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Some remarks on Finsler vector bundles

By TADASHI AIKOU (Kagoshima)

Abstract. Let (E, F) be a Finsler vector bundle modeled on a Minkowski space $(\mathbb{V}, f)$. Then, by definition, the structure group of E is reducible to the isometry group G_f of f. In this paper, we shall give a condition for a Finsler vector bundle (E, F) to be a Riemannian vector bundle in terms of G_f.

1. Introduction

Recently Finsler geometry became an important branch in differential geometry, and numbers of articles in this field have been published (cf. Abate–Patrizio [1], Bao–Chern [5], Chern [6], Kobayashi [8], Shen [10], e.t.c.). In particular, the notion of Finsler manifolds modeled on a Minkowski space due to Ichijyō [7] is an important object in Finsler geometry. This is a natural generalization of so-called Berwald spaces (or affinely connected spaces)(cf. Matsumoto [9]). In the present paper, we shall extend this notion to a vector bundles over a smooth manifold. In the case of complex Finsler manifolds, we have investigated in Aikou [2], [3].

Let $\mathbb{V}$ be an n-dimensional vector space with a Minkowski norm f. If we denote by $GL(\mathbb{V})$ the general linear transformation group of $\mathbb{V}$, we can define a natural action of $GL(\mathbb{V})$ to f. If we denote by G_f the isometry group of f, we know that G_f is isomorphic to a closed subgroup of the orthogonal group $O(n)$ (cf. [7]).

Let $\pi : E \to M$ be a vector bundle of rank $E = n$ over a connected C^∞-manifold M. A *Finsler structure* $F : E \to \mathbb{R}$ on E is a smooth

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assignment for each point $x \in M$ a Minkowski norm F_x on each fibre $\pi^{-1}(x) := E_x$. Roughly speaking, a Finsler vector bundle (E, F) is said to be *modeled on a Minkowski space* $(\mathbb{V}, f)$ if E admits a G_f-structure and, moreover, each fibre $E_x = \pi^{-1}(x)$ is isometric to $(\mathbb{V}, f)$ (Proposition 3.1). We note that a Riemannian vector bundle is modeled on an inner product space. Since the Lie group G_f is isomorphic to a closed subgroup of $O(n)$, it is a natural question whether G_f can be maximum, that is, $G_f \cong O(n)$.

In this short note, we shall give a condition for (E, F) to be a Riemannian vector bundle in terms of the isometry group G_f.

2. Minkowski space

We shall recall the notion of Minkowski space. For an introduction to Minkowski space and some examples, see Thompson [12].

A vector space $\mathbb{V}$ is called a *Minkowski space* if it admits a function $f : \mathbb{V} \to \mathbb{R}$ satisfying the following conditions:

1. $f(\xi) \geq 0$, and $f(\xi) = 0$ if and only if $\xi = \mathbb{O}$.
2. $f(\xi)$ is C^∞ on $\mathbb{V} - \{\mathbb{O}\}$, and f is continuous on $\mathbb{V}$.
3. $f(\lambda\xi) = \lambda f(\xi)$ for $\forall \xi \in \mathbb{V}$ and $\forall \lambda > 0$.
4. f is strongly convex.

If we denote by $(\xi^1, \dots, \xi^n)$ the coordinate on $\mathbb{V}$ with respect to a fixed bases $\{e_1, \dots, e_n\}$, the strong convexity means that

$$\frac{1}{2} \sum_{A,B=1}^{n} \frac{\partial^2 f^2}{\partial \xi^A \partial \xi^B} X^A X^B > 0$$

for $\forall \xi \in \mathbb{V} - \{\mathbb{O}\}$ and $\forall (X^1, \dots, X^n) \neq \mathbb{O}$. The norm $\|\xi\|$ of $\forall \xi \in \mathbb{V}$ is defined by $\|\xi\| = f(\xi)$. For a Minkowski norm f, we define the *indicatrix* $\mathcal{D}_f \subset \mathbb{V}$ of f by

$$\mathcal{D}_f = \{\xi \in \mathbb{V};\ \|\xi\| < 1\}.$$

$\mathcal{D}_f$ is a bounded and strictly convex domain centered at the origin $\mathbb{O}$. Since

$$[f(\xi)]^2 = \frac{1}{2} \sum_{A,B=1}^{n} \frac{\partial^2 f^2}{\partial \xi^A \partial \xi^B} (\lambda\xi) \xi^A \xi^B > 0$$

for $\forall \lambda \in \mathbb{R}$, it is easily shown that the following conditions are equivalent:

