic formulae. □

Let us now assume that System (2) has an (h,k)-dichotomy. Since the pair of functions (h,k) is compensated, System (2) has both an h and a k-dichotomy. If (14) is satisfied, Theorem 1 can be applied to this k-dichotomy, and therefore the k-bounded solutions of Systems (2) and (1) are in bijective correspondence with the k-bounded solutions of System (1); this correspondence is obtained by the k-bounded solution of the integral equation

$$\begin{aligned} y(t) = x(t) + &\int_{t_0}^{t} \Phi(t) Q_{k,0} \Phi^{-1}(s) B(s) y(s) ds \\ &- \int_{t}^{\infty} \Phi(t)(I - Q_{k,0}) \Phi^{-1}(s) B(s) y(s) ds, \end{aligned}$$

from where the asymptotic formula

$$y(t) = x(t) + o(k(t)) \tag{19}$$

follows [14]. This asymptotic formula yields the asymptotic correspondence

$$\Psi(t)\widetilde{Q}_k = (\Phi(t) + o(k(t)))Q_k. \tag{20}$$

As it is shown in [14], in general the function h of the pair (h,k) stands for the asymptotic equivalence of bounded solutions, while the function k generates the asymptotic equivalence of unbounded solutions. Since $V_h \subset V_k$, the h-bounded solutions of System (2) can be considered as k-bounded solutions and therefore for an h-bounded solution the asymptotic representations (15) and (19) are valid. Simple examples show that for h-bounded solutions formula (15) is more precise than (19), and therefore we assign formula (15) to h-bounded solutions and formula (19) to k-bounded solutions that are not h-bounded. In order to make this construction precise, let us denote $W_k = (Q_k - Q_h)[V]$ and $S_k = Q_k - Q_h$. Thus (13) implies

$$V_k = V_h \bigoplus W_k.$$

The property of compensation of the (h,k)-dichotomy implies $Q_h[V] \subset Q_k[V]$, from where it follows $Q_k Q_h = Q_h$. But the projections Q_h, Q_k can be defined so as to have the following property: $\operatorname{Kernel} Q_k \subset \operatorname{Kernel} Q_h$. Thus, $Q_h Q_k = Q_h$ and S_k is a projection.

Theorem 2. *If* (14) *is satisfied and System* (2) *has an* (h,k)*-dichotomy, then the fundamental matrix* $\Psi(t)$ $(\Psi(t_0)=I)$ *of System* (1) *has the asymptotic representation*

$$\Psi(t)(\widetilde{Q}_h Q_h + \widetilde{Q}_k S_k) = (\Phi(t)+o(h(t)))Q_h + (\Phi(t)+o(k(t)))S_k. \tag{21}$$

Proof. The proof of this theorem is obtained by adding (16) and (20), previously multiplied from the right by Q_h and S_k, respectively. □

The asymptotic formula (21) synthesizes the results of [14] concerning the asymptotic equivalence of h-bounded and k-bounded solutions of Systems (2) and (1). A useful consequence of the above theorem is obtained in the case $Q_k = I$ (in this case we will call this (h,k)-dichotomy exhaustive):

Theorem 3. *Under the conditions of Theorem 2, if* $V_k = V$, *then there exists* $\widetilde{\Phi}(t)$, *a fundamental matrix of System,* (1) *with the asymptotic representation*

$$\widetilde{\Phi}(t) = (\Phi(t)+o(h(t)))Q_h + (\Phi(t)+o(k(t)))S_k. \tag{22}$$

Proof. Let us define $\widetilde{\Phi}(t) = \Psi(t)E$, where $E = \widetilde{Q}_h Q_h + \widetilde{Q}_k S_k$. We have to show that the matrix E is invertible. Let $E\xi = 0$. We decompose $\xi = \xi_1 + \xi_2$, $\xi_1 \in Q_h[V]$, $\xi_2 \in (Q_k - Q_h)[V]$. Therefore

$$0 = \Phi(t)\xi_1 + o(h(t))\xi_1 + \Phi(t)\xi_2 + o(k(t))\xi_2.$$

This identity shows that $\xi_1+\xi_2 \in V_{k,0}$, but Theorem A implies $\xi_1+\xi_2 \in V_h$. Thus $\xi_2 = 0$. Since $\Phi(t)\xi_1 + o(h(t))\xi_1 = 0$ is an h-bounded solution of System (1), and the h-bounded solutions of Systems (1) and (2) are in bijective linear correspondence as given by the matrix (18), $\xi_1 = 0$. □

Multiplying (22) from the left by Q_h and S_k respectively, we obtain

$$\widetilde{\Phi}(t)Q_h = (\Phi(t)+o(h(t)))Q_h,$$

and

$$\widetilde{\Phi}(t)S_k = (\Phi(t)+o(k(t)))S_k.$$

These formulas say not only that (22) is the asymptotic integration of a fundamental matrix of solutions of System (1), but they also establish the asymptotic integration of the h and k solutions of System (1).

Instead of (22), an asymptotic formula having the form

$$\widetilde{\Phi}(t) = (I + o(1))\Phi(t), \tag{23}$$

(for example formula (8)) would be more attractive for its apparent simplicity, but the precision of (22) yields consequences for System (1), that cannot be obtained from (23); namely, the asymptotic formulas for $\widetilde{\Phi}Q_h$ and $\widetilde{\Phi}S_k$ follow from (22), but they cannot be obtained from an asymptotic formula like (23).

