Publ. Math. Debrecen
57 / 3-4 (2000), 411–419

Symmetric words in free nilpotent groups of class 4

By WALDEMAR HOŁUBOWSKI (Gliwice)

Abstract. A word $w(X_1,\dots,X_n)$ is called n-symmetric for a given group G if $w(g_1,\dots,g_n) = w(g_{\sigma(1)},\dots,g_{\sigma(n)})$ for all $g_1,\dots,g_n$ in G and all permutations σ from the symmetric group S_n. In this note we describe n-symmetric words in the free nilpotent groups of class 4.

1. Preliminaries and main results

The problem of characterizing the n-symmetric words in the given group G was initiated by PŁONKA [8]–[10] who gave a complete description of the n-symmetric words in nilpotent groups of class ≤ 3. For results for metabelian and other groups we refer to [1]–[6].

Let F_n denote the absolutely free group on $X_1,\dots,X_n$.

Definition. A word $w(X_1,\dots,X_n) \in F_n$ is called n-symmetric word for a group G if $w(g_{\sigma(1)},\dots,g_{\sigma(n)}) = w(g_1,\dots,g_n)$ for all $g_1,\dots,g_n \in G$ and all permutations σ from the symmetric group S_n.

It follows from the definition that we can restrict ourselves to relatively free groups with n free generators and to natural action of S_n on them. Let $F_n(G)$ be the relatively free group on $x_1,\dots,x_n$ in a variety generated by the group G. Let A be the group of automorphisms of $F_n(G)$ induced by the mappings $x_i \longrightarrow x_{\sigma(i)}$, $1 \leq i \leq n$, $(\sigma \in S_n)$. The group

$$S^{(n)}(G) = \{w \in F_n(G) : w = \alpha(w) \text{ for every } \alpha \in A\}$$

is called a group of n-symmetric words for G.

Mathematics Subject Classification: 20F18, 20F12.
Key words and phrases: symmetric words, nilpotent groups, commutator calculus.

In this paper we describe $S^{(n)}(G)$ in the case of G, the free nilpotent group of class 4 which we denote shortly by $S^{(n)}(\mathfrak{N}_4)$ ($\mathfrak{N}_c$ – variety of nilpotent groups of class c). Our results extend these from [8], [10] and give a correction to one statement in [5].

We denote by $[x,y] = x^{-1}y^{-1}xy$ a commutator of elements x, y. Commutators of higher weight are defined as left-normed.

Let $u_1(x,y) = [y,x,x][y,x,y]^{-1}$, $u_2(x,y) = [y,x,x,x][y,x,y,y]^{-1}$.

Theorem 1. *The group $S^{(2)}(\mathfrak{N}_4)$ is a free nilpotent group of class 2 generated by $u_1(x,y)$, $u_2(x,y)$ and $u_3 = x^4y^4[y,x]^8[y,x,x]^{24}[y,x,x,x]^{16} \times [y,x,x,y]^{18}$.*

The Theorem 1 answers affirmatively a question raised in [9]. We note here that all groups $S^{(n)}(\mathfrak{N}_c)$ are abelian if $c \leq 3$.

Theorem 2. *The group $S^{(3)}(\mathfrak{N}_4)$ is a free abelian group generated by*
$w_1(x,y,z) = u_1(x,y)u_1(x,z)u_1(y,z)$,
$w_2(x,y,z) = u_2(x,y)u_2(x,z)u_2(y,z)$.

Theorem 3. *The group $S^{(4)}(\mathfrak{N}_4)$ is a free abelian group generated by*
$w_3(x,y,z,t) = u_1(x,y)u_1(x,z)u_1(x,t)u_1(y,z)u_1(y,t)u_1(z,t)$,
$w_4(x,y,z,t) = u_2(x,y)u_2(x,z)u_2(x,t)u_2(y,z)u_2(y,t)u_2(z,t)$.

