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Bornologicity of certain spaces of bounded linear operators

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Abstract. The example of a nuclear Fréchet space E is given, for which the space $L(E, E)$ of bounded linear operators on E, when endowed with the topology of uniform convergence on bounded sets, is not bornological. It is shown further that for E the subspace of the product $\mathbb{R}^I$ over an uncountable index set I formed by the sequences with countable support, this locally convex topology on the space $L(E, E)$ is bornological.

1. Introduction and preliminaries

We consider spaces E, carrying a locally convex structure and hence a canonical convex bornological structure (a collection of bounded sets) given by the associated von Neumann bornology (the scalarly bounded sets). The space $L(E, E)$ of bounded linear operators on E may then be equipped with a natural locally convex structure, namely the one of uniform convergence on bounded sets. The bornologification of this topology plays an important role in connection with approximation problems studied in [4] that arise in different contexts of differential calculus in infinite dimensions, e.g. approximation of vector valued smooth functions (see [1]) and the question whether operational and kinematic tangent bundle of a given manifold modeled on an infinite dimensional vector space coincide ([9, Chapter VI]). As in general the zero neighbourhoods of the bornologification do not admit an explicit description, one is interested in the case where the natural topology of $L(E, E)$ is bornological. Trivially, the latter

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is true for the locally convex spaces $\mathbb{R}^I$ and $\mathbb{R}^{(I)}$, where the index set I is of non measurable cardinality. A less obvious example is given by the space $\mathbb{R}^I_{\aleph_0} \subseteq \mathbb{R}^I$ of sequences with countable support, as we will prove in Section 3.

If the topology of uniform convergence on bounded sets is bornological, then the same must hold for the direct summands E and $L(E, \mathbb{R})$, where the latter carries the topology of uniform convergence on bounded subsets of E, i.e. the strong topology. The question, whether this is sufficient, is answered to the negative, since in Section 2 we show the existence of a nuclear Fréchet space E, for which $L(E, E)$ is not bornological. That there exist Fréchet spaces with non bornological strong dual, was established by A. Grothendieck in 1954 ([5]).

Our example is the space $C^\infty(\mathbb{R})$ of infinitely often differentiable real-valued functions on $\mathbb{R}$ and a topological isomorphism

$$L(C^\infty(\mathbb{R}), C^\infty(\mathbb{R})) \cong C^\infty(\mathbb{R}, C^\infty(\mathbb{R})')$$

yields non-bornologicity of the former space once we know the same is true for the latter, obviously involving some concept of smoothness of maps into infinite dimensional locally convex spaces. An appropriate setting for this is the differential calculus mappings as developed in [3], of which we will give a brief outline in this section following the lines of the most recent exposition given in [9]. There exist approaches also for nonlinear objects ([2]) but we will essentially restrict to mappings between (manifolds modeled on open subsets of) vector spaces: For a curve $c : \mathbb{R} \to E$ into a locally convex space E we can define difference quotients and derivatives in each point in an obvious way. If all iterated derivatives exist then c is said to be *smooth* or C^∞. The set $C^\infty(\mathbb{R}, E)$ of smooth curves into a locally convex space E does not depend on the topology of E but only on the system of its bounded sets, its bornology, because the difference quotients of a smooth curve are even *Mackey convergent*, i.e., convergent in E_B for some bounded disk $B \subseteq E$, where E_B is the linear span of B in E endowed with the Minkowski seminorm p_B associated to B (see [6] for the theory of bornological spaces). A mapping between locally convex spaces is called smooth if it carries smooth curves to smooth curves. This definition of smoothness implies existence, smoothness and linearity of derivatives and the chain rule. The most important result is the *exponential law*

$$C^\infty(\mathbb{R} \times \mathbb{R}, E) \cong C^\infty(\mathbb{R}, C^\infty(\mathbb{R}, E))$$

which holds much more generally for smooth maps between smooth spaces, i.e. sets for which a family of smooth curves is given. The smooth maps between smooth spaces are again defined as those that carry smooth curves to smooth curves.

If E is *Mackey complete*, i.e. each sequence that is Cauchy in E_B for some bounded disk $B \subseteq E$ is converging Mackey in E, then scalarly smooth curves are smooth. In case E is a space of smooth mappings it even suffices to test by point evaluations. This is called the *differentiable uniform boundedness principle*. A *convenient vector space* is a locally convex space which is Mackey complete. Multilinear maps are smooth if and only if they are bounded. Furthermore, if we denote by $L(E_1, \dots, E_n; F)$ the space of bounded (smooth) n-linear maps between the convenient vector spaces $E_1 \times \cdots \times E_n$ and F, we have the corresponding exponential law

$$L(E \times F, G) \cong L(E, L(F, G))$$

for any three convenient vector spaces E, F, G. The bornology consisting of sets which are bounded on bounded subsets defines the structure of a convenient vector space on $L(E, F)$ for each two convenient vector spaces E and F and the exponential law is true also with respect to these structures. Instead of $L(E, \mathbb{R})$ we will write E'.

By Con we denote the category of convenient vector spaces and bounded linear maps. Note that the objects of this category are determined by their bornology, whereas their locally convex topology may vary over all locally convex topologies having the same system of bounded sets. The finest among them is bornological and we will call it the *bornological locally convex topology* of E. A convenient vector space E is said to be *reflexive* if $E \cong E''$ in Con by means of the natural map. As by [3, 5.4.7] for Fréchet spaces this is equivalent to reflexivity in the usual locally convex sense, $C^\infty(\mathbb{R})$ is reflexive as a convenient vector space.

On the space $C^\infty(\mathbb{R}^n, E)$, the topology of uniform convergence on compact sets in each partial derivative determines a convenient vector space structure for any convenient vector space and the exponential law holds for this structure. Since smoothness is tested along curves it is natural to consider on a convenient vector space E the topology induced by its C^∞-curves. This topology is called c^∞-topology and it is finer than any locally convex topology compatible with the bornology of E. Accordingly, a set which is dense with respect to this topology, is said to be c^∞-*dense*. The adherence of a subset $A \subseteq E$ with respect to the c^∞-topology coincides with the set of limits of sequences in the subset which are Mackey convergent (see [3, 2.3.10]) and is called the *Mackey adherence* of A and is denoted by A'. More generally, if an ordinal α is given, we can define the α-*th Mackey adherence* $A^{(\alpha)}$ of A inductively by $A^{(\alpha)} := \bigcup_{\beta<\alpha} A^{(\beta)}$ if α is a limit ordinal and $A^{(\alpha)} := (A^{(\alpha-1)})'$ if α has a predecessor. The closure of subsets in the c^∞-topology coincides with the sequential closure, i.e. the union of all adherences.

In the holomorphic setting, the basic spaces are the sets of all holomorphic mappings from the unit disk $\mathbb{D} \subseteq \mathbb{C}$ into a complex convenient vector space. Holomorphic mappings between complex convenient vector spaces are then defined as those that carry holomorphic curves to holomorphic curves. Again, multilinear mappings are holomorphic if and only if they are bounded and the exponential laws hold (c.f. [10] for the theory of holomorphic mappings). As in the real case, the dual of a complex convenient vector space will be denoted by E'.