4. Linked dichotomies

Let us consider two ordered sets of positive continuous functions

$$\mathcal{H} = \{h_1, h_2, \ldots, h_r\}, \quad \mathcal{K} = \{k_1, k_2, \ldots, k_r\},$$

and an ordered collection of projection matrices

$$\mathcal{P} = \{P_1, P_2, \ldots, P_r\}.$$

Definition 3. We shall say that the triad $(\mathcal{H}, \mathcal{K}, \mathcal{P})$ is a chain of linked dichotomies for System (2) (briefly: a linked dichotomy) iff

L1: For $j = 1, \ldots, r$, the System (2) has a dichotomy (h_j, k_j, P_j).

L2:
$$V_{k_1} \subset V_{k_2} \subset \cdots \subset V_{k_r}.$$

Throughout, we will assume that all the dichotomies (h_j, k_j, P_j), $j = 1, \ldots, r$, are defined with the same constant K.

We will employ the abbreviation $(\mathcal{H}, \mathcal{P}) = (\mathcal{H}, \mathcal{H}, \mathcal{P})$. In applications, a convenient algebraic condition implying **L2** is given by

L2': For some constant D we have $k_j(t) \leq D k_{j+1}(t)$, $j = 1, 2, \ldots, r-1$.

A more stringent condition than **L2'**, useful in applications (see the second example of the next section), is the uniform condition

L2": For some constant D we have

$$k_j(t)k_j(s)^{-1} \leq Dk_{j+1}(t)k_{j+1}(s)^{-1}, \quad j = 1, 2, \dots, r-1,\ t \geq s.$$

We observe that condition $B(t) \in L^1$ implies that for each $j = 1, \dots, r$, System (1) has a dichotomy $(h_j, k_j, \widetilde{P}_j)$, where $\widetilde{P}_j$ is a projection similar to the projection P_j. We may assume that all these dichotomies are accomplished with the same constant $\widetilde{K}$. We will give conditions under which the triad $(\mathcal{H}, \mathcal{K}, \mathcal{P}')$ is a linked dichotomy for System (1).

Lemma 1. *Let h, g be positive functions. If System* (2) *has an h-dichotomy and $V_h \subset V_g$ properly ($V_h \neq V_g$), then the function $g(t)^{-1}h(t)$ is bounded.*

Proof. Let $\eta \in V_g \setminus V_h$ and let P be the projection defining the h-dichotomy. Then

$$\eta = P\eta + (I - P)\eta.$$

From Theorem A we have $P\eta \in V_h$. Therefore $\xi = (I - P)\eta \in V_g$, $\xi \neq 0$. From the estimate $|\Phi(t)(I-P)\Phi^{-1}(s)| \leq Kh(t)h(s)^{-1}$, $s \geq t$, we obtain $h(t)^{-1}|\Phi(t)\xi| \geq \alpha > 0$ for some constant α. Thus we can write

$$\alpha \leq h(t)^{-1}|\Phi(t)\xi| \leq h(t)^{-1}g(t)g(t)^{-1}|\Phi(t)\xi| \leq h(t)^{-1}g(t)|\Phi\xi|_g,$$

from where the assertion of the lemma follows. □

Lemma 2. *Let h and g be positive functions. If System* (2) *has an h-dichotomy and $V_h \subset V_g$ properly, then condition* (14) *implies $\widetilde{V}_h \subset \widetilde{V}_g$.*

Proof. Let $\eta \in \widetilde{V}_h \setminus \widetilde{V}_g$. From Theorem 1, there exists $\xi \in V_h$, such that

$$\Psi(t)\eta = \Phi(t)\xi + o(h)(t).$$

Since $\xi \in V_g$, Lemma 1 implies that the right hand side of the last identity is g-bounded, whereas the left hand side is not. This contradiction shows that $\widetilde{V}_h \setminus \widetilde{V}_g$ is empty. □

Theorem 4. *If System* (2) *has a linked dichotomy* $(\mathcal{H}, \mathcal{K}, \mathcal{P})$, *where* $V_{k_j} \subset V_{k_{j+1}}$ *properly,* $j = 1, 2, \ldots, r-1$, *then condition* (14) *implies that the triplet* $(\mathcal{H}, \mathcal{K}, \mathcal{P}')$ *is a dichotomy linked to the System* (1).