Since we have isomorphisms $S^{(n)}(\mathfrak{N}_4) \cong S^{(4)}(\mathfrak{N}_4)$ (for $n > 4$ [9]), our theorems give a full description of n-symmetric words for any natural n.

A map $w(x_1,\dots,x_n,x_{n+1}) \to w(x_1,\dots,x_n,1)$ induces homomorphism

$$\delta_n^{n+1}(\mathfrak{N}_c) : S^{(n+1)}(\mathfrak{N}_c) \to S^{(n)}(\mathfrak{N}_c).$$

It is clear that $\delta_n^{n+1}(\mathfrak{N}_4)$ is an isomorphism for $n \geq 3$. However, $\delta_2^3(\mathfrak{N}_4)$ is a monomorphism, which contradicts a second part of Theorem 3 from [5] which states that $\delta_n^{n+1}(\mathfrak{N}_{n+2})$ is not a monomorphism for any n. In fact, a sketch of the proof given in [5] shows that $\delta_n^{n+1}(\mathfrak{N}_{n+2})$ is not monomorphism for $n > 2$. This raise a question of checking the validity of this statement from [5] for other nilpotent groups of class 4.

2. Identities in nilpotent groups

We use a standard definitions from [7] without explanations.

We need some well known identities:

(1) $[x^{-1}, y] = [x, y]^{-1}[y, x, x^{-1}]$, (2) $[x, y^{-1}] = [x, y]^{-1}[y, x, y^{-1}]$

(3) $[xy, z] = [x, z][x, z, y][y, z]$, (4) $[x, yz] = [x, z][x, y][x, y, z]$

valid in arbitrary groups. We use notation $\binom{n}{i} = \frac{1}{i!} \cdot n(n-1)\cdots(n-i+1)$.

Now we list identities of nilpotent groups of class 4 which we use in next sections to rewrite some words as the products of basic commutators. We fix a natural order of basic commutators:

$$x < y < z < t < [y, x] < [z, x] < [t, x] < [z, y] < [t, y] < [t, z] < \dots$$

Lemma 1. *The following identities hold in a nilpotent group G of class four for any x, y, z, $t \in G$ and all integers n, m, k, l.*

$$\begin{aligned}(5)\qquad [y^n, x^m] &= [y, x]^{nm}[y, x, x]^{n\binom{m}{2}}[y, x, y]^{\binom{n}{2}m}[y, x, x, x]^{n\binom{m}{3}}\\ &\quad\times [y, x, x, y]^{\binom{n}{2}\binom{m}{2}}[y, x, y, y]^{\binom{n}{3}m},\end{aligned}$$

$$\begin{aligned}(6)\qquad [y^n, x^m, z^k] &= [y, x, z]^{nmk}[y, x, y, z]^{\binom{n}{2}mk}\\ &\quad\times [y, x, x, z]^{\binom{m}{2}nk}[y, x, z, z]^{\binom{k}{2}nm},\end{aligned}$$

$$(7)\qquad [y^n, x^m, z^k, t^l] = [y, x, z, t]^{nmkl}.$$

Proof. Using (1)–(4) one can prove that

$$[y^{-1}, x] = [y, x]^{-1}[y, x, y][y, x, y, y]^{-1},$$
$$[y, x^{-1}] = [y, x]^{-1}[y, x, x][y, x, x, x]^{-1}$$

and by induction the following identities for all natural n, m

$$[y^n, x] = [y, x]^n[y, x, y]^{\binom{n}{2}}[y, x, y, y]^{\binom{n}{3}},$$
$$[y, x^m] = [y, x]^m[y, x, x]^{\binom{m}{2}}[y, x, x, x]^{\binom{m}{3}}.$$