2. Two isomorphy results and their consequences

Proposition 2.1 and Definition ([3, 5.1.1]). *For every smooth space X there exists a convenient vector space λX and a smooth map $\iota_X : X \to \lambda X$ with the property that every smooth map $g : X \to E$ into a convenient vector space factors as $g = \overline{g} \circ \iota_X$ with a unique linear* <u>Con</u>*-morphism $\overline{g} : \lambda X \to E$. The space λX is called the free convenient vector space over X. It can be constructed as the c^∞-closure of the linear subspace of $C^\infty(X, \mathbb{R})'$ generated by the point evaluations $\mathrm{ev}_x =: \iota_X(x)$ for $x \in X$.*

Proposition 2.2 ([3, 5.1.8]). *Let X be a finite-dimensional separable smooth manifold. Then the free convenient vector space λX over X equals $C^\infty(X, \mathbb{R})'$.*

We will obtain as a corollary of Theorem 2.6 the generalization of (2.2) to manifolds with edges, modelled on *generalized half spaces*, i.e. finite products of copies of $\mathbb{R}$ and the half line.

In the context of holomorphic spaces, we have a very similar situation, as shown in [13], where the analogon of Proposition 2.2 is established for Riemann surfaces.

Before giving the main result of this section in Theorem 2.7, some preparations will be necessary.

Proposition 2.3. *Let E be a convenient vector space, $H \subseteq \mathbb{R}^n$ a generalized half space of dimension n and the space $C^\infty(H, E)$ of functions carrying smooth curves with image in H to smooth curves be endowed with the topology of uniform convergence of each derivative on relatively compact subsets of H with respect to the bornological locally convex topology of E, where the space $L(\mathbb{R}^n, \dots, \mathbb{R}^n; E)$ carries the topology of uniform convergence on relatively compact sets and in $C^\infty(H, E)$ the restrictions of the derivatives of one (any) smooth extension to $\mathbb{R}^n$ is taken. (These extensions exist by* [11]*.) Then $C^\infty(H, E)$ is a complemented subspace of $C^\infty(\mathbb{R}^n, E)$ in the category $\underline{LCS}$ of locally convex spaces.*

Proof. *Step 1.* $C^\infty(H, E)$ is a complemented $\underline{\text{Con}}$-subspace of $C^\infty(\mathbb{R}^n, E)$: The assertion for the case $E = \mathbb{R}$ and $H = [0, \infty)$ is shown in [3], where a right inverse σ of the restriction map $\rho : C^\infty(\mathbb{R}) \to C^\infty([0, \infty), \mathbb{R})$ is given by using a construction due to Seeley (c.f. [14]). The same construction works also in case E is an arbitrary convenient vector space. Finally, let E be a convenient vector space and the statement hold for any generalized half space of dimension $k = n - 1$. Let $H \subseteq \mathbb{R}^n$ a generalized half space of dimension n. Then we may write either $H \cong H_{n-1} \times \mathbb{R}$ or $H \cong H_{n-1} \times [0, \infty)$, where $H_{n-1} \subseteq \mathbb{R}^{n-1}$ is a generalized half space. The second case may be reduced to the first by means of the exponential law which shows that $C^\infty(H_{n-1} \times [0, \infty), E)$ is a complemented subspace of $C^\infty(H_{n-1} \times \mathbb{R}, E)$. So it remains to show that $C^\infty(H_{n-1} \times \mathbb{R}, E)$ is a direct summand of $C^\infty(\mathbb{R}^n, E)$. This can be done by using the exponential law once more and by induction hypothesis.

Step 2. For E a Banach space the topologies of $C^\infty(H, E)$ and $C^\infty(\mathbb{R}^n, E)$ are Fréchet and hence bornological so that the morphisms σ, ρ explained in Step 1 are continuous. If E is an arbitrary convenient vector space then its bornological locally convex topology is embedded in a projective limit of Banach spaces so that we can reduce this case to the case where E is a Banach space by means of Lemma 2.4 below which is obtained very easily. □

Lemma 2.4. *For any generalized half space $H \subseteq \mathbb{R}^n$, the functor $C^\infty(H, \_) : \underline{LCS} \to \underline{LCS}$ and for any convex bornological space E, the functor $L(E, \_) : \underline{LCS} \to \underline{LCS}$ given by means of the topology of uniform convergence on bounded sets preserves limits.*

Lemma 2.5. *Let the finite product $E := \prod_{i=1}^k E_i$ of the convenient vector spaces $E_1, \dots, E_k$, $x := (x_1, \dots, x_k) \in E$, a finite sequence $\alpha_1, \dots, \alpha_k$ of ordinals and subsets $A_i \subseteq E_i$ for $i = 1, \dots, k$ be given such that $x_i \in A_i^{(\alpha_i)}$ for all $i = 1, \dots, k$. Then*

$$x \in \left(\prod_{i=1}^k A_i\right)^{(\max_{i=1}^k \alpha_i)}.$$

Proof. We proceed by induction on the number k of the factors E_i, $i = 1, \dots, k$: Since for the product consisting of one factor there is nothing to show, let the assertion hold for every product of $k \in \mathbb{N}$ factors and a product $E := \prod_{i=1}^k E_i$ as well as $x \in E$, subsets $A_i \subseteq E_i$, $i = 1, \dots, k+1$ and a finite sequence $\alpha_1, \dots, \alpha_{k+1}$ of ordinals with the properties formulated above be given. By induction hypothesis and since $\max_{i=1}^{k+1} \alpha_i = \max\{\max_{i=1}^k \alpha_i, \alpha_{k+1}\}$, we may assume $k = 1$. We show the assertion by transfinite induction on the ordinal $\alpha := \max\{\alpha_1, \alpha_2\}$:

If α has a predecessor, then we may assume the same to hold for both α_1, α_2, since if α_i is a limit ordinal for $i = 1$ or $i = 2$, then for obvious reasons $\alpha_i < \alpha$ so that we can simply replace α_i by its successor $\alpha_i + 1$. Now $x_i \in A_i^{(\alpha_i)}$ implies that there exist sequences $(x_i^n)_{n\in\mathbb{N}}$ in $A_i^{(\alpha_i-1)}$ converging Mackey to x_i. By induction hypothesis,

$$(x_1^n, x_2^n) \in A_1^{(\alpha_1-1)} \times A_2^{(\alpha_2-1)} \subseteq (A_1 \times A_2)^{(\max\{\alpha_1-1,\alpha_2-1\})} = (A_1 \times A_2)^{(\alpha-1)}$$

and of course the sequence $(x_1^n, x_2^n)_{n\in\mathbb{N}}$ is Mackey convergent to $x = (x_1, x_2)$.

If α is a limit ordinal, then there exist $\beta_i \le \alpha_i \le \alpha$ such that $\beta_i < \alpha$ and $x_i \in A_i^{(\beta_i)}$ for $i = 1, 2$ so that we get

$$(x_1, x_2) \in A_1^{(\beta_1)} \times A_2^{(\beta_2)} \subseteq (A_1 \times A_2)^{(\max\{\beta_1,\beta_2\})} \subseteq (A_1 \times A_2)^{(\alpha)}. \qquad \square$$

Theorem 2.6.
(1) *The map*

$$\iota_X^* : L(C^\infty(X, \mathbb{R})', E) \to C^\infty(X, E)$$

is a topological isomorphism for any convenient vector space E, $X = M$ a separable finite dimensional smooth manifold with edges, where $C^\infty(X, E)$

is endowed with the initial topology with respect to the pullbacks $u_n^* : C^\infty(M,E) \to C^\infty(U_n,E)$ *along the charts* (u_n, U_n) *of a countable atlas* $\mathcal{A}$, $C^\infty(U_n,E)$ *is endowed with the topology of uniform convergence on compact sets in each derivative for each chart domain* U_n *with respect to the bornological locally convex topology of* E *and* $L(C^\infty(X,\mathbb{R})', E)$ *carries the topology of uniform convergence on bounded sets with respect to the bornological locally convex topology of* E.