Proof. From the compensation property of each dichotomy (h_j, k_j, P_j), we obtain that System (2) has a k_j-dichotomy with projection P_j. From Lemma 2 we have $\widetilde{V}_{k_1} \subset \widetilde{V}_{k_2} \subset \cdots \subset \widetilde{V}_{k_r}$. Thus conditions **L1**, **L2** are satisfied for the triplet $(\mathcal{H}, \mathcal{K}, \mathcal{P}')$. □

Assuming that System (2) has the linked dichotomy $(\mathcal{H}, \mathcal{K}, \mathcal{P})$, we will perform the following construction: Let us define $U_{h_1} = V_{h_1}$. Further, if $V_{h_1} = V_{k_1}$ we define $W_{k_1} = \{0\}$. If V_{h_1} is properly contained in V_{k_1}, then we define W_{k_1} as a complementary subspace to V_{h_1} in the space V_{k_1}. In both cases we can write the disjoint sum $V_{k_1} = \{0\} + U_{h_1} + W_{k_1}$. Thus in the space U_{h_1} we keep all the initial conditions corresponding to the h_1-bounded solutions of System (2). To the space W_{k_1} we assign the initial conditions of k_1-bounded solutions that are not h_1-bounded. We repeat this process for the space V_{k_2} in the following manner: If $V_{k_2} = V_{k_1}$, we define $U_{h_2} = W_{k_2} = \{0\}$. If V_{k_1} is properly contained in V_{k_2}, then we define U_{h_2} as the subspace of the initial conditions of the h_2-bounded solutions not contained in V_{h_1} and the subspace W_{k_2} groups the initial conditions of k_2-solutions not included in U_{h_2}; therefore V_{k_2} can be written as a disjoint sum $V_{k_2} = V_{k_1} + U_{k_2} + W_{k_2}$. Carrying out this process further, we obtain the decomposition:

$$
\begin{array}{lllll}
V_{k_1} = & \{0\} & + U_{h_1} & + W_{k_1} \\
V_{k_2} = & V_{k_1} & + U_{h_2} & + W_{k_2} \\
\vdots & \vdots & \vdots & \vdots \\
V_{k_r} = & V_{k_{r-1}} & + U_{h_r} & + W_{k_r}.
\end{array}
\tag{24}
$$

In applications the table (24) does not give a good decomposition of the subspaces of initial conditions corresponding to solutions with different growths; for example if $k_1 = k_2 = \ldots = k_r$, all subspaces of table (24) would be trivial, except U_{h_1} and maybe W_{k_1}. This situation can be improved by asking from the linked dichotomy the property defined as follows:

Definition 4. We say that the linked dichotomy $(\mathcal{H}, \mathcal{K}, \mathcal{P})$ is stratified iff

$$U_{h_1} \subset V_{k_1} \subset U_{h_2} \subset V_{k_2} \subset \cdots \subset U_{h_r} \subset V_{k_r},$$

where V_{k_j} and U_{h_j} are defined in (24).

This property holds if for some constant D we have

$$k_j(t) \leq D h_{j+1}(t), \quad j = 1, 2, \ldots, r-1.$$

An $(\mathcal{H}, \mathcal{P})$ linked dichotomy is stratified. All examples considered in Section 7 correspond to stratified linked dichotomies.

5. Asymptotic integration

In this section we generalize the asymptotic formula (22) under the existence of a linked dichotomy for System (2). Let us consider a linked dichotomy $(\mathcal{H}, \mathcal{K}, \mathcal{P})$. According to (24), we define the projections R_j, S_j, such that

$$R_j[V] = U_{h_j}, \quad S_j[V] = W_{k_j}.$$

R_j is a projection that chooses in V, exactly, the initial conditions of all h_j-bounded solutions of System (2). The projection S_j plays a similar role. From the construction of the subspaces U_{h_j} and V_{k_j} we have

$$R_j R_i = 0, \; S_j S_i = 0, \text{ if } i \neq j, \quad R_j S_i = 0 \text{ for all indexes } i, j.$$

Moreover, since the ranges of the projections R_j and S_j are contained in V_{h_j} and V_{k_j}, respectively we have the identities

$$Q_{h_j} R_j = R_j, \quad Q_{k_j} S_j = S_j. \tag{25}$$

Theorem 5. *If System (2) has a linked dichotomy $(\mathcal{H}, \mathcal{K}, \mathcal{P})$ and condition (14) is satisfied, then the fundamental matrix Ψ of System (1), $\Psi(t_0) = I$, has the property*

$$\Psi(t)E = \sum_{j=1}^{r} \left(\Phi(t) + o(h_j(t))\right) R_j + \sum_{j=1}^{r} \left(\Phi(t) + o(k_j(t))\right) S_j, \tag{26}$$

where

$$E = \sum_{j=1}^{r} (\widetilde{Q}_{h_j} R_j + \widetilde{Q}_{k_j} S_j) \tag{27}$$

and some of the projections R_j or S_j in (26) *could be equal zero.*

Proof. Applying Theorem 1 to each (h_j, k_j)-dichotomy we obtain from (16) and (20) the decompositions

$$\Psi(t)\widetilde{Q}_{h_j} = (\Phi(t) + o(h_j(t)))Q_{h_j}, \quad \Psi(t)\widetilde{Q}_{k_j} = (\Phi(t) + o(k(t)))Q_{k_j}.$$

Multiplying these formulas by R_j and S_j and using (25) we obtain

$$\Psi(t)\widetilde{Q}_{h_j} R_j = (\Phi(t) + o(h_j(t)))R_j, \quad \Psi(t)\widetilde{Q}_{k_j} S_j = (\Phi(t) + o(k(t)))S_j,$$

whence (26) follows. □

Definition 5. A linked dichotomy $(\mathcal{H}, \mathcal{K}, \mathcal{P})$ is called exhaustive iff $V_{k_r} = V$.

