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On Chen immersions into Lorentzian space forms with nonflat normal space

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Abstract. Let $f : M^m \to \widetilde{M}_1^{m+2}(c)$ be a smooth totally geodesic isometric immersion from an m-dimensional connected Riemannian manifold M^m into an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$. Let ξ be a nonparallel time-like normal vector field on M^m. By using the normal exponential map we define, for some $t \in \mathbb{R}$, a push-out map $f_t(x) = \exp(x, t\xi(x))$ into the Lorentzian space form $\widetilde{M}_1^{m+2}(c)$, where $x \in M$. We show that the map f_t is a nontrivial Chen immersion with nonflat normal bundle under some conditions on the components of the normal connection form. We construct some examples.

1. Introduction

The notion of an $\mathcal{A}$-submanifold of a Riemannian space was introduced by B.Y. Chen in [1, p. 203] and an $\mathcal{A}$-submanifold later became known as a Chen submanifold. Chen submanifolds and the generalized Chen submanifolds (in [5]) of Riemannian spaces were investigated by many mathematicians.

Chen surfaces of an 4-dimensional Minkowski space were first studied by B. Rouxel in [4]. Later, in [4], a pseudo-Riemannian version of the definition of a Chen submanifold of a pseudo-Riemannian space was given and a characterization of Chen submanifolds in pseudo-Riemannian spaces was obtained. Then generalized Chen submanifolds of pseudo-Riemannian spheres and pseudo-hyperbolic spaces were investigated and many examples were given in [2].

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There are many examples of Chen surfaces of pseudo-Riemannian spaces with nonflat normal bundle (see [2], [4]). However, one can construct higher dimensional Chen submanifolds of pseudo-Riemannian spaces with flat normal bundle or nonflat normal bundle by using certain product submanifolds.

In [3], by using push-out maps in a nonparallel normal direction, we studied the construction of Chen immersions which are not product, into space forms with nonflat normal bundle. It is natural to investigate the same problem in Lorentzian space forms by using a specific normal direction. More precisely, our aim is to determine Chen immersions into Lorentzian space forms with nonflat normal bundle which are not product immersions. We start with a totally geodesic immersion f from an m-dimensional Riemannian manifold into an $(m+2)$-dimensional Lorentzian space form and a nonparallel time-like unit normal vector field to define a push-out map f_t into the Lorentzian space form. We show that the push-out map f_t is a nontrivial Chen immersion with nonflat normal bundle under some conditions on the components of the normal connection form of f. We also build up some examples.

2. Preliminaries

Let $\widetilde{M}_q^m$ be an m-dimensional pseudo-Riemannian manifold with pseudo-Riemannian metric tensor $\tilde{g}$ of index q. Denoting by $\langle\, ,\, \rangle$ the associated nondegenerate inner product on $\widetilde{M}_q^m$, a tangent vector X to $\widetilde{M}_q^m$ is said to be *space-like* if $\langle X, X\rangle > 0$ (or $X = 0$), *time-like* if $\langle X, X\rangle < 0$ or *light-like* (*null*) if $\langle X, X\rangle = 0$ and $X \neq 0$.

A smooth isometric immersion $f : M^m \to \widetilde{M}_q^{m+n}$ from an m-dimensional Riemannian manifold M into an $(m+n)$-dimensional pseudo-Riemannian manifold $\widetilde{M}_q^m$ is called a space-like immersion. Let X and Y be vector fields on M^m and let ξ be a normal vector field on M^m in $\widetilde{M}_q^{m+n}$. Then the *Gauss* formula and the *Weingarten* formula are, given as

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y) \quad \text{and} \quad \widetilde{\nabla}_X \xi = -A_\xi(X) + \nabla^\perp_X \xi,$$

respectively, where $\widetilde{\nabla}$ is the Riemannian connection of $\widetilde{M}_q^{m+n}$, ∇ and $\nabla^\perp$ are the induced Riemannian connection of M and the normal connection of M^m in $\widetilde{M}_q^{m+n}$, h is the second fundamental form of M in $\widetilde{M}_q^{m+n}$ and A_ξ

is the shape operator of M with respect to the normal vector ξ. However, the Gauss and Weingarten formulas yield

$$\langle A_\xi(X), Y\rangle = \langle h(X,Y), \xi\rangle. \tag{1}$$

A normal vector field ξ is said to be parallel in the normal space if $\nabla^\perp_X \xi = 0$ identically on M.