(2) *The map*

$$\iota_X^* : L(\mathcal{H}(X,\mathbb{C})', E) \to \mathcal{H}(X,E)$$

is a topological isomorphism for any complex convenient vector space E, $X = M$ *a separable complex manifold modelled on polycylinders, where* $\mathcal{H}(X,\mathbb{C})$ *is endowed with the initial topology with respect to the pullbacks along charts of a countable atlas and* $L(\mathcal{H}(X,\mathbb{C})', E)$ *carries the topology of uniform convergence on bounded sets with respect to the bornological locally convex topology of* E.

Proof. **Claim 1.** The map $\iota_{\mathbb{R}}^*$ is injective: This is equivalent to density of the linear span of $\operatorname{Im} \iota_{\mathbb{R}}$ with respect to the bornological locally convex topology of $C^\infty(\mathbb{R})'$. The claimed density follows from the algebraic isomorphism $C^\infty(\mathbb{R})'' \cong C^\infty(\mathbb{R})$.

Claim 2. The map $\iota_{\mathbb{R}}^*$ is surjective: In an obvious way, we obtain a canonical map

$$C^\infty(\mathbb{R}, E) \to L(E', C^\infty(\mathbb{R})) \cong L(C^\infty(\mathbb{R})', E'').$$

Let us show that it has image in $L(C^\infty(\mathbb{R})', E)$: If E_c^* denotes the topological dual E^* of E endowed with the topology of uniform convergence on compact subsets of E, then $E \cong (E_c^*)^*$ algebraically by the Mackey–Arens theorem ([8, 8.5.5]). Since E is bornologically embedded in its bidual E'', our claim follows if we can show the following: For each $f \in C^\infty(\mathbb{R}, E)$ and $l \in C^\infty(\mathbb{R})'$, the map $l \circ (f)^* : E_c' \to \mathbb{R}$ is continuous. Indeed, there exists $K \in \mathbb{R}$ compact, $\epsilon > 0$ and $n \in \mathbb{N}$ such that $l \in V(K,\epsilon,n)^o$ where $V(K,\epsilon,n) := \{g \in C^\infty(\mathbb{R}) : |g^{(i)}(x)| \leq \epsilon \ \forall x \in K, \ i \leq n\}$. Further, there exist compact sets $K_i \subseteq L^i(\mathbb{R}, E)$ such that $f^{(i)}(K) \subseteq K_i \ \forall i \leq n$ and a compact subset $K' \subseteq E$ with $\bigcup_{i \leq n} K_i \subseteq K'$ (where we identify E with $L^i(\mathbb{R}, E)$). Thus $|l(x' \circ f)| \leq 1 \ \forall x' \in \epsilon(K')^o$ and we are done.

Claim 3. The map $\iota_{\mathbb{R}}^*$ is continuous: For this, let

$$V(K,W,n) := \{c \in C^\infty(\mathbb{R},E) : c^{(k)}(x) \in W \ \forall x \in K,\ i \leq n\}$$

be a typical neighbourhood of zero in $C^\infty(\mathbb{R},E)$ with $n \in \mathbb{N}$, $K \subseteq \mathbb{R}$ compact and $W \subseteq E$ an arbitrary closed and absolutely convex zero neighbourhood. It suffices to show the existence of a bounded subset $B \subset C^\infty(\mathbb{R})'$ with the property that $\iota_{\mathbb{R}}^*(N_{B,W}) \subseteq V(K,W,n)$, where

$$N_{B,W} := \{l \in L(C^\infty(\mathbb{R})',E) : l(B) \subseteq W\}.$$

We set $B := (V(K,1,n))^o$, using the simplified notation given in the proof of Claim 2 for our zero neighbourhood base in $C^\infty(\mathbb{R})$. Obviously we have

$$l \in N_{B,W} \Leftrightarrow l(B) \subseteq W \Rightarrow l\left(\iota_{\mathbb{R}}^{(i)}\left(\left(\iota_{\mathbb{R}}^{(i)}\right)^{-1}(B)\right)\right) \subseteq W \quad \text{for all } i \leq n.$$

Note furthermore, that

$$\left(\iota_{\mathbb{R}}^{(i)}\right)^*(l) = l \circ \iota_{\mathbb{R}}^{(i)} = (l \circ \iota_{\mathbb{R}})^{(i)} = (\iota_{\mathbb{R}}^*(l))^{(i)} \quad \text{for all } i \in \mathbb{N}$$

as l is a morphism. So $(\iota_{\mathbb{R}}^*(l))^{(i)}((\iota_{\mathbb{R}}^{(i)})^{-1}(B)) \subseteq W$ and in order to deduce our claim it will suffice to show that $\iota_{\mathbb{R}}^{(i)}(K) \subseteq B$ for $i \leq n$. Indeed, we have

$$\begin{aligned} \iota_{\mathbb{R}}^{(i)} &: \mathbb{R} \to C^\infty(\mathbb{R})' \\ x &\mapsto \big(f \mapsto f^{(i)}(x)\big), \end{aligned}$$

i.e. $\iota_{\mathbb{R}}^{(i)} = (D^{(i)})^* \circ \mathrm{ev}$: Again, this holds since $\mathrm{ev}_f : C^\infty(\mathbb{R})' \to \mathbb{R}$ is a morphism and thus $\mathrm{ev}_f \circ (\iota_{\mathbb{R}}^{(i)}) = (\mathrm{ev}_f \circ \iota_{\mathbb{R}})^{(i)} = f^{(i)}$.

Claim 4. The map $\iota_{\mathbb{R}}^*$ has a continuous inverse: With the notations of the previous claim, it suffices to show that for each compact subset $K \subseteq \mathbb{R}$, $n \in \mathbb{N}$ and any closed and absolutely convex zero neighbourhood in E we have

$$\iota_{\mathbb{R}}^*(N_{B,W}) \supseteq V(K,W,n)$$

where $B := (V(k,1,n))^o$. It suffices to prove

$$B = \overline{\left\langle \bigcup_{i \leq n} \left(\iota_{\mathbb{R}}^{(i)}\right)(K) \right\rangle_{acx}},$$

where the latter denotes the closed absolutely convex hull of the set $\bigcup_{i\le n}(\iota_{\mathbb{R}}^{(i)})(K)$ with respect to any locally convex topology on $C^\infty(\mathbb{R})'$, for which the elements of $L(C^\infty(\mathbb{R})', E)$ are continuous.

Indeed, in this case

$$l \in N_{B,W} \Leftrightarrow l(B) \subseteq W \Leftrightarrow l\Big(\bigcup_{i\le n}\left(\iota_{\mathbb{R}}^{(i)}\right)(K)\Big) \subseteq W$$

$$\Leftrightarrow \left(\iota_{\mathbb{R}}^{(i)}\right)^*(l)(K) \subseteq W \ \forall i \le n \Leftrightarrow (\iota_{\mathbb{R}}^*(l))^{(i)}(K) \subseteq W \ \forall i \le n$$

$$\Leftrightarrow \iota_{\mathbb{R}}^*(l) \in V(K, W, n).$$

From now on we will w.l.o.g. assume $K = I$ for some compact real interval I. By (2.3), $C^\infty(I,\mathbb{R})$ is a Con-direct summand of $C^\infty(\mathbb{R})$. By dualizing we get in particular that $(C^\infty(I,\mathbb{R}))'$ is a direct summand (whence a closed subspace) of $C^\infty(\mathbb{R},\mathbb{R})'$ for the topologies of uniform convergence on the bounded sets. Since all $l \in B$ vanish on $\{f \in C^\infty(\mathbb{R}) : f|_I = 0\}$, we may consider B as (by the Alaoğlu–Bourbaki theorem weakly compact) subset of $(C^\infty(I,\mathbb{R}))'$. Actually, the elements of B are continuous for the topology $C^\infty_n(I,\mathbb{R})$ induced on $C^\infty(I,\mathbb{R})$ by the inclusion

$$\iota : C^\infty(I,\mathbb{R}) \hookrightarrow \prod_{i=1}^{n} C(I,\mathbb{R})$$

$$f \mapsto (f^{(k)})_{i=1}^n,$$

where the product is endowed with the maximum of the respective supremum norms and B is nothing else than the image of the unit ball of $\coprod_{i=1}^n (C(I,\mathbb{R}))'$ under the restriction map

$$\iota^* : \coprod_{i=1}^{n} (C(I,\mathbb{R}))' \to (C^\infty_n(I,\mathbb{R}))'.$$