1. $\mathcal{D}_f$ is linear isomorphic to the unit ball, that is, there exists $\exists(g^A_B) \in GL(n, \mathbb{R})$ satisfying

$$\mathcal{D}_f = \left\{ \sum_{A,B=1}^{n} (g^A_B \xi^B)^2 < 1 \right\}.$$

2. f is C^∞ on $\mathbb{V}$.
3. $f(\xi) = \sqrt{(\xi, \xi)}$ for an inner product $(\,\cdot\,,\,\cdot\,)$ on $\mathbb{V}$.

For a Minkowski norm f on $\mathbb{V}$, we denote by G_f the isometry group of f:

$$G_f = \{A \in GL(\mathbb{V});\ f(A\xi) = f(\xi) \text{ for } \forall \xi \in \mathbb{V}\}.$$

Because of the continuity of f, it is proved that G_f is a compact Lie group, and so it is isomorphic to a closed subgroup of the orthogonal group $O(n)$. Since $f(g\xi) = f(\xi) = 1$ for $\forall \xi \in \mathbb{S}_f = \partial\mathcal{D}_f$ and $\forall g \in G_f$, G_f acts on the f-sphere $\mathbb{S}_f$. The action is transitive if and only if $(\mathbb{V}, f)$ is an inner product space(cf. [2]). We shall give another characterization of inner product spaces.

Proposition 2.1. *Let $(\mathbb{V}, f)$ be Minkowski space. Then $(\mathbb{V}, f)$ is an inner product space if and only if the isometry group G_f is isomorphic to the orthogonal group $O(n)$.*

Proof. Since G_f is compact, there exists a bi-invariant Haar measure dg. Then, for an arbitrary inner product $(\,\cdot\,,\,\cdot\,)$, we define a G_f-invariant inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ by

$$\langle \xi, \eta \rangle = \int_{G_f} (g\xi, g\eta) dg. \tag{2.1}$$

If we put $f_0(\xi) = \sqrt{\langle \xi, \xi \rangle}$, the indicatrix $\mathcal{D}_{f_0}$ of f_0 is the open unit ball centered at the origin $\mathbb{O}$ with $G_{f_0} \cong O(n)$. The group G_{f_0} acts on the unit sphere $\mathbb{S}^{n-1} = \partial\mathcal{D}_{f_0}$ transitively. We can assume without loss of generality that $\mathcal{D}_f \cap \mathcal{D}_{f_0} \neq \phi$, because if it is necessary we multiply the inner product $(\,\cdot\,,\,\cdot\,)$ in (2.1) by a positive constant. Let ξ_0 be a fixed point in $\mathcal{D}_f \cap \mathcal{D}_{f_0}$.

We suppose that $G_f \cong O(n)$. Then G_f also acts on $\mathbb{S}^{n-1}$ transitively. For an arbitrary point $\eta \in \mathbb{S}^{n-1}$, there exists a $g \in G_f$ satisfying $\eta = g\xi_0$.

Then we have $f(\eta) = f(g\xi_0) = f(\xi_0) = 1$. Hence $\eta \in \mathbb{S}_f$, from which we have $f(\xi) = \sqrt{\langle \xi, \xi \rangle}$. The converse is true. $\square$

The group $GL(\mathbb{V})$ also acts on the set of Minkowski norms on $\mathbb{V}$ as follows. For $\forall A \in GL(\mathbb{V})$ and a Minkowski norm f, we define a Minkowski norm A^*f by $(A^*f)(\xi) = f(A\xi)$ for $\xi \in \mathbb{V}$. By this action, a Minkowski norm f is invariant by G_f, and moreover by Proposition 2.1, a Minkowski space $(\mathbb{V}, f)$ is an inner product space if and only if $G_f \cong O(n)$.