For an exhaustive linked dichotomy we have the property

$$I = Q_{k_r} = \sum_{j=1}^{r} R_j + \sum_{j=1}^{r} S_j.$$

From this identity, we can establish the following abstract version of the Levinson asymptotic theorem for the nondiagonal System (1)–(2):

Theorem 6. *Under the conditions of Theorem 1, let us assume that System* (2) *has an exhaustive chain of linked dichotomies $(\mathcal{H}, \mathcal{K}, \mathcal{P})$. Then System* (1) *has a fundamental matrix $\widetilde{\Phi}$ allowing the asymptotic representation*

$$\widetilde{\Phi}(t) = \sum_{j=1}^{r} \left(\Phi(t) + o(h_j(t))\right) R_j + \sum_{j=1}^{r} \left(\Phi(t) + o(k_j(t))\right) S_j.$$

Proof. We will prove that the matrix (27) is nonsingular. Let $E\xi = 0$, then $0 = \sum_{j=1}^{r} \left(\Phi(t)R_j + o(h_j(t))\right) R_j\xi + \sum_{j=1}^{r} \left(\Phi(t)S_j + o(k_j(t))\right) S_j\xi$. From the construction of table (24) we obtain that the solution $\Phi(t)S_r\xi$

of System (2) satisfies $\Phi(t)S_r\xi = o(k_r(t))$; therefore $S_r\xi \in V_{k_r,0}$. Applying Theorem A to the dichotomy (h_r, k_r, P_r) we obtain $S_r\xi \in V_{h_r}$. Since $S_r\xi \in W_{k_r}$, the last row of table (24) says that $S_r\xi = 0$. Hence

$$0 = \sum_{j=1}^{r} (\Phi(t) + o(h_j(t)))\, R_j\xi + \sum_{j=1}^{r-1} (\Phi(t) + o(k_j(t)))\, S_j\xi.$$

The right hand side of this last equation is an h_r-bounded solution of System (1). But under condition (14), the h_r-bounded solutions of Systems (1) and (2) are in bijective corespondence. Therefore

$$0 = \sum_{j=1}^{r} \Phi(t)R_j\xi + \sum_{j=1}^{r-1} \Phi(t)S_j\xi + \Phi(t)R_r\xi. \tag{28}$$

Since $\sum_{j=1}^{r-1} \Phi(t)R_j\xi + \sum_{j=1}^{r-1} \Phi(t)S_j\xi \in V_{k_{r-1}}$ and $\Phi(t)R_r\xi \in U_{h_r}$, we obtain from the last row of table (24) $R_r\xi = 0$. Inasmuch as $R_r\xi = 0$ and $S_r\xi = 0$, we obtain from (28)

$$0 = \sum_{j=1}^{r-1} (\Phi(t) + o(h_j(t)))\, R_j + \sum_{j=1}^{r-1} (\Phi(t) + o(k_j(t)))\, S_j.$$

If we repeat this reasoning we will obtain $R_j\xi = 0$, $S_j\xi = 0$, $\forall j$, implying

$$\xi = \sum_{j=1}^{r} (R_j + S_j)\xi = 0.$$

Since E is nonsingular, $\widetilde{\Phi}(t) = \Psi(t)E$ is a fundamental matrix of System (1). □

6. A linked dichotomy for diagonal systems

In this section we show that the asymptotic formula (8) can be obtained from the notion of a linked dichotomy. First we recall the notion of a Levinson dichotomy [7], where we will use the notation $N = \{1, 2, \ldots, n\}$.

Definition 6. We shall say that the diagonal System (6) allows a Levinson dichotomy iff for any $j \in N$, the set N can be partitioned as

$N = N_1^j \cup N_2^j$ such that for any $i \in N_1^j$

$$\int_s^t (Re\lambda_i(u) - Re\lambda_j(u))du \le L_1, \quad t \ge s, \tag{29}$$

and

$$\lim_{t\to\infty} \int_{t_0}^t (Re\lambda_i(u) - Re\lambda_j(u))du = -\infty \tag{30}$$

is satisfied, and for any $i \in N_2^j$

$$\int_s^t (Re\lambda_i(u) - Re\lambda_j(u))du \ge L_2, \quad t \ge s, \tag{31}$$

where L_1, L_2 are constants.

Let $P_j = \mathrm{diag}\{p_{11}, p_{22}, \dots, p_{nn}\}$, $j = 1, 2, \dots, n$, be a diagonal projection defined as follows: $p_{ii} = 1$ if $i \in N_1^j$, $p_{ii} = 0$, if $i \in N_2^j$. It is easy to prove the following

Proposition 1. *If System* (6) *satisfies conditions* (29)–(31), *then for each* $j \in N$ *System* (6) *allows an* (h_j, P_j)*-dichotomy, where*

$$h_j(t) = \exp\left\{\int_{t_0}^t Re\lambda_j(u)du\right\}.$$

Hence, the conditions of a Levinson dichotomy imply the existence of a family of n dichotomies (h_j, P_j) for System (6). For each fixed j let us define

$$N_3^j = \left\{i \in N; \limsup_{t\to\infty} \int_{t_0}^t Re(\lambda_i(u) - \lambda_j(u))du < \infty\right\}.$$

Note that $j \in N_3^j$.