We claim that the extremal points of the unit sphere of $\coprod_{i=1}^n (C(I,\mathbb{R}))'$ are given by evaluations of single components: First, let $x = (x_i)_{i=1}^n \in \coprod_{i=1}^n (C(I,\mathbb{R}))'$ have at least two non vanishing components. Without loss of generality we may assume $x_i \neq 0$ for $i = 1, 2$. But then

$$x = \frac{\|x_1\|}{\|x_1\| + \|x_2\|} \left(\frac{\|x_1\| + \|x_2\|}{\|x_1\|} x_1, 0, x_3, \ldots, x_n \right)$$
$$+ \frac{\|x_2\|}{\|x_1\| + \|x_2\|} \left(0, \frac{\|x_1\| + \|x_2\|}{\|x_2\|} x_2, \ldots, x_n \right)$$

is a nontrivial convex combination of two elements of the unit sphere so that it cannot be extremal. Thus if an extremal point of the unit sphere is given, it must have at most one non vanishing component, which itself cannot be a nontrivial convex combination of two elements of the unit sphere of $C(I, \mathbb{R})'$, whence must be an extremal point. But it follows from Riesz's representation theorem (e.g. [8, 7.6.1]) that the extremal points of the unit sphere of $C(I, \mathbb{R})'$ are exactly the measures with support consisting of one point: If μ is a measure the support of which contains at least two points, then separate these by a partition of unity $\{\psi_1, \psi_2\}$ and

$$\mu = \int \psi_1 \left(\frac{\operatorname{mult}^*_{\psi_1}(\mu)}{\int \psi_1} \right) + \int \psi_2 \left(\frac{\operatorname{mult}^*_{\psi_2}(\mu)}{\int \psi_2} \right)$$

is a desired nontrivial convex combination of elements of the unit sphere provided μ is. Conversely, let $t \in I$ with $\alpha \neq 0, \beta \neq 0$ and y, z in the unit sphere of $\coprod_{i=1}^n (C(I, \mathbb{R}))'$ such that $x = (\operatorname{ev}_t, 0, \ldots, 0) = \alpha y + \beta z$. Then in particular $\operatorname{ev}_t = \alpha y_1 + \beta z_1$. Since $\|y_1\| \leq \sum_{i=1}^n \|y_k\| = \|y\| \leq 1$, we get that y_1 (and for the same reason also z_1) are elements of the unit sphere of $C(I, \mathbb{R})$. It follows by extremality of ev_t that $y_1 = z_1 = \operatorname{ev}_t$ but then $y_i = x_i = 0$ for all $i \neq 1$ since $\| \operatorname{ev}_t \| = 1$. By the Krein–Milman theorem ([8, 7.5]) the unit sphere of the Banach space $\coprod_{i=1}^n (C(I, \mathbb{R}))'$ coincides with the weak*-closure of the absolutely convex hull of its extremal points. Since

$$\operatorname{id} \circ \iota^* : \coprod_{i=1}^n C(I, \mathbb{R})' \to C^\infty_n(I, \mathbb{R})' \to C^\infty(I, \mathbb{R})'$$

is linear and weak*-continuous, its image B of the unit sphere is contained in the closed absolutely convex hull of evaluations of derivatives in some point in I and the theorem is proved for $M = \mathbb{R}$.

Claim 5. The map $\iota^*_{[0,\infty)}$ is a topological isomorphism: Consider the diagram

$$\begin{array}{ccc} L(C^\infty(\mathbb{R})', E) & \xrightarrow{\iota^*_{\mathbb{R}}} & C^\infty(\mathbb{R}, E) \\ \sigma^{**}\Big| & & \sigma\Big| \\ L(C^\infty([0,\infty), \mathbb{R})', E) & \xrightarrow{\iota^*_{[0,\infty)}} & C^\infty([0,\infty), E) \end{array}$$

where σ is as in (2.3). Then

$$\iota^*_{[0,\infty)}\big(L(C^\infty([0,\infty))', E)\big) \subseteq C^\infty([0,\infty), E)$$

and $\iota^*_{[0,\infty)}$ is a morphism since $\iota_{[0,\infty)} : [0,\infty) \to C^\infty([0,\infty), \mathbb{R})'$ is welldefined and smooth as restriction of the smooth map $\iota_{\mathbb{R}}$. By commutativity of the diagram, $\iota^*_{[0,\infty)}$ admits the inverse $(\rho^*)^* \circ (\iota^*_{\mathbb{R}})^{-1} \circ \sigma$, where $\rho : C^\infty(\mathbb{R}) \to C^\infty([0,\infty), \mathbb{R})$ is the restriction map.

Claim 6. The map ι^*_H is a topological isomorphism for any generalized half space $H \subseteq \mathbb{R}^n$: In the case where $H \cong \mathbb{R} \times H_{n-1}$ for some generalized half space $H_{n-1} \subseteq \mathbb{R}^{n-1}$ we have the sequence of topological isomorphisms

$$\begin{array}{c} L(C^\infty(\mathbb{R} \times H_{n-1}, \mathbb{R})', E) \\ \cong \Big| (1) \\ L(C^\infty(\mathbb{R}, C^\infty(H_{n-1}, \mathbb{R}))', E) \\ \cong \Big| (2) \\ L(L(C^\infty(\mathbb{R})', C^\infty(H_{n-1}, \mathbb{R}))', E) \\ \cong \Big| (3) \\ L(L(C^\infty(\mathbb{R})', C^\infty(H_{n-1}, \mathbb{R})'')', E) \\ \cong \Big| (4) \\ L(L(C^\infty(\mathbb{R})' \hat{\otimes}_\pi C^\infty(H_{n-1}, \mathbb{R})'; \mathbb{R})', E) \\ \cong \Big| (5) \end{array}$$

$$\cong \Big| (5)$$

$$L(C^\infty(\mathbb{R})'\hat{\otimes}_\pi C^\infty(H_{n-1},\mathbb{R})', E)$$

$$\cong \Big| (6)$$

$$L((C^\infty(\mathbb{R})', L(C^\infty(H_{n-1},\mathbb{R})', E))$$

$$\cong \Big| (7)$$

$$L(C^\infty(\mathbb{R})', C^\infty(H_{n-1}, E))$$

$$\cong \Big| (8)$$

$$C^\infty(\mathbb{R}, C^\infty(H_{n-1}, E))$$

$$\cong \Big| (9)$$

$$C^\infty(\mathbb{R} \times H_{n-1}, E).$$

It is obvious that the maps given in (1), (2), (3) and (8) constitute topological isomorphisms. Further, we have the Con-isomorphism

$$L(C^\infty(\mathbb{R})', C^\infty(H_{n-1},\mathbb{R})';\mathbb{R}) \cong \mathcal{L}(C^\infty(\mathbb{R})', C^\infty(H_{n-1},\mathbb{R})';\mathbb{R}),$$

where the latter denotes the space of bilinear continuous maps from $C^\infty(\mathbb{R})' \times C^\infty(H_{n-1},\mathbb{R})'$ to $\mathbb{R}$ with respect to the bornological locally convex topology, since the topology of $C^\infty(\mathbb{R})'$ is bornological DF and bilinear bounded maps on products of such spaces are continuous by [8, 15.6.7]. The bijection

$$\mathcal{L}(C^\infty(\mathbb{R})', C^\infty(\mathbb{R})';\mathbb{R}) \cong \mathcal{L}(C^\infty(\mathbb{R})'\hat{\otimes}_\pi C^\infty(\mathbb{R})';\mathbb{R})$$

is an isomorphism for the topologies of uniform convergence on bounded sets and in particular for the associated bornologies since by [8, 15.6] each bounded subset of $C^\infty(\mathbb{R})'\hat{\otimes}_\pi C^\infty(\mathbb{R})'$ is contained in the bipolar of the image of some bounded subset of $C^\infty(\mathbb{R})' \times C^\infty(\mathbb{R})'$. Moreover, the locally convex space $C^\infty(\mathbb{R})'\hat{\otimes}_\pi C^\infty(\mathbb{R})'$ is bornological by [8, 15.6.8] so that we obtain the isomorphism given in (5).