Let $f : M^m_p \to \widetilde{M}^{m+n}_q$ be a smooth immersion from an m-dimensional pseudo-Riemannian manifold M^m_p of index p into an $(m+n)$-dimensional pseudo-Riemannian manifold $\widetilde{M}^{m+n}_q$. Let ξ be a normal vector field on M such that $\langle \xi, \xi\rangle \neq 0$ which means that ξ is a nonzero and nonnull vector. We choose an orthonormal local basis $\xi_1, \ldots, \xi_n$ for $T^\perp M$ such that $\xi_1 = \xi/\|\xi\|$. Then the *allied vector field* $\mathcal{A}(\xi)$ of ξ is defined by

$$\mathcal{A}(\xi) = \sum_{i=2}^{n} \varepsilon_i \operatorname{trace}(A_\xi A_{\xi_i})\xi_i,$$

where $\varepsilon_i = \langle \xi_i, \xi_i\rangle = \pm 1$. In particular, $\mathcal{A}(H)$ is the allied vector of the mean curvature vector H and it is called the *allied mean curvature vector* of M^m_p in $\widetilde{M}^{m+n}_q$. Especially, if $n = 2$ then the allied mean curvature vector of M^m_p in $\widetilde{M}^{m+n}_q$ is

$$\mathcal{A}(H) = \varepsilon_2 \operatorname{trace}(A_H A_{\xi_2})\xi_2. \tag{2}$$

An immersion $f : M^m_p \to \widetilde{M}^{m+n}_q$ is called an *$\mathcal{A}$-immersion* or a *Chen immersion* if $H = 0$ or $\langle H, H\rangle \neq 0$ and $\mathcal{A}(H)$ vanishes identically. This definition was studied in [4] as a pseudo-Riemannian version of Chen's definition given in [1, p. 203] for Riemannian spaces. The definition of the generalized Chen submanifolds in a pseudo-Riemannian manifold can be found in [2]. It is clear that the class of Chen immersions into a pseudo-Riemannian manifold $\widetilde{M}^{m+n}_q$ contains all minimal immersions and pseudo-umbilical immersions, and also all immersions for which $\dim N_1 \leq 1$, where N_1 is the first normal space of M^m_p in $\widetilde{M}^{m+n}_q$, in particular it includes all hypersurfaces. These Chen immersions are said to be *trivial Chen immersions*. Many examples of Chen immersions into pseudo-Riemannian manifolds were given in [2], [4].

Let $\widetilde{M}^m_q(c)$ be an m-dimensional connected pseudo-Riemannian manifold of index q and of constant curvature c, which is called an *indefinite* *space form*. According as to $c > 0$, $c = 0$ or $c < 0$, it is a pseudo-Riemannian sphere $\mathbb{S}^m_q(c)$, a pseudo-Euclidean space $\mathbb{R}^m_q$, or a pseudo-hyperbolic space $\mathbb{H}^m_q(c)$, respectively. For the index $q = 1$, $\mathbb{S}^m_1(c)$, $\mathbb{R}^m_1$ and $\mathbb{H}^m_1(c)$ are, respectively, called the de Sitter space-time, the Minkowski space-time and the anti-de Sitter space-time. The indefinite space form $\widetilde{M}^m_1(c)$ is called a *Lorentzian space form*. If $q = 0$ then $\widetilde{M}^m_q(c)$ is a Riemannian space form. Without loss of generality we can suppose that the constant curvature c of $\widetilde{M}^m_1(c)$ is equal to 1, 0, -1 according to whether $c > 0$, $c = 0$ or $c < 0$.