Lemma 3. *For* $i, j \in N$ *either* $N_3^j \subset N_3^i$, *or* $N_3^i \subset N_3^j$.

Proof. We have two possibilities. Let $i \in N_3^j$; then for $r \in N_3^i$, we have

$$\int_{t_0}^t Re(\lambda_r - \lambda_j)(u)du = \int_{t_0}^t Re(\lambda_r - \lambda_i)(u)du + \int_{t_0}^t Re(\lambda_i - \lambda_j)(u)du,$$

implying $r \in N_3^j$. If $i \notin N_3^j$, then

$$\int_{t_0}^{t} Re(\lambda_i(u) - \lambda_j(u))du$$

is not bounded from above. (31) implies $j \in N_3^i$, from where $N_3^j \subset N_3^i$. $\square$

Definition 7. We say that the indexes i and j are equivalent iff $i, j \in N_3^j \cap N_3^i$.

It is easy to verify that the indexes i and j are equivalent iff

$$K_1 \leq h_j(t)^{-1} h_i(t) \leq K_2$$

for some positive constants K_1, K_2. Moreover, for equivalent indexes i and j we have $N_3^i = N_3^j$. According to Lemma 3, we can order the sets N_3^j as a chain

$$N_3^{j_1} \subset N_3^{j_2} \subset \cdots \subset N_3^{j_r},$$

where we agree to drop out repeated sets. This last chain implies

$$K_{j_1} h_{j_1} \leq K_{j_2} h_{j_2} \leq \cdots \leq K_{j_r} h_{j_r}$$

for some positive constants K_{j_i}. In order to avoid composed indexes we denote $M_i = N_3^{j_i}$ and $\hat{h}_i = h_{j_i}$. Thus we have

$$M_1 \subset M_2 \subset \cdots \subset M_r, \quad \hat{K}_1 \hat{h}_1 \leq \hat{K}_2 \hat{h}_2 \leq \cdots \leq \hat{K}_r \hat{h}_r. \tag{32}$$

From Lemma 3 it follows that all indexes contained in $M_i \backslash M_{i-1}$ are equivalent. Let us define the diagonal projections $Q_i = \{a_{11}, a_{22}, \dots, a_{nn}\}$, where $a_{mm} = 1$ if $m \in M_i$, and $a_{mm} = 0$, if $m \notin M_i$. From the definition of the set M_i we have $V_{\hat{h}_i} = Q_i[V]$. Since $i \in M_i$, we obtain $M_r = N$. This formula implies

$$I = \sum_{i=1}^{r} Q_i = Q_r. \tag{33}$$

From (32) and $V_{\hat{h}_i} = Q_i[V]$ we have $V_{\hat{h}_1} \subset V_{\hat{h}_2} \subset \cdots \subset V_{\hat{h}_r}$. Moreover, from the definition of the projection Q_i, we have for

$$\Phi(t) = \exp\left\{ \int_{t_0}^{t} \Lambda(u)du \right\}$$

the estimates (where we will use the definition $Q_0 = 0$)

$$|\Phi(t)\Phi^{-1}(s)(Q_i - Q_{i-1})| \le K\hat{h}_i(t)\hat{h}_i(s)^{-1}, \forall t, s, \quad i = 1, \dots, r, \tag{34}$$

and

$$\begin{aligned} |\Phi(t)Q_i\Phi^{-1}(s)| &\le K\hat{h}_i(t)\hat{h}_i(s)^{-1}, \quad t \ge s, \\ |\Phi(t)(I - Q_i)\Phi^{-1}(s)| &\le K\hat{h}_i(t)\hat{h}_i(s)^{-1}, \quad s \ge t. \end{aligned} \tag{35}$$

The properties (32) and (35) imply that System (6) has the linked dichotomy $(\mathcal{H}, \mathcal{Q})$, where

$$\mathcal{H} = \left\{\hat{h}_1, \hat{h}_2, \dots, \hat{h}_r\right\}, \quad \mathcal{Q} = \{Q_1, Q_2, \dots, Q_r\}.$$

From (33), this linked dichotomy is exhaustive. Using Theorem 6, we deduce the following formula for the fundamental matrix of System (7):

$$\begin{aligned} \widetilde{\Phi}(t) = {}& (\Phi(t) + o(\hat{h}_1))Q_1 + (\Phi(t) + o(\hat{h}_2))(Q_2 - Q_1) \\ & + \cdots + (\Phi(t) + o(\hat{h}_r))(Q_r - Q_{r-1}). \end{aligned} \tag{36}$$

Lemma 4.

$$(\Phi(t) + o(\hat{h}_j))(Q_j - Q_{j-1}) = (Q_j - Q_{j-1} + o(1))\Phi(t).$$

PROOF. Since the matrices $\Phi(t)$ and $Q_j - Q_{j-1}$ commute,

$$(\Phi(t)+o(\hat{h}_j))(Q_j-Q_{j-1}) = \left(Q_j - Q_{j-1} + o(\hat{h}_j)\Phi^{-1}(t)(Q_j - Q_{j-1})\right)\Phi(t).$$

From (34) we obtain $|o(\hat{h}_j)\hat{\Phi}^{-1}(t)(Q_j - Q_{j-1})| \le o(\hat{h}_j)\hat{h}_j(t)^{-1}$. This estimate implies the proof of the lemma. □

The property (33), Lemma 4 and (36) imply (8).