The locally convex space $C^\infty(\mathbb{R})'\hat{\otimes}_\pi C^\infty(\mathbb{R})'$ is nuclear by [8, 21.2.3] and hence

$$C^\infty(\mathbb{R})'\hat{\otimes}_\pi C^\infty(\mathbb{R})' \cong L(C^\infty(\mathbb{R})'\hat{\otimes}_\pi C^\infty(\mathbb{R})', \mathbb{R})'$$

in $\underline{\mathrm{Con}}$ by [7, p. 140], which implies that the map given in (5) is an isomorphism. The isomorphism (6) follows as in the case of (4). The map given in (7) is a topological isomorphism since

$$C^\infty(H_{n-1}, E) \cong L(C^\infty(H_{n-1}, \mathbb{R})', E)$$

by induction hypothesis. Finally, we know by the exponential law that (9) is a $\underline{\mathrm{Con}}$-isomorphism and hence a topological one for Banach spaces. If E is an arbitrary convenient vector space, then its locally convex topology is dense in a topological projective limit of Banach spaces. Now by 2.4 the statement follows.

As for the case $H \cong [0, \infty) \times H_{n-1}$, it is clear that up to obvious modifications we obtain the desired isomorphism by the same sequence.

Claim 7. The map ι_M^* is a topological isomorphism for any separable C^∞-manifold modelled on a generalized half space $H \subseteq \mathbb{R}^n$:

Let $\mathcal{A} = \{(u_n, U_n); n \in \mathbb{N}\}$ a countable atlas, where $U_n \subseteq H$ is c^∞-open. Then the diagram below commutes:

$$\begin{array}{ccc} L(C^\infty(M,\mathbb{R})', E) & \xrightarrow{\quad \iota_M^* \quad} & C^\infty(M, E) \\ {\scriptstyle (u_n^{***})_{n\in\mathbb{N}}} \Big| & & {\scriptstyle (u_n^{*})_{n\in\mathbb{N}}} \Big| \\ \prod_{n\in\mathbb{N}} L(C^\infty(U_n,\mathbb{R})', E) & \xrightarrow{\quad \cong \quad} & \prod_{n\in\mathbb{N}} C^\infty(U_n, E) \end{array}$$

where the map at the bottom is given by $\prod_{n\in\mathbb{N}} \iota_{U_n}^*$. Since the atlas $\mathcal{A}$ is countable, the map

$$(u_n^*)_{n\in\mathbb{N}} : C^\infty(M, \mathbb{R}) \to \prod_{n\in\mathbb{N}} C^\infty(U_n, \mathbb{R})$$

is an embedding for the bornological locally convex topologies. Dualizing yields a bounded linear map

$$\coprod_{n\in\mathbb{N}} C^\infty(U_n, \mathbb{R})' \to C^\infty(M, \mathbb{R})'$$

which is surjective by the Hahn–Banach theorem and has the property that each bounded set in $C^\infty(M,\mathbb{R})'$ is contained in the image of some bounded subset of $\coprod_{n\in\mathbb{N}} C^\infty(U_n,\mathbb{R})'$. Dualizing again yields a topological embedding

$$L(C^\infty(M,\mathbb{R})',E) \to L\bigg(\coprod_{n\in\mathbb{N}} C^\infty(U_n,\mathbb{R})',E\bigg).$$

Furthermore there is a topological isomorphism

$$L\bigg(\coprod_{n\in\mathbb{N}} C^\infty(U_n,\mathbb{R})',E\bigg) \cong \prod_{n\in\mathbb{N}} L(C^\infty(U_n,\mathbb{R})',E).$$

That these two spaces are isomorphic in $\underline{\mathrm{Con}}$, follows by rightadjointness of $L(\_,E) : \underline{\mathrm{Con}}^{op} \to \underline{\mathrm{Con}}$. This isomorphism is easily seen to be topological for the given topologies so that we get that the map (u_n^{***}) given in the diagram above is a topological embedding. Note that, since we may choose the chart domains so small that each U_n is isomorphic to some (not necessarily fixed) generalized half space, isomorphism of the map $\prod_{n\in\mathbb{N}} \iota^*_{U_n}$ follows by what we have already established and it suffices to show that ι^*_M is a bijection. We have mentioned in (2.1) that the free convenient vector space of a general smooth space X may be constructed by taking the Mackey closure of the linear span of $\iota_X(X) \subseteq C^\infty(X,\mathbb{R})'$ so that we have to show Mackey denseness of the linear span of $\iota_M(M)$ in $C^\infty(M,\mathbb{R})'$. For this, let $l \in C^\infty(M,\mathbb{R})'$. As already mentioned above, l admits an extension

$$\bar{l} \in \bigg(\prod_{n\in\mathbb{N}} C^\infty(U_n,\mathbb{R})\bigg)' \cong \coprod_{n\in\mathbb{N}} C^\infty(U_n,\mathbb{R})'.$$

Let $F \subset \mathbb{N}$ denote the (finite) support of $\bar{l}$. By what we have shown for generalized half spaces, for each $n \in F$, there exists a countable ordinal α_n such that $\bar{l}_n$ is contained in the α_n-th Mackey adherence $(\langle \iota_{U_n}(U_n)\rangle_{VS})^{(\alpha_n)}$ of the linear span of $\iota_{U_n}(U_n)$. By (2.5),

$$\bar{l} \in \bigg(\coprod_{n\in F} \langle \iota_{U_n}(U_n)\rangle_{VS}\bigg)^{(\alpha)} \subseteq \coprod_{n\in\mathbb{N}} C^\infty(U_n,\mathbb{R})',$$

where $\alpha := \max\{\alpha_n; n \in F\}$. Clearly,

$$\big((u_n^*)_{n\in\mathbb{N}}\big)^*\bigg(\coprod_{n\in\mathbb{N}} \langle \iota_{U_n}(U_n)\rangle_{VS}\bigg) \subseteq \langle \iota_M(M)\rangle_{VS},$$

so that

$$l = \big((u_n^*)_{n\in\mathbb{N}}\big)^*(\bar{l}) \in \big((u_n^*)_{n\in\mathbb{N}}\big)^*\left(\left(\coprod_{n\in\mathbb{N}}\langle\iota_{U_n}(U_n)\rangle_{VS}\right)^{(\alpha)}\right)$$

$$\subseteq \left(\big((u_n^*)_{n\in\mathbb{N}}\big)^*\left(\coprod_{n\in\mathbb{N}}\langle\iota_{U_n}(U_n)\rangle_{VS}\right)\right)^{(\alpha)} \subseteq \big(\langle\iota_M(M)\rangle_{VS}\big)^{(\alpha)}.$$