Let $\mathbb{R}^m_q$ be an m-dimensional pseudo-Euclidean space with metric tensor given by

$$\tilde{g} = -\sum_{i=1}^{q}(dx_i)^2 + \sum_{i=q+1}^{m}(dx_i)^2,$$

where $(x_1, \dots, x_m)$ is a rectangular coordinate system of $\mathbb{R}^m_q$. So $(\mathbb{R}^m_q, \tilde{g})$ is a flat pseudo-Riemannian manifold of index q. For the pseudo-Riemannian sphere and the pseudo-hyperbolic space we put

$$\mathbb{S}^m_q(1) = \{x \in \mathbb{R}^{m+1}_q \mid \langle x, x\rangle = 1\}$$

and

$$\mathbb{H}^m_q(-1) = \{x \in \mathbb{R}^{m+1}_{q+1} \mid \langle x, x\rangle = -1\}.$$

Hence the hyperbolic space $\mathbb{H}^m(-1)$ is defined by

$$\mathbb{H}^m(-1) = \{x \in \mathbb{R}^{m+1}_1 \mid \langle x, x\rangle = -1 \text{ and } x_1 > 0\},$$

where x_1 is the first coordinate in $\mathbb{R}^{m+1}_1$.

Let $f : M^m \to \widetilde{M}^{m+2}_1(c)$ be a smooth isometric immersion from an m-dimensional connected Riemannian manifold M^m into an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}^{m+2}_1(c)$. Let ξ, η be a local orthonormal normal basis of M^m in $\widetilde{M}^{m+2}_1(c)$ with signature $\varepsilon_1 = \langle \xi, \xi\rangle = -1$ and $\varepsilon_2 = \langle \eta, \eta\rangle = 1$. Let $X_1, \dots, X_m$ be a local tangent basis on M and s the normal connection form for $\nabla^\perp$ defined by $s(X_i) = \langle \nabla^\perp_{X_i}\xi, \eta\rangle$. Considering the signatures of ξ and η and $\langle \xi, \eta\rangle = 0$, we see that $\nabla^\perp_{X_i}\xi = s(X_i)\eta$ and $\nabla^\perp_{X_i}\eta = s(X_i)\xi$. Here it is observed that if either ξ or η is parallel in the normal space then the normal connection form for $\nabla^\perp$ is zero. We therefore suppose that ξ and η are nonparallel.

Denoting by s_i the components of the connection form s, the covariant derivative of the 1-form s is defined by $s_{ij} = (\nabla_{X_j} s)(X_i) = X_j(s_i) - s(\nabla_{X_j} X_i)$. As the ambient space is a space form, the Ricci equation can be written as

$$\langle R^\perp(X,Y)\xi,\eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle,$$

where $R^\perp$ denotes the normal curvature tensor of the normal connection $\nabla^\perp$ and $[A_\xi, A_\eta] = A_\xi A_\eta - A_\eta A_\xi$. Also, we can see that $s_{ij} = \langle \nabla^\perp_{X_j}\nabla^\perp_{X_i}\xi - \nabla^\perp_{\nabla_{X_j}X_i}\xi, \eta\rangle$. So we can express the Ricci equation as

$$s_{ji} - s_{ij} = \langle R^\perp(X_i, X_j)\xi, \eta\rangle = \langle [A_\xi, A_\eta]X_i, X_j\rangle.$$

If the normal curvature tensor $R^\perp$ of the normal connection $\nabla^\perp$ vanishes then the normal connection is said to be flat.

From now on, for simplicity of the calculations we take a local isothermal coordinate system $(x_1, \dots, x_m)$ of M such that $\partial_i = \frac{\partial}{\partial x_i} = \varphi X_i$, $i = 1, \dots, m$, where $X_1, \dots, X_m$ forms an orthonormal tangent basis on M and φ is a positive function on some open set in M. Thus the components of the first fundamental form g on M are $\langle f_i, f_j\rangle = \varphi^2\delta_{ij}$, $i, j = 1, \dots, m$. In terms of the chosen tangent basis it is easily seen that

$$\nabla_{X_j} X_i = \sum_{k=1}^{m} \gamma_{ij}^k X_k, \tag{3}$$

where $\gamma_{ij}^k = -\frac{1}{\varphi}(X_j(\varphi)\delta_i^k - \Gamma_{ij}^k)$ and Γ_{ij}^k are the Christoffel symbols of M, and hence $s(\nabla_{X_j} X_i) = \sum_{k=1}^m \gamma_{ij}^k s_k$. So we have