In the following, we shall identify $GL(\mathbb{V})$ with the general linear group $GL(n, \mathbb{R})$ with respect to a fixed bases $\{e_1, \ldots, e_n\}$ of $\mathbb{V}$. Under this identification, we also denote by G_f the closed subgroup of $O(n)$.

3. Finsler vector bundles

Let $\pi : E \to M$ be a vector bundle with $\operatorname{rank} E = n$ over a connected C^∞-manifold M. We denote by (x, y) a point of the total space E, where x means the point of the base manifold M and y the point of the fibre $\pi^{-1}(x)$.

On each fibre E_x, if a Minkowski norm $F_x(y)$ is assigned smoothly, the function $F(x, y)$ defined by $F(x, y) = F_x(y)$ is called a Finsler structure on E.

Definition 3.1. A function $F : E \to \mathbb{R}$ is called a *Finsler structure* if it satisfies the following conditions:

1. $F(x, y) \geq 0$, and $F(x, y) = 0$ if and only if $y = \mathbb{O}$,
2. $F(x, y)$ is C^∞ on the outside of the zero-section and continuous on E,
3. $F(x, \lambda y) = \lambda F(x, y)$ for $\forall \lambda > 0$ and $\forall y \in E_x$.
4. The restriction F_x of F to each fibre E_x is strongly convex for all $x \in M$.

A Finsler structure F is said to be *Riemannian structure* if F is the norm function defined by an inner product g on E: $F(x, y) = \sqrt{g(y, y)}$.

Let $\{s_1, \ldots, s_n\}$ be a local frame field of E. If we denote by $(y^1, \ldots, y^n)$ the fibre coordinate of E_x, the strong convexity condition is written as

$$\frac{1}{2} \sum_{A,B=1}^{n} \frac{\partial^2 F^2}{\partial y^A \partial y^B} X^A X^B > 0$$

for $\forall y \in E_x - \{\mathbb{O}\}$ and $\forall (X^1, \ldots, X^n) \neq \mathbb{O}$.

We suppose that the structure group $GL(n, \mathbb{R})$ of E is reducible to the Lie group G_f. Let $\{t_A\} = \{t_1, \ldots, t_n\}$ be a frame field on E adapted to the G_f-structure. Identifying each fibre E_x $(x \in M)$ with $\mathbb{V}$, we define an isomorphism $T_x : E_x \to \mathbb{V}$ by

$$T_x y = \sum_A y^A e_A$$

for $\forall y = \sum y^A t_A \in E_x$.

Definition 3.2 ([6]). If a Finsler vector bundle (E, F) satisfies the following conditions, then (E, F) is said to be *modeled on a Minkowski space* $(\mathbb{V}, f)$.

1. The structure group $GL(n, \mathbb{R})$ of E is reducible to G_f.
2. The norm $\|y\|_F = F(x, y)$ of $y \in E_x$ is given by $\|y\|_F = f(T_x y)$.

Remark 3.1. Since $\{t_A\}$ is an adapted frame field of the G_f-structure and f is G_f-invariant, the definition of $\|y\|_F$ is well-defined.

Let ∇ be a connection on E. The connection forms $\Pi = (\Pi^A_B)$ and $\Theta = (\Theta^A_B)$ with respect to the frames $\{s_A\}$ and $\{t_A\}$ are defined y

$$\nabla s_B = \sum_A \Pi^A_B s_A, \quad \nabla t_B = \sum_A \Theta^A_B t_A,$$

respectively. If we denote by $s_A = \sum t_B T^B_A(x)$ the change of local frame field, the connection forms Π and Θ satisfy the relation

$$\Theta^A_B = \sum_C T^A_C \left(dS^C_B + \sum_D \Pi^B_D S^D_C \right), \tag{3.1}$$

where (S^A_B) is the inverse of (T^A_B).