It remains to show the assertion on complex manifolds:

Claim 8. The map $\iota_{\mathbb{D}}^*$ is a topological isomorphism: According to [9] we have topological embeddings $\mathcal{H}(\mathbb{D}, E)_{\mathbb{R}} \hookrightarrow C^\infty(\mathbb{D}_{\mathbb{R}}, E_{\mathbb{R}})$ whenever E is a convenient vector space over $\mathbb{C}$ and $\mathcal{H}(\mathbb{D}, E\otimes\mathbb{C}) \hookrightarrow C^\infty(\mathbb{D}, E)\otimes\mathbb{C}$ whenever E is a convenient vector space over $\mathbb{R}$. Thus we get the following sequence of ($\mathbb{R}$-linear) morphisms:

$$\begin{array}{c}
L(\mathcal{H}(\mathbb{D},\mathbb{C})', E)_{\mathbb{R}} \\
(1)\big|\cong \\
L_{\mathbb{C}}(\mathcal{H}(\mathbb{D},\mathbb{C})'_{\mathbb{R}}, E_{\mathbb{R}}) \\
(2)\big\| \\
L_{\mathbb{C}}(L(\mathcal{H}(\mathbb{D},\mathbb{C}),\mathbb{C})_{\mathbb{R}}, E_{\mathbb{R}}) \\
(3)\big| \\
L_{\mathbb{C}}(L(C^\infty(\mathbb{D}_{\mathbb{R}},\mathbb{R})\otimes\mathbb{C},\mathbb{C})_{\mathbb{R}}, E_{\mathbb{R}}) \\
(4)\big|\cong \\
L_{\mathbb{C}}((L(C^\infty(\mathbb{D}_{\mathbb{R}},\mathbb{R}),\mathbb{R})\otimes\mathbb{C})_{\mathbb{R}}, E_{\mathbb{R}}) \\
(5)\big|\cong \\
L(C^\infty(\mathbb{D},\mathbb{R})', E_{\mathbb{R}})
\end{array}$$

where $L_{\mathbb{C}}(F_{\mathbb{R}}, G_{\mathbb{R}}) \subseteq L(F_{\mathbb{R}}, G_{\mathbb{R}})$ denotes the convenient subspace of $\mathbb{C}$-linear maps, i.e. the isomorphic image of $L(F, G)_{\mathbb{R}}$ in $L(F_{\mathbb{R}}, G_{\mathbb{R}})$ whenever F, G are complex convenient vector spaces. Hence the map given in (1) is a topological isomorphism by definition. By metrizability, the Con-embedding

$$\mathcal{H}(\mathbb{D},\mathbb{C}) \hookrightarrow C^\infty(\mathbb{D}_\mathbb{R},\mathbb{R}) \otimes \mathbb{C}$$

is an embedding for the bornological locally convex topologies and by dualizing we get a surjective bornological map

$$L(C^\infty(\mathbb{D}_\mathbb{R},\mathbb{R}) \otimes \mathbb{C},\mathbb{C})_\mathbb{R} \to L(\mathcal{H}(\mathbb{D},\mathbb{C}),\mathbb{C})_\mathbb{R}$$

with the property that each bounded subset of $L(\mathcal{H}(\mathbb{D},\mathbb{C}),)_\mathbb{R}$ is contained in the image of a bounded subset of $L(C^\infty(\mathbb{D}_\mathbb{R},\mathbb{R}) \otimes \mathbb{C},\mathbb{C}))_\mathbb{R}$. Dualizing a second time yields the topological embedding (3). The map given in (4) is a topological isomorphism since clearly, the algebraic bijection

$$L(C^\infty(\mathbb{D}_\mathbb{R},\mathbb{R}) \otimes \mathbb{C},\mathbb{C}) \to L(C^\infty(\mathbb{D}_\mathbb{R},\mathbb{R}),\mathbb{R}) \otimes \mathbb{C}$$

constitutes a Con-isomorphism. Similarly, the map given in (5) is a topological isomorphism. So we get the following commuting diagram

$$\begin{array}{ccc} L(C^\infty(\mathbb{D},\mathbb{R})^\times, E_\mathbb{R}) & \xrightarrow[\iota_\mathbb{D}^*]{\cong} & C^\infty(\mathbb{D},E_\mathbb{R}) \\ \Big| & & \Big| \\ L(\mathcal{H}(\mathbb{D},\mathbb{C})', E)_\mathbb{R} & \xrightarrow[\iota_\mathbb{D}^*]{} & \mathcal{H}(\mathbb{D},E)_\mathbb{R} \end{array}$$

where the vertical arrows denote the respective natural embeddings, the left hand side defined by composition of the morphisms given in the sequence above. Like in the smooth case, holomorphy of $\iota_\mathbb{D} : \mathbb{D} \to \mathcal{H}(\mathbb{D},\mathbb{C})'$ follows by the exponential law so that the map $\iota_\mathbb{D}^*|_{L(\mathcal{H}(\mathbb{D},\mathbb{C})',E)_\mathbb{R}}$: $L(\mathcal{H}(\mathbb{D},\mathbb{C})', E)_\mathbb{R} \to \mathcal{H}(\mathbb{D},E)_\mathbb{R}$ is well defined and injective as restriction of an injective morphism and it remains to show surjectivity.

Let $l \in L_\mathbb{C}(L(C^\infty(\mathbb{D}_\mathbb{R},\mathbb{R}) \otimes \mathbb{C},\mathbb{C})_\mathbb{R}, E_\mathbb{R})$ and $l \circ \iota_\mathbb{D} \in \mathcal{H}(\mathbb{D},E)_\mathbb{R} \subseteq C^\infty(\mathbb{D}_\mathbb{R},E_\mathbb{R})$. We have to show that $\bar{l}$ factors over $\mathcal{H}(\mathbb{D},\mathbb{C})'_\mathbb{R}$. Let

$$\lambda_1, \lambda_2 \in L(C^\infty(\mathbb{D}_\mathbb{R},\mathbb{R}) \otimes \mathbb{C},\mathbb{C})_\mathbb{R}$$

with $\lambda_1|_{\mathcal{H}(\mathbb{D},\mathbb{C})} = \lambda_2|_{\mathcal{H}(\mathbb{D},\mathbb{C})}$. Since the $\mathbb{C}$-linear span of $\iota_\mathbb{D}(\mathbb{D})$ is Mackey dense in $L(C^\infty(\mathbb{D}_\mathbb{R},\mathbb{R}) \otimes \mathbb{C},\mathbb{C})$, there exist nets $(\sum_{i \in F_\alpha} a_i^\alpha \operatorname{ev}_{x_i})_{\alpha \in A}$ and