$$X_j(s_i) = s_{ij} + \sum_{k=1}^{m} \gamma_{ij}^k s_k. \tag{4}$$

Let ξ be a unit time-like normal vector field on M^m in $\widetilde{M}_1^{m+2}(c)$. The normal exponential mapping of M^m in $\widetilde{M}_1^{m+2}(c)$ in the direction ξ can be given by

$$\exp(x, t\xi) = a(t)f(x) + b(t)\xi(x),$$

where $x \in M$ and $t \in \mathbb{R}$. The functions $a(t)$ and $b(t)$ are given by $a(t) = 1$ $b(t) = t$ if $c = 0$; $a(t) = \cosh t$, $b(t) = \sinh t$ if $c = 1$, and $a(t) = \cos t$, $b(t) = \sin t$ if $c = -1$.

3. Chen immersions

Let $f : M^m \to \widetilde{M}_1^{m+2}(c)$ be a smooth totally geodesic isometric immersion from an m-dimensional connected Riemannian manifold M^m into an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$. For some $t \in \mathbb{R}$, we define a map $f_t : M^m \to \widetilde{M}_1^{m+2}(c)$, which is the push-out of M^m in a nonparallel time-like unit normal direction ξ, by

$$f_t(x) = \exp(x, t\xi).$$

Let us set

$$R^* = \begin{cases} \mathbb{R} & \text{if } c = 0, 1 \\ \mathbb{R} \setminus \left\{ (2n-1)\dfrac{\pi}{2} : \forall n \in \mathbb{Z} \right\} & \text{if } c = -1. \end{cases}$$

The tangent vectors to the push-out f_t at $(x_1, \dots, x_m)$ are expressed as

$$(f_t)_i = \frac{\partial f_t}{\partial x_i} = af_i + b\xi_i, \quad i = 1, \dots, m,$$

where f_i, $\xi_i, \dots$ denote the derivatives of f and ξ with respect to x_i. As f is totally geodesic we have $A_\xi \equiv 0$ and $A_\eta \equiv 0$. Therefore

$$(f_t)_i = \varphi(aX_i + bD_{X_i}\xi) = \varphi(aX_i + b\nabla^\perp_{X_i}\xi) = \varphi(aX_i + bs_i\eta), \quad i = 1, \dots, m,$$

where D is the covariant differentiation in $\mathbb{R}_1^{m+2}$ or $\mathbb{R}_1^{m+3}$. Hence

$$\langle (f_t)_i, (f_t)_j \rangle = \varphi^2(a^2\delta_{ij} + b^2 s_i s_j). \tag{5}$$

Here it is seen that the tangent vectors $(f_t)_i$, $i = 1, \dots, m$, are all space-like. So the metric tensor g_t with the components $(g_t)_{ij} = \varphi^2(a^2\delta_{ij} + b^2 s_i s_j)$ induced by f_t is positive definite, that is, f_t is a space-like immersion.

In order to calculate the reciprocal of g_t we need the following

Lemma 3.1 ([3]). *Let $E = I + v^T v$ be an $m \times m$ matrix, where I is the $m \times m$ identity matrix and $v = (v_1, \dots, v_m) \in \mathbb{R}^m$. Then E has two distinct eigenvalues 1 and $1 + \|v\|^2$ with multiplicities $m-1$ and 1, respectively, and further $\det E = 1 + \|v\|^2$, and the matrix $I - \frac{1}{\det E} v^T v$ is the inverse of E.*

One can show that the push-out map f_t is an immersion for each $t \in R^*$. By considering Lemma 3.1 the determinant and the components of reciprocal tensor of g_t are obtained as $\det g_t = a^{2(m-1)}\varphi^{2m}(a^2+b^2\hat{s}^2)$ and $(g_t)^{-1}_{ij} = \frac{1}{\alpha^2\varphi^2 a^2}(\alpha^2\delta_{ij} - b^2 s_i s_j)$, respectively, where $\hat{s}^2 = s_1^2 + \cdots + s_m^2$ and $\alpha^2 = a^2 + b^2\hat{s}^2$.