A connection ∇ on E said to be *adapted* to the G_f-structure, if its connection form $\Theta = (\Theta^A_B)$ with respect to an adapted frame $\{t_A\}$ is a 1-form with values in the Lie algebra $\mathfrak{g}$ of G_f. Since Θ is a 1-form with values in $\mathfrak{g}$, we have $\exp(t\Theta) \in G_f$ for $\forall t \in \mathbb{R}$. Hence we have $f(\exp(t\Theta)\xi) = f(\xi)$ for $\forall \xi \in \mathbb{V}$. Differentiating at $t = 0$, we have

$$\sum_{A,B} \frac{\partial f}{\partial \xi^A} \Theta^A_B \xi^B = 0. \tag{3.2}$$

The second condition in Definition 3.2 implies that each fibre (E_x, F_x) is isometric to (E_{x_0}, F_{x_0}). In fact, the equation (3.1) and (3.2) imply

$$\frac{d}{dt}F(c(t), y(t)) = 0. \tag{3.3}$$

for any parallel field $y(t)$ with respect to ∇ along a curve $c(t)$. Hence the connection ∇ preserves the metric F. By the well-known theorem due to Szabó [11], the connection ∇ is a metrical connection of a Riemannian structure. We shall show an example of Finsler vector bundle which is modeled on a Minkowski space.

Example 3.1. Let (E_1, g_1) and (E_2, g_2) be Riemannian vector bundles over a manifold M of rank $E_1 = n_1$ and rank $E_2 = n_2$ respectively. (E_1, g_1) and (E_2, g_2) are modeled on an inner product space $(\mathbb{V}_1, f_1)$ and $(\mathbb{V}_2, f_2)$ respectively. The direct sum $E = E_1 \oplus E_2$ with the Riemannian structure $g = \begin{pmatrix} g_1 & O \\ O & g_2 \end{pmatrix}$ admits an $O(n_1) \times O(n_2)$-structure. We define a Finsler structure F on E by

$$[F(x,y)]^2 = \frac{1}{2}\left\{\|y\|^2 + \sqrt{\|y\|^4 + 4t\,\|y_2\|^4}\right\}$$

for $y = y_1 \oplus y_2 \in E$, where $t \geq 0$ is sufficiently small so that F is strong convex, and we put $\|y\|^2 = g(y,y)$ and $\|y_2\|^2 = g_2(y_2, y_2)$. This bundle (E, F) is modeled on $(\mathbb{V}_1 \oplus \mathbb{V}_2, f)$ with

$$[f(\xi)]^2 = \frac{1}{2}\left\{\|\xi\|^2 + \sqrt{\|\xi\|^4 + 4t\,\|\xi_2\|^4}\right\},$$

where we put $\|\xi\|^2 = [f_1(\xi_1)]^2 + [f_2(\xi_2)]^2$ and $\|\xi_2\|^2 = [f_2(\xi_2)]^2$ for $\xi = \xi_1 \oplus \xi_2 \in \mathbb{V}_1 \oplus \mathbb{V}_2$. (E, F) is reducible to a Riemannian vector bundle if and only if $t = 0$.

Suppose that (E, F) is modeled on a Minkowski space $(\mathbb{V}, f)$. As stated in the previous section, the group G_f is isomorphic to a closed subgroup of $O(n)$ for an arbitrary Minkowski norm f. The main result of this paper, a characterization of Riemannian vector bundle in terms of the Lie group G_f, is the following:

Theorem 3.1. *Let (E, F) be a Finsler vector bundle modeled on a Minkowski space $(\mathbb{V}, f)$. Then (E, F) is a Riemannian vector bundle if and only if the isometry group G_f is isomorphic to the orthogonal group $O(n)$.*

Proof. Since each fibre (E_x, F_x) is isometric to a fixed Minkowski space (E_{x_0}, F_{x_0}), the proof of this theorem is obtained from Proposition 2.1. $\square$

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TADASHI AIKOU
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
FACULTY OF SCIENCE
KAGOSHIMA UNIVERSITY
KORIMOTO 1–21–35
KAGOSHIMA 890–0065
JAPAN

E-mail: aikou@sci.kagoshima-u.ac.jp

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