$(\sum_{j\in F_\beta} b_j^\beta \operatorname{ev}_{y_j})_{\beta\in B}$ converging to λ_1 and λ_2, respectively, with respect to the c^∞-topology, hence a fortiori pointwise so that

$$\sum_{i\in F_\alpha} a_i^\alpha \operatorname{ev}_{x_i}(\lambda\circ \bar{l}\circ \iota_{\mathbb{D}}) \to \lambda_1(\lambda\circ \bar{l}\circ\iota_{\mathbb{D}})$$

and

$$\sum_{j\in F_\beta} b_j^\beta \operatorname{ev}_{y_j}(\lambda\circ l\circ \iota_{\mathbb{D}}) \to \lambda_2(\lambda\circ l\circ\iota_{\mathbb{D}})$$

for any $\lambda \in E^\times$ since then $\lambda\circ\bar{l}\circ\iota_{\mathbb{D}} \in \mathcal{H}(\mathbb{D},\mathbb{C}) \subseteq C^\infty(\mathbb{D}_{\mathbb{R}},\mathbb{R}^2)$. On the other hand, we have

$$\begin{aligned}\sum_{i\in F_\alpha} a_i^\alpha \operatorname{ev}_{x_i}(\lambda\circ\bar{l}\circ\iota_{\mathbb{D}}) &= \sum_{i\in F_\alpha} a_i^\alpha (\lambda\circ\bar{l})(\operatorname{ev}_{x_i})\\ &= (\lambda\circ\bar{l})\Big(\sum_{i\in F_\alpha} a_i^\alpha \operatorname{ev}_{x_i}\Big) \to (\lambda\circ\bar{l})(\lambda_1)\end{aligned}$$

and likewise $\sum_{j\in F_\beta} b_j^\beta \operatorname{ev}_{y_j}(\lambda\circ\bar{l}\circ\iota_{\mathbb{D}}) \to (\lambda\circ\bar{l})(\lambda_1)$ since

$$\lambda\circ\bar{l} \in L_{\mathbb{C}}(L(C^\infty(\mathbb{D}_{\mathbb{R}},\mathbb{R})\otimes\mathbb{C},\mathbb{C})_{\mathbb{R}},\mathbb{R}^2).$$

It follows that $(\lambda\circ\bar{l})(\lambda_1) = \lambda_1(\lambda\circ\bar{l}\circ\iota_{\mathbb{D}}) = \lambda_2(\lambda\circ\bar{l}\circ\iota_{\mathbb{D}}) = (\lambda\circ\bar{l})(\lambda_2)$. Thus $\bar{l}$ factors algebraically, whence as a morphism by finality of

$$L(C^\infty(\mathbb{D}_{\mathbb{R}},\mathbb{R})\otimes\mathbb{C},\mathbb{C})_{\mathbb{R}} \to L(\mathcal{H}(\mathbb{D},\mathbb{C}),\mathbb{C})_{\mathbb{R}}.$$

Claim 9. The map ι_M^* is a topological isomorphism for M a complex separable manifold modelled on polycylinders: We may proceed as we did in the real case. ☐

Remark 2.7. Another way to prove Claims 1 to 4 of (2.6) is the following: By [3, 5.1.3] the linear isomorphism $\iota_X^* : C^\infty(X,E) \cong L(\lambda X, E)$ obtained according to (2.1) is even a <u>Con</u>-isomorphism and hence $\iota_{\mathbb{R}}^*$ is a topological one for E Fréchet. The rest follows by proving Claim 2 as we did and applying Lemma 2.4 to the representation of an arbitrary convenient vector space as subspace of a projective limit of Banach spaces (compare the proof of (2.3)).

Corollary 2.8. *The natural topology of $L(C^\infty(\mathbb{R})', C^\infty(\mathbb{R})')$ is not bornological and the same holds for $L(C^\infty(\mathbb{R}), C^\infty(\mathbb{R}))$.*

Proof. The topology of uniform convergence on compact sets in each derivative on $C^\infty(\mathbb{R}, \mathbb{R}^{(\mathbb{N})})$ is not bornological: The functional $f \mapsto \sum_{k\in\mathbb{N}} (\mathrm{pr}_k \circ f)^{(k)}(0)$ is bounded but not continuous (see [3]). On the other hand, $\mathbb{R}^{(\mathbb{N})}$ is a complemented locally convex subspace of $C^\infty(\mathbb{R})'$ by means of the defining sequence

$$\mathbb{R}^{(\mathbb{N})} \xrightarrow{\iota} C^\infty(\mathbb{R}, \mathbb{R})' \xrightarrow{\rho} \mathbb{R}^{(\mathbb{N})}$$

where $\iota : (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n\in\mathbb{N}} x_n \,\mathrm{ev}_n$ and $\rho : l \mapsto (l(\psi_n))_{n\in\mathbb{N}}$ with $\psi_n \in C^\infty(\mathbb{R})$ defined as follows: Choose a smooth function $\psi \in C^\infty(\mathbb{R})$ with carrier contained in $[-1, 1]$, $\psi(0) = 1$ and set $\psi_n := \psi(\_ - n)$. Clearly, ρ is left inverse to ι and both maps are <u>Con</u>-morphisms, so that they are also continuous for the bornological locally convex topologies. So $C^\infty(\mathbb{R}, \mathbb{R}^{(\mathbb{N})})$ is a topological direct summand of

$$C^\infty(\mathbb{R}, C^\infty(\mathbb{R})') \cong L(C^\infty(\mathbb{R})', C^\infty(\mathbb{R})').$$

The latter cannot be bornological, since complemented subspaces of a bornological locally convex space must be bornological. By reflexivity of $C^\infty(\mathbb{R})$,

$$L(C^\infty(\mathbb{R}), C^\infty(\mathbb{R})) \cong L(C^\infty(\mathbb{R})', C^\infty(\mathbb{R})')$$

topologically, as can be shown with the methods of the proof of (2.6), Claim 6. $\square$

3. The space $L(\mathbb{R}^I_{\aleph_0}, \mathbb{R}^I_{\aleph_0})$

In the following, let I be an arbitrary set and for $i \in I$, $E_i \neq \{0\}$ a vector space. Let furthermore $\mathcal{Z} := \{z = (z_i)_{i\in I} \in \prod_{i\in I} E_i : z_i \neq 0 \ \forall i \in I\}$. If $z \in \prod_{i\in I} E_i$ and $M \subseteq I$ an arbitrary subset, set

$$(x_M)_i := \begin{cases} x_i & \text{for } i \in M \\ 0 & \text{otherwise.} \end{cases}$$

A subspace $F \subseteq \prod_{i\in I} E_i$ is said to be *invariant under projection* if $x_M \in F$ for each $x \in F$ and each subset $M \subseteq I$. Proposition 3.1 is an adaption of [8, 2.5].

Proposition 3.1. *Let E_i be topological vector spaces, $E \subseteq \prod_{i\in I} E_i$ a topological linear subspace invariant under projection and F a second topological vector space. Then a linear map $T : E \to F$ is continuous if and only its restrictions to E_i and $E \cap \prod_{i\in I}\langle z_i\rangle$ for each $z = (z_i)_{i\in I} \in \mathcal{Z}$ are continuous.*

Let I be a set with $|I| \geq \aleph_0$. We set $\mathbb{R}^I_{\aleph_0} := \{x \in \mathbb{R}^I : \exists M \subset I, |M| \leq \aleph_0 : x_{I\setminus M} = 0\}$ and endow the space $\mathbb{R}^I_{\aleph_0}$ with the trace of the product topology on $\mathbb{R}^I$.

Proposition 3.2. *For any index set I of non measurable cardinality, the topology of uniform convergence on bounded sets on $L(\mathbb{R}^I_{\aleph_0}, \mathbb{R}^I_{\aleph_0})$ is bornological.*