If we use (3) and (4), f, being totally geodesic the second derivatives of f_t are obtained as

$$\begin{aligned}(f_t)_{ij} &= \frac{\partial^2 f_t}{\partial x_i \partial x_j}\\ &= \frac{\partial\varphi}{\partial x_j}(aX_i + bs_i\eta) + \varphi^2\left(aD_{X_j}X_i + bX_j(s_i)\eta + bs_i\nabla^{\perp}_{X_j}\eta\right)\\ &= X_j(\varphi)(f_t)_i + \varphi^2\left\{\sum_{k=1}^m \gamma_{ij}^k(aX_k + bs_k\eta) - ac\delta_{ij}f + b(s_{ij}\eta + s_is_j\xi)\right\}\\ &= \sum_{k=1}^m \left(X_j(\varphi)\delta_{ik} + \varphi\gamma_{ij}^k\right)(f_t)_k + b\varphi^2(s_{ij}\eta + s_is_j\xi) - ac\varphi^2\delta_{ij}f\\ &= \sum_{k=1}^m \Gamma_{ij}^k(f_t)_k + b\varphi^2(s_{ij}\eta + s_is_j\xi) - ac\varphi^2\delta_{ij}f, \qquad i,j = 1,\dots,m.\end{aligned} \tag{6}$$

Here we will show that the immersion f_t is Chen in the space form $\widetilde{M}_1^{m+2}(c)$ under some conditions on the components of the normal connection form s. For a fixed $t \in R^*$, since f_t has codimension two in $\widetilde{M}_1^{m+2}(c)$, we choose a local orthonormal normal basis to f_t in $\widetilde{M}_1^{m+2}(c)$ as

$$\zeta_1 = cbf - a\xi, \quad \zeta_2 = \frac{1}{\alpha}\left(\sum_{i=1}^m bs_iX_i - a\eta\right).$$

By considering $a(t)$ and $b(t)$, the normal vector ζ_1 is time-like and ζ_2 is space-like. We assume that, for the chosen $t \in R^*$, $b(t) \neq 0$, otherwise the map f_t is nothing but f which is a trivial Chen immersion because of being totally geodesic.

Now we will calculate the second fundamental forms of the push-out map f_t in the directions ζ_1 and ζ_2 which are represented by the matrices $(h_{ij}^{\zeta_1})$ and $(h_{ij}^{\zeta_2})$, respectively. Their entries are defined by $h_{ij}^{\zeta_1} = \langle (f_t)_{ij}, \zeta_1\rangle$ and $h_{ij}^{\zeta_2} = \langle (f_t)_{ij}, \zeta_2\rangle$. So, using (6) we get

$$h_{ij}^{\zeta_1} = cb\langle (f_t)_{ij}, f\rangle - a\langle (f_t)_{ij}, \xi\rangle = -ab\varphi^2(c\delta_{ij} + s_is_j)$$

and

$$h_{ij}^{\zeta_2} = \frac{1}{\alpha}\left\{b\sum_{\ell=1}^{m} s_\ell\langle (f_t)_{ij}, X_\ell\rangle - a\langle (f_t)_{ij}, \eta\rangle\right\}$$
$$= \frac{ab}{\alpha}\left\{\sum_{\ell=1}^{m} \varphi s_\ell \Gamma_{ij}^k \delta_{k\ell} - \left(\varphi\Gamma_{ij}^k s_k + \varphi^2 s_{ij}\right)\right\} = -\frac{ab\varphi^2}{\alpha} s_{ij}.$$

Theorem 3.1. *Let $f : M^m \to \widetilde{M}_1^{m+2}(c)$ be a smooth totally geodesic isometric immersion from an m-dimensional connected Riemannian manifold M^m into an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$. Then, for a fixed $t \in R^*$, the immersion $f_t : M^m \to \widetilde{M}_1^{m+2}(c)$ is Chen if the components s_i, of the normal connection form s of f satisfy the equations*

$$\sum_{i=1}^{m} s_{ii} = 0 \quad \textit{and} \quad \sum_{i,j=1}^{m} s_i s_j s_{ji} = 0. \tag{7}$$