Formula (36) apparently is new and therefore has not been used in problems of asymptotic integration of System (7). This formula differs from the coordinate formulation of the Levinson theorem [4], [12]. Multiplying (36) by the projection $Q_j - Q_{j-1}$ we obtain

$$\widetilde{\Phi}(t)(Q_j - Q_{j-1}) = \Phi(t)(Q_j - Q_{j-1}) + o(\hat{h}_j)(Q_j - Q_{j-1}),$$

a formula that not only involves all the solutions of System (7) with the same growth h_j, but includes the projection $Q_j - Q_{j-1}$, which chooses the initial conditions giving these solutions. Such considerations are important in problems of asymptotic integration of nonlinear systems

$$y' = [\Lambda(t) + B(t)]\, y + f(t, y),$$

where we are interested in the asymptotic equivalence of $\hat{h}_j$-bounded solutions of System (7) and those of this nonlinear system [17].

7. Examples

7.1 A 2×2 system

Let us consider the asymptotic integration of the following system:

$$y' = [\mathrm{diag}\{-1, 0\} + B(t)]\, y. \tag{37}$$

The fundamental matrix of $x' = \mathrm{diag}\{-1, 0\}x$, is $\Phi(t) = \mathrm{diag}\{\mathrm{e}^{-t}, 1\}$. In order to obtain (8), the dichotomy of Levinson requires the construction of two dichotomies, namely an e^{-t}-dichotomy with projection $P_1 = 0$ and an 1-dichotomy with projection $P_2 = \mathrm{diag}\{1, 0\}$. The application of Theorem 3 groups these two dichotomies in the exhaustive $(e^{-t}, 1)$-dichotomy with projection $P = \mathrm{diag}\{1, 0\}$ and gives the following asymptotic formula for Ψ the fundamental matrix of System (37):

$$\Psi(t) = (\Phi(t) + o(e^{-t}))P + (\Phi(t) + o(1))(I - P). \tag{38}$$

If we order the dichotomies given by the Levinson dichotomy as is done in Section 6, then the asymptotic formula (36) coincides with (38).

Further, we observe that the adjoint system $z' = z\,\mathrm{diag}\{1, 0\}$ has a dichotomy $(1, \mathrm{e}^t, I - P)$. Therefore, we obtain for $\Psi^{-1}(t)$ the formula

$$\Psi^{-1}(t) = P(\Phi^{-1}(t) + o(e^t)) + (I - P)(\Phi^{-1}(t) + o(1)). \tag{39}$$

In this example, the formula for the inverse Ψ^{-1} can be obtained also from the Levinson asymptotic theorem; here the interesting fact is that formula (39) was not obtained from the Levinson theory, and therefore (39) suggests the possibility of extending the asymptotic formula (39) for a general System (1) under suitable conditions on the linked dichotomy of System (2).

7.2 A diagonal example

Let us consider (7), where

$$\Lambda(t) = \operatorname{diag}(-t^{-1}, 0, i, t^{-1}),\ t \geq 1, \tag{40}$$

whose fundamental matrix is $\Phi(t) = \operatorname{diag}\left\{t^{-1}, 1, \mathrm{e}^{it}, t\right\}$. Let us define the diagonal projections $P_1 = \operatorname{diag}(1,0,0,0)$, $P_2 = \operatorname{diag}(1,1,0,0)$, and let $\mathcal{H} = \{h_1, h_2\} = \left\{t^{-1}, 1,\right\}$, $\mathcal{K} = \{k_1, k_2\} = \{1, t\}$, $\mathcal{P} = \{P_1, P_2\}$. It is easy to see that System (2)–(40) has the linked dichotomy $(\mathcal{H}, \mathcal{K}, \mathcal{P})$. In what follows $\langle e_j \rangle$ denotes the subspace generated by the coordinate vector j of the canonical basis. In this case the table (24) is given by

$$U_{h_1} = \langle e_1 \rangle, \quad W_{k_1} = \langle e_2, e_3 \rangle, \quad U_{h_2} = \{0\}, \quad W_{k_2} = \langle e_4 \rangle.$$

These calculations show that $(\mathcal{H}, \mathcal{K}, \mathcal{P})$ is an exhaustive chain of linked dichotomies. This decomposition and the definition of the functions $(\mathcal{H}, \mathcal{K})$ say that the System (7)–(40) has a t^{-1}-bounded solution generated by the initial condition e_1, two 1-bounded solutions defined by the initial conditions e_2 and e_3, and a t-bounded solution given by the initial condition e_4. According to Theorem 6, the system

$$y' = \left(\operatorname{diag}(-t^{-1}, 0, i, t^{-1}) + B(t)\right) y \tag{41}$$

has a fundamental matrix Ψ with the following asymptotic representation:

$$\begin{aligned}\Psi(t) = {} & (\Phi(t) + o(1/t))P_1 + (\Phi(t) + o(1))(P_2 - P_1) \\ & + (\Phi(t) + o(t))(I - P_2).\end{aligned}$$

This asymptotic integration has been obtained with two (h,k)-dichotomies. Applying the Levinson theory to this example we require four dichotomies. A third possibility for asymptotic integration of System (41) is given by the formula (36); in this case three different dichotomies will be required. This example says that the decomposition of (35) of a Levinson dichotomy as linked dichotomy is not optimal. We can order a Levinson dichotomy not as a chain of h-dichotomies, but as a chain of (h,k)-dichotomies (this can be done for a general diagonal Levinson system), then it is easy to verify that only two dichotomies, exactly the dichotomies of the present examples, are required to obtain the asymptotic integration of the corresponding System (1).