Proof. **Claim 1.** $L(\mathbb{R}^I_{\aleph_0}, \mathbb{R}^I_{\aleph_0})$ is topologically embedded in $(\mathbb{R}^{(I)})^I$: By [3, 6.2.9] the topology of $\mathbb{R}^I_{\aleph_0}$ is bornological so that its inclusion into $\mathbb{R}^I$ is a dense embedding for the bornological locally convex topologies. Hence $\mathbb{R}^I$ is the locally convex completion of $\mathbb{R}^I_{\aleph_0}$ so that for any complete locally convex space E we have an algebraic isomorphism $L(\mathbb{R}^I_{\aleph_0}, E) \cong L(\mathbb{R}^I, E)$ which is in fact a topological one for uniform convergence on bounded sets: Let $B \subseteq \mathbb{R}^I$ be bounded and $b \in B$. Without loss of generality, we may assume $B = \prod_{i\in I} B_i$, where $B_i \subset \mathbb{R}$ is an absolutely convex bounded subset of $\mathbb{R}$. Then the net $(b_F)_{F\subset I \text{ finite}}$ is convergent to b in $\mathbb{R}^I$ and by our assumption on the shape of B, $b_F \in B$ for all $F \subset I$ finite. Since $b \in B$ was arbitrary, we conclude $B \subseteq \overline{B \cap \mathbb{R}^I_{\aleph_0}}$ so that each bounded subset of $\mathbb{R}^I$ is contained in the closure of some bounded subset of $\mathbb{R}^I_{\aleph_0}$. Conversely, each bounded subset of $\mathbb{R}^I_{\aleph_0}$ has bounded closure in $\mathbb{R}^I$. Let now $N_{\overline{B},V} \subseteq L(\mathbb{R}^I, E)$ be a typical neighbourhood of zero, where $B \subseteq \mathbb{R}^I_{\aleph_0}$ is closed and bounded and $V \subseteq E$ an arbitrary absolutely convex closed neighbourhood of zero. Then $N_{B,V}$ is contained in the inverse image in $L(\mathbb{R}^I_{\aleph_0}, F)$ under the above isomorphism which we have now shown to be continuous. Its inverse is the restriction map and as such obviously continuous. Hence we may consider $L(\mathbb{R}^I_{\aleph_0}, \mathbb{R}^I_{\aleph_0})$ as topological subspace of $\left(\mathbb{R}^{(I)}\right)^I$ by means of

$$L(\mathbb{R}^I_{\aleph_0}, \mathbb{R}^I_{\aleph_0}) \subseteq L(\mathbb{R}^I_{\aleph_0}, \mathbb{R}^I) \cong L(\mathbb{R}^I, \mathbb{R}^I) \cong \left(\mathbb{R}^{(I)}\right)^I.$$

Claim 2. $L(\mathbb{R}^I_{\aleph_0}, \mathbb{R}^I_{\aleph_0})$ is quasibarrelled: Let $\mathcal{B} \subseteq L(\mathbb{R}^I, \mathbb{R}^I)$ be bounded. Then for any $B_1 \subset \mathbb{R}^I_{\aleph_0}$ closed and bounded there exists $B_2 \subset \mathbb{R}^I_{\aleph_0}$

bounded with $\mathcal{B}(\overline{B_1}) \subseteq \overline{B_2}$. Without loss of generality, we may assume B_2 to be of form $\{x = (x_i)_{i\in I} \in \mathbb{R}^I_{\aleph_0}; x_i \leq \epsilon_i\}$ for some family $(\epsilon_i)_{i\in I}$ of positive real numbers. Let $f \in \mathcal{B}$. Then $(\mathrm{pr}_F)_*(f)$ is convergent to f uniformly on $\mathbb{R}^I$ and $(\mathrm{pr}_F)_* f(B_1) \subseteq \mathrm{pr}_F(\overline{B_2}) \subseteq B_2$. Hence each bounded subset of $L(\mathbb{R}^I, \mathbb{R}^I)$ is contained in the closure of some bounded subset of $L(\mathbb{R}^I_{\aleph_0}, \mathbb{R}^I_{\aleph_0})$. Let V be a bornivorous barrel in $L(\mathbb{R}^I_{\aleph_0}, \mathbb{R}^I_{\aleph_0})$. Then the same holds for $\overline{V} \subseteq L(\mathbb{R}^I, \mathbb{R}^I)$. Since $L(\mathbb{R}^I, \mathbb{R}^I)$ is bornological, $\overline{V}$ is a zero neighbourhood in $L(\mathbb{R}^I, \mathbb{R}^I)$ and so its trace V is a zero neighbourhood in the trace topology.

Claim 3. E is bornological: Note first that the image E of $L(\mathbb{R}^I_{\aleph_0}, \mathbb{R}^I_{\aleph_0})$ under its embedding into $\left(\mathbb{R}^{(I)}\right)^I$ is given by

$$F = \left\{ (f_j)_i \in \prod_{j\in I}\left(\coprod_{i\in I}\mathbb{R}\right);\ |J_i(f)| \leq \aleph_0 \right\},$$

where $(f_j)_i := (f(e_i))_j$, e_i the i-th standard unit vector in $\mathbb{R}^I_{\aleph_0}$ and $J_i(f) := \{j \in I : (f_j)_i \neq 0\}$.

For $j \in I$, let $f_j \in \mathbb{R}^{(I)}$ be non zero. In view of (3.1), it suffices to show that every bounded linear functional on $E \cap \prod_{j\in I}\langle f_j\rangle$ is continuous. For this, we adapt part of the proof of [3, 6.2.9], which we include here for the sake of completeness: Let $l : E \cap \prod_{j\in I}\langle f_j\rangle \to \mathbb{R}$ be bounded linear. Note, that

$$l : E \cap \prod_{j\in I}\langle f_j\rangle \cong \{(\lambda_j)_{j\in I} \in \mathbb{R}^I;\\ |\{j \in J_i(f) : \lambda_j \neq 0\}| \leq \aleph_0 \text{ for all } i \in I\}$$

and hence $\mathbb{R}^I_{\aleph_0} \subseteq E \cap \prod_{j\in I}\langle f_j\rangle \subseteq \mathbb{R}^I$. The restriction $l_0 := l|_{\mathbb{R}^I_{\aleph_0}} : \mathbb{R}^I_{\aleph_0} \to \mathbb{R}$ is sequentially continuous: Being bounded on $\mathbb{R}^I_{\aleph_0}$, l_0 carries Mackey convergent sequences to convergent sequences in $\mathbb{R}$. On the other hand, each convergent sequence in $\mathbb{R}^I_{\aleph_0}$ is contained in a metrizable complemented subspace $\mathbb{R}^A$ with $A \subset I$ countable and is therefore even Mackey convergent (therein). By [12] l_0 factors over some countable subset of the index set I, i.e. there exists $A_0 \subset I$ countable such that $l_0(x) = l_0(x_{A_0})$ for all $x \in \mathbb{R}^I_{\aleph_0}$.

Our claim is that $l(x) = l(x_{A_0})$ for arbitrary $x \in E \cap \prod_{j\in I}\langle f_j\rangle$: Indeed, let us consider the map $\varphi_x : 2^I \to \mathbb{R}$ defined by $A \mapsto l(x_{A\cap A_0}) - l(x_{A_0})$. The definition makes sense as $E \cap \prod_{j\in I}\langle f_j\rangle$ is defined by carrier conditions and hence $x_A \in E \cap \prod_{j\in I}\langle f_j\rangle$ for arbitrary subsets A of the index set. Clearly $\varphi_x(A) = 0$ for $A \in 2^I$ countable. Furthermore, φ_x is sequentially continuous: Let $(A_n)_{n\in\mathbb{N}}$ be convergent to A in 2^I, i.e. for all $i \in I$ there exists $n(i) \in \mathbb{N}$ such that $\chi_{A_n}(i) = \chi_A(i)$ for all $n \in \mathbb{N}$ with $n > n(i)$. Then $\{n(x_{A_n} - x_A) : n \in \mathbb{N}\} \subset E \cap \prod_{j\in I}\langle f_j\rangle$ is bounded since each coordinate eventually equals zero. But this exactly means that $(x_{A_n})_{n\in\mathbb{N}}$ is Mackey convergent to x_A and, since l is bounded, $\varphi_x(A_n)$ is convergent to $\varphi_x(A)$ in $\mathbb{R}$. Now the conclusion follows by another result of [12] stating that each sequentially continuous realvalued function on 2^I vanishing on all countable subsets vanishes on the whole of 2^I, provided the index set I is of nonmeasurable cardinality. □

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