Proof. Since $(g_t)_{ij}^{-1} = \frac{1}{\alpha^2\varphi^2 a^2}(\alpha^2\delta_{ij} - b^2 s_i s_j)$, by using (1) the shape operators of f_t in the directions ζ_1 and ζ_2 are obtained as

$$A_{\zeta_1} = g_t^{-1} h^{\zeta_1} = -\frac{b}{a\alpha^2}\left(\sum_{k=1}^{m}(\alpha^2\delta_{ik} - b^2 s_i s_k)(c\delta_{kj} + s_k s_j)\right)$$
$$= -\frac{b}{a\alpha^2}(c\alpha^2\delta_{ij} + s_i s_j)$$

and

$$A_{\zeta_2} = g_t^{-1} h^{\zeta_2} = -\frac{b}{a\alpha^3}\left(\sum_{k=1}^{m}(\alpha^2\delta_{ik} - b^2 s_i s_k)s_{kj}\right)$$
$$= -\frac{b}{a\alpha^3}\left(\alpha^2 s_{ij} - b^2\sum_{k=1}^{m} s_i s_k s_{kj}\right).$$

However, when we take the trace of the shape operators we get

$$\operatorname{trace} A_{\zeta_1} = -\frac{b}{a\alpha^2}(cm\alpha^2 + \hat{s}^2)$$

and

$$\operatorname{trace} A_{\zeta_2} = -\frac{b}{a\alpha^3}\left(\alpha^2\sum_{i=1}^{m} s_{ii} - b^2\sum_{i,j=1}^{m} s_i s_j s_{ji}\right) = 0$$

because of (7). For $c = 0$ and $c = 1$ $\operatorname{trace} A_{\zeta_1} \neq 0$ and for $c = -1$ $\operatorname{trace} A_{\zeta_1} \not\equiv 0$. Thus, at a point $x \in M$ such that $\operatorname{trace} A_{\zeta_1} \neq 0$ the mean curvature vector $\tilde{H}$ of the push-out f_t is parallel to ζ_1 and the allied mean curvature vector is $\mathcal{A}(\tilde{H}) = \|\tilde{H}\| \operatorname{trace}(A_{\zeta_1}A_{\zeta_2})\zeta_2$. We calculate $A_{\zeta_1}A_{\zeta_2}$ as

$$\begin{aligned} A_{\zeta_1}A_{\zeta_2} &= \frac{b^2}{a^2\alpha^5}\left(\sum_{\ell,k=1}^{m}(c\alpha^2\delta_{i\ell} + s_i s_\ell)(\alpha^2\delta_{\ell k} - b^2 s_\ell s_k)s_{kj}\right) \\ &= \frac{b^2}{a^2\alpha^5}\left(c\alpha^4 s_{ij} + (a^2 - cb^2\alpha^2)\sum_{k=1}^{m} s_i s_k s_{kj}\right). \end{aligned}$$

Taking trace it is seen that $\operatorname{trace} A_{\zeta_1}A_{\zeta_2} = 0$ because of (7). Therefore f_t is a nontrivial Chen immersion. By using the shape operators A_{ζ_1}, A_{ζ_2} and the Ricci equation, it is easily seen that the normal curvature of f_t is not identically zero. □

4. Construction of examples

We construct here some examples of Chen immersion, defined as in the previous section, into space forms $\widetilde{M}_1^{m+2}(c)$. To do this we consider a totally geodesic isometric immersion $f : M^m(c) \to \widetilde{M}_1^{m+2}(c)$ from an m-dimensional Riemannian space form $M^m(c)$ into an $(m+2)$-dimensional Lorentzian space form $\widetilde{M}_1^{m+2}(c)$ defined by

$$f(x_1, \dots, x_m) = \begin{cases} (0, x_1, \dots, x_m, 0) & \text{if } c = 0, \\ \dfrac{1}{r^2}(0, c(r^2 - 2), 2x_1, \dots, 2x_m, 0) & \text{if } c = \mp 1, \end{cases}$$

where $x_1, \dots, x_m \in \mathbb{R}$, $r^2 = 1 + c(x_1^2 + \cdots + x_m^2)$ and for $c = -1$, $x_1^2 + \cdots + x_m^2 < 1$.