7.3 Asymptotic integration of block-diagonal systems

We ask for the asymptotic integration of (1), where (2) is a block diagonal system

$$A(t) = \mathrm{diag}\{A_1(t), \dots, A_r(t)\}. \tag{42}$$

We will suppose that the system

$$x' = A_i(t)x \tag{43}$$

is an h_i-system [21], that is Φ_i, the fundamental matrix of (43), satisfies

$$|\Phi_i(t)\Phi_i^{-1}(s)| \le Kh_i(t)h_i(s)^{-1} \quad \text{for all } t \text{ and } s; \tag{44}$$

moreover let us assume that

$$h_i(t)h_i(s)^{-1} \le Ch_{i+1}(t)h_{i+1}(s)^{-1}, \quad t \ge s,\ C = \text{constant}. \tag{45}$$

The fundamental matrix of (42) is $\Phi(t) = \mathrm{diag}\{\Phi_1(t), \Phi_2(t), \dots, \Phi_r(t)\}$. Let us define the projection matrices $Q_i = \mathrm{diag}\{P_1, P_2, \dots, P_r\}$, where

$$P_j = \begin{cases} I_j, & 0 \le j \le i \\ 0, & i+1 \le j \le r, \end{cases}$$

where I_i are identity matrices, dimension$[I_i]$ = dimension$[A_i]$. The bounds (44) imply

$$\begin{aligned} |\Phi(t)Q_i\Phi^{-1}(s)| &\le Kh_i(t)h_i(s)^{-1}, \quad t \ge s, \\ |\Phi(t)(I - Q_i)\Phi^{-1}(s)| &\le Kh_i(t)h_i(s)^{-1}, \quad s \ge t. \end{aligned} \tag{46}$$

The property (45) and the estimates (46) imply that System (42) has the exhaustive linked dichotomy

$$\mathcal{H} = \{h_1, h_2, \dots, h_r\}, \quad \mathcal{Q} = \{Q_1, Q_2, \dots, Q_r\}.$$

By Theorem 6 we can accomplish the asymptotic integration of the system

$$y' = (\mathrm{diag}\{A_1, A_2, \dots, A_r\} + B(t))y. \tag{47}$$

This class of systems is important in the applications; we limit ourselves to pointing out the asymptotic integration of the second order system

$$y'' = (\Lambda(t) + B(t))\, y,$$

a system that can be reduced to the form (47), where

$$A_i(t) = \begin{pmatrix} 0 & 1 \\ \lambda_i(t) & 0 \end{pmatrix}.$$

Concerning this problem, we refer to [18] and the forthcoming paper [17].

7.4 Asymptotic integration of a nondiagonal system

Let us consider the System (2)–(42), with $A(t) = \mathrm{diag}\{A_1(t), A_2(t)\}$ defined as follows:

$$A_1(t) = \begin{pmatrix} -1 & t^{-1}\phi(t) \\ 0 & t^{-1} \end{pmatrix}, \quad A_2(t) = \begin{pmatrix} 0 & t^{-1} \\ 0 & -t^{-1} \end{pmatrix}, \tag{48}$$

where $\phi(t)$ is a continuous function, such that $|\phi(t)| \le 1$ for all $t \ge 1$. The corresponding systems $x' = A_i(t)x$ have the fundamental matrices

$$\Phi_1(t) = \begin{pmatrix} e^{-(t-1)} & \int_1^t e^{-(t-s)}\phi(s)ds \\ 0 & t \end{pmatrix}, \quad \Phi_2(t) = \begin{pmatrix} 1 & -t^{-1} \\ 0 & t^{-1} \end{pmatrix}.$$

For the projection matrix $P = \mathrm{diag}\{1, 0\}$ we have the following estimates:

$$|\Phi_1(t)P\Phi_1^{-1}(s)| \le 3e^{-(t-s)}, \quad |\Phi_1(s)(I-P)\Phi_1^{-1}(t)| \le st^{-1}, \ \forall t, s \ge 1$$

and

$$|\Phi_2(t)P\Phi_2^{-1}(s)| = 1, \qquad |\Phi_2(s)(I-P)\Phi_2^{-1}(s)| = s^{-1}t, \ \forall\, t, s \ge 1,$$

(it is worthwhile mentioning that Systems (2)–(48) are examples of nondiagonal, nonautonomous systems, respectively possessing an (e^{-t}, t) and a $(t^{-1}, 1)$-dichotomy). The fundamental matrix of the diagonal system

$$x' = \mathrm{diag}\{A_1(t), A_2(t)\}x \tag{49}$$

is $\Phi(t) = \mathrm{diag}\{\Phi_1(t), \Phi_2(t)\}$. This system allows, among many others, the following family of dichotomies:

– The (e^{-t}, t^{-1})-dichotomy with projection $Q_1 = \mathrm{diag}\{1, 0, 0, 0\}$,

– The $(t^{-1}, 1)$-dichotomy with projection $Q_2 = \operatorname{diag}\{1, 0, 0, 1\}$,

– The $(1, t)$-dichotomy with projection $Q_3 = \operatorname{diag}\{1, 0, 1, 1\}$.

Defining $\mathcal{H} = \{e^{-t}, t^{-1}, 1\}$, $\mathcal{K} = \{t^{-1}, 1, t\}$, $\mathcal{P} = \{Q_1, Q_2, Q_3\}$, we obtain that System (49) allows the exhaustive linked dichotomy $(\mathcal{H}, \mathcal{K}, \mathcal{Q}\}$. The nonzero subspaces given by the table (24) are the following: $U_{h_1} = \langle e_1 \rangle$, $W_{k_1} = \langle e_4 \rangle$, $U_{h_1} = \langle 0 \rangle$, $W_{k_2} = \langle e_3 \rangle$, $U_{h_3} = \langle 0 \rangle$, $W_{k_3} = \langle e_2 \rangle$. Thus, according to Theorem 6, we can assure that the asymptotic representation of the fundamental matrix $\Psi(t)$ of the corresponding System (47) has the columns given by the formulas

$$\Psi_1(t) = e^{1-t} e_1 + o(e^{1-t}), \qquad \Psi_2(t) = (1 - e^{1-t}) e_1 + t e_2 + o(t),$$

$$\Psi_3(t) = e_3 + o(1), \qquad \Psi_4(t) = t^{-1} e_4 + o(t^{-1}).$$

We emphasize that the asymptotic representation displayed for this example cannot be obtained from Levinson's results expounded in [11], [12].

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References

[1] R. Bellman, Stability theory of differential equations, *International Series in Pure and Applied Mathematics*, *McGraw-Hill, New York*, 1953.

[2] F. Brauer and J. S Wong, On the asymptotic relationships between solutions of two systems of ordinary differential equations, *J. Diff. Eqns.* **9** (1969), 527–543.

[3] Jr W. A. Harris and D. A. Lutz, A unified theory of asymptotic integration, *J. Math. Anal. and Appl.* **57** (1977), 571–586.

[4] E. A. Coddington and N. Levinson, Theory of ordinary differential equations, *New York, McGraw-Hill*, 1995.

[5] W. A. Coppel, Dichotomies in stability theory, Lecture notes in mathematics, 629, *Springer Verlag*, *Berlin*, 1978.

[6] W. A. Coppel, Stability and asymptotic behavior of differential equations, *D. C. Heath and Company*, *Boston*, 1965.

[7] M. S. P. Eastham, The asymptotic solution of linear differential system, Applications of the Levinson Theorem, *Clarendon Press*, *Oxford*, 1989.

[8] J. K. Hale and N. Onuchic, On the asymptotic behavior of a class of differential equations, *Contrb. Diff.* **2** (1963), 61–75.

[9] P. Hartman and A. Wintner, Asymptotic integration of linear differential equations, *Amer. Jour. Math.* **77** (1955), 48–86.

[10] J. L. Massera and J. J. Schäffer, Linear differential equations and functions spaces, *Academic Press*, *New York*, 1966.

[11] N. Levinson, The asymptotic behavior of system of linear differential equations, *Amer. Jour. Math.* **68** (1946), 1–6.

[12] N. Levinson, The asymptotic nature of solutions of linear differential equations, *Duke Math. J.* **15** (1948), 111–126.

[13] R. Naulin and M. Pinto, Projections for dichotomies in linear differential equations, *Applicable Analysis* **69** (3–4) (1998), 239–255.

[14] R. Naulin and M. Pinto, Dichotomies and asymptotic solutions of nonlinear differential systems, *Nonlinear Analysis and Applications, T.M.A.* **23** no. 7 (1994), 871–882.

[15] R. Naulin and M. Pinto, Roughness of (h, k)-dichotomies, *J. Diff. Eqns.* **118** no. 1 (1995), 20–35.

[16] R. Naulin and M. Pinto, Stability of discrete dichotomies for linear difference systems, *J. of Difference Eqns. and Appl.* **3** (1995), 101–123.

[17] R. Naulin and M. Pinto, Asymptotic integration of second order differential systems, (1996), (*preprint*).

[18] N. Onuchic, Asymptotic relationships at infinity between the solutions of two systems of ordinary differential equations, *J. Diff. Eqns.* **3** (1967), 47–58.

[19] K. J. Palmer, Exponential separation, exponential dichotomy and spectral theorems for linear systems of ordinary diferential equations, *J. Diff. Eqns.* **45** (1982), 324–345.

[20] M. Pinto, Dichotomy and asymptotic integration, *Contributions USACH* (1992), 13–22.

[21] M. Pinto, Asymptotic integration of a system resulting from the perturbation of an h-system, *J. Math. Anal. and App.* **131** (1988), 194–216.

[22] M. Pinto, Dichotomies and asymptotic formulas for the solutions of differential equations, *J. Math Anal. Appl.* **195** (1995), 16–31.

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