From now on we will do all calculations for $c = \mp 1$. By calculation it is seen that the components of the induced metric tensor on $M^m(c)$ are obtained as $\langle f_i, f_j \rangle = \frac{4}{r^4}\delta_{ij}$, $i, j = 1, \dots, m$. This shows that the chosen coordinate system on M is isothermal and $\varphi = \frac{2}{r^2}$. Thus, $X_i = \frac{r^2}{2}\frac{\partial}{\partial x_i}$,

$i = 1, \ldots, m$, is a local orthonormal tangent basis on $M^m(c)$. In terms of this metric the Christoffel symbols are obtained as

$$\Gamma_{ij}^k = -\frac{2c}{r^2}(x_i\delta_{kj} + x_j\delta_{ik} - x_k\delta_{ij}). \tag{8}$$

For the normal space of $M^m(c)$ in $\widetilde{M}_1^{m+2}(c)$ an orthonormal local basis can, generally, be chosen as

$$\xi = (\cosh\theta, 0, \ldots, 0, \sinh\theta), \qquad \eta = (\sinh\theta, 0, \ldots, 0, \cosh\theta),$$

where $\theta = \theta(x_1, \ldots, x_m)$. We will find a θ which determines the unit normal vector ξ on $M^m(c)$ so that the immersion f_t as defined in previous sections is Chen. To do this we need to calculate the components s_i of the normal connection s and their covariant derivatives s_{ij}. So,

$$s_i = \left\langle \nabla^\perp_{X_i}\xi, \eta \right\rangle = \langle D_{X_i}\xi, \eta\rangle = \frac{r^2}{2}\left\langle \frac{\partial \xi}{\partial x_i}, \eta \right\rangle = \frac{r^2}{2}\frac{\partial\theta}{\partial x_i}, \tag{9}$$

that is, $s_i = \frac{r^2}{2}\theta_i$, $i = 1, \ldots, m$, and hence

$$X_j(s_i) = \frac{r^2}{2}\frac{\partial}{\partial x_j}\left(\frac{r^2}{2}\theta_i\right) = \frac{r^2}{2}\left(cx_j\theta_i + \frac{r^2}{2}\theta_{ij}\right).$$

Using (3) and (8) we have $\gamma_{ij}^k = -c(x_i\delta_{kj} - x_k\delta_{ij})$. Therefore, by using (4) we get

$$\begin{aligned} s_{ij} &= \frac{r^2}{2}\left(cx_j\theta_i + \frac{r^2}{2}\theta_{ij} + c\sum_{k=1}^m (x_i\delta_{kj} - x_k\delta_{ij})\theta_k\right) \\ &= \frac{r^4}{4}\left(\theta_{ij} - \sum_{k=1}^m \Gamma_{ij}^k\theta_k\right). \end{aligned}$$

Here it is clear that $s_{ij} = s_{ji}$ if and only if $\theta_{ij} = \theta_{ji}$. Thus, taking the trace of s_{ij}, the first equation of (7) turns out to be

$$\begin{aligned} \sum_{i=1}^m s_{ii} &= \frac{r^4}{4}\left(\sum_{i=1}^m \theta_{ii} - \sum_{i,k=1}^m \Gamma_{ii}^k\theta_k\right) \\ &= \frac{r^4}{4}\left(\sum_{i=1}^m \theta_{ii} + \frac{2c}{r^2}(2-m)\tilde{\theta}\right) = 0, \end{aligned}$$

where $\tilde{\theta} = \sum_{i=1}^m x_i \theta_i$, that is,

$$2c(2-m)\tilde{\theta} + r^2 \sum_{i=1}^m \theta_{ii} = 0. \tag{10}$$

Similarly, the second equation of (7) becomes

$$\begin{aligned}\sum_{i,j=1}^m s_i s_j s_{ji} &= \left(\frac{r^4}{4}\right)^2 \left(\sum_{i,j=1}^m \theta_i\theta_j\theta_{ji} - \sum_{i,j,k=1}^m \Gamma_{ij}^k \theta_i\theta_j\theta_k\right) \\ &= \left(\frac{r^4}{4}\right)^2 \left(\sum_{i,j=1}^m \theta_i\theta_j\theta_{ji} + \frac{2c}{r^2}\tilde{\theta}\hat{\theta}^2\right) = 0,\end{aligned}$$

where $\hat{\theta}^2 = \sum_{i=1}^m \theta_i^2$, so we have

$$2c\tilde{\theta}\hat{\theta}^2 + r^2 \sum_{i,j=1}^m \theta_i\theta_j\theta_{ji} = 0. \tag{11}$$

For $c = 0$, we can see by straightforward calculation that the equations (10) and (11) are valid. Thus the equations (10) and (11) are true for $c = -1, 0, 1$.

Finally, the solutions of the partial differential equations (10) and (11) which were studied for $m = 2$ in [6] give us the function θ, that is, the normal vector fields on $M^m(c)$ which are not parallel in the normal space of f in $\widetilde{M}_1^{m+2}(c)$ unless θ is a constant function on $M^m(c)$. However, some special solutions of the equations (10) and (11), for $m \geq 2$, were studied in [3]. Here we give some examples for θ.

Let ℓ be a positive integer such that $\ell \leq m/2$, $m \geq 2$. For $c = -1, 0, 1$, the function

$$\theta(x_1, \ldots, x_m) = \sum_{i=1}^{\ell} C_i \arctan \frac{x_{2i}}{x_{2i-1}}, \tag{12}$$

where $C_1, \ldots, C_\ell \in \mathbb{R}$, is a solution of (10) and (11) in the open domain $D = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_{2i-1} \neq 0,\ i = 1, \ldots, \ell\}$.

Also, for $c = 0$ we have another very simple solution

$$\theta(x_1, \ldots, x_m) = \sum_{i=1}^m C_i x_i \tag{13}$$

that gives a trivial Chen immersion f_t.

For $c = 0$, a linear combination of the solutions (12) and (13)

$$\theta(x_1, \ldots, x_{2\ell}, x_{2\ell+1}, \ldots, x_m) = \sum_{i=1}^{\ell} C_i \arctan \frac{x_{2i}}{x_{2i-1}} + \sum_{i=2\ell+1}^{m} C_i x_i$$

is also a solution of (1) and (11) in the open domain D, see [3]. If we change the coordinate system (for $c = 0$) to $x_{2i-1} = u_i \cos v_i$, $x_{2i} = u_i \sin v_i$ for $i = 1, \ldots, \ell$ then we have

$$\theta(u_1, v_1, \ldots, u_\ell, v_\ell, x_{2\ell+1}, \ldots, x_m) = \sum_{i=1}^{\ell} C_i v_i + \sum_{i=2\ell+1}^{m} C_i x_i$$

and for this θ the push-out map f_t is a nontrivial Chen immersion. For instance, for $m = 3$ we can rearrange the immersion f_t into Minkowski space as

$$f_t(u, v, w) = (t \cosh(c_1 v + c_2 w), u \cos v, u \sin v, w,\ t \sinh(c_1 v + c_2 w)).$$

However, from the solution (12), for $c = \pm 1$ and $m = 3$ we rearrange the immersion f_t as

$$f_t(u, v, w) = \Bigg(b(t) \cosh Cv,\ \frac{a(t)(u^2 + w^2 - c)}{1 + c(u^2 + w^2)},\ \frac{2a(t) u \cos v}{1 + c(u^2 + w^2)},$$
$$\frac{2a(t) u \sin v}{1 + c(u^2 + w^2)},\ \frac{2a(t) w}{1 + c(u^2 + w^2)},\ b(t) \sinh Cv \Bigg),$$

where $C \in \mathbb{R}$; $a(t) = \cosh t$, $b(t) = \sinh t$ if $c = 1$ and $a(t) = \cos t$, $b(t) = \sin t$ and $u^2 + w^2 < 1$ if $c = -1$.

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