Now we have $[y,x^{-m}]=[y,(x^m)^{-1}]=[y,x^m]^{-1}[y,x^m,x^m]\times$ $[y,x^m,x^m,x^m]^{-1}=[y,x]^{-m}[y,x,x]^{\binom{-m}{2}}[y,x,x,x]^{\binom{-m}{3}}$ so, this identity is valid for all integers. Similarly we obtain the expression for $[y^{-n},x]$. Finally, for all integers n, m, we have

$$[y^n,x^m]=[y^n,x]^m[y^n,x,x]^{\binom{m}{2}}[y^n,x,x,x]^{\binom{m}{3}}=\prod_{i,j>0}^{i+j<5}[y,_i\,x,_{(j-1)}\,y]^{\binom{n}{i}\binom{m}{j}}.$$

Using this identity one can easily prove (6); (7) is easy to check directly. □

Lemma 2. *The following identities hold in any nilpotent group of class four:*

(8) $[x,y,z]=[y,x,z]^{-1}$,

(9) $[z,y,x]=[z,x,y][y,x,z]^{-1}[[z,x],[y,x]][[z,y],[y,x]][[z,y],[z,x]]$,

(10) $[x,y,z,t]=[y,x,z,t]^{-1}$,

(11) $[y,x,t,z]=[y,x,z,t][[z,t],[y,x]]$,

(12) $[z,y,x,t]=[z,x,y,t][y,x,z,t]^{-1}$,

(13) $[t,y,x,z]=[t,x,y,z][y,x,z,t]^{-1}[[t,z],[y,x]]$,

(14) $[z,y,t,x]=[z,x,y,t][y,x,z,t]^{-1}[[t,x],[z,y]]^{-1}$,

(15) $[t,y,z,x]=[t,x,y,z][y,x,z,t]^{-1}[[t,y],[z,x]][[t,z],[y,x]]$.

Proof. (8) and (10) follow easily from (1)–(4). (9) is the Jacobi identity. We have

$$\begin{aligned}[xy,zt]&=[xy,t][xy,z][xy,z,t]\\&=[x,t][x,t,y][y,t][x,z][x,z,y][y,z][x,z,t][x,z,y,t][y,z,t]\end{aligned}$$

and similarly

$$\begin{aligned}[xy,zt]&=[x,zt][x,zt,y][y,zt]\\&=[x,t][x,z][y,t][y,z][x,z,t][y,z,t][x,t,y][x,z,y][x,z,t,y]\end{aligned}$$

which implies (11). By Jacobi identity we have

$$\begin{aligned}[z,y,x,t] &= [z,y,x]^{-1}t^{-1}[z,y,x]t = [y,x,z][z,x,y]^{-1}[[z,y],[z,x]]^{-1}\\ &\quad\times [[z,y],[y,x]]^{-1}[[z,x],[y,x]]^{-1}t^{-1}[z,x,y][y,x,z]^{-1}\\ &\quad\times [[z,y],[z,x]][[z,y],[y,x]][[z,x],[y,x]]t\\ &= [y,x,z][z,x,y]^{-1}t^{-1}[z,x,y]t[y,x,z]^{-1}[y,x,z,t]^{-1}\\ &= [y,x,z][z,x,y,t][y,x,z]^{-1}[y,x,z,t]^{-1}\end{aligned}$$

which gives us (12). (13) follows from

$$[t,y,x,z] \overset{(11)}{=} [t,x,y,z][y,x,t,z]^{-1} \overset{(12)}{=} [t,x,y,z][y,x,z,t]^{-1}[[z,t],[y,x]]^{-1}.$$

Similarly we have

$$[z,y,t,x] \overset{(11)}{=} [z,y,x,t][[t,x],[z,y]]^{-1} \overset{(12)}{=} [z,x,y,t][y,x,z,t]^{-1}[[t,x],[z,y]]^{-1}$$

and

$$\begin{aligned}[t,y,z,x] &\overset{(11)}{=} [t,y,x,z][[t,y],[z,x]] \overset{(12)}{=} [t,x,y,z][y,x,t,z]^{-1}[[t,y],[z,x]]\\ &\overset{(11)}{=} [t,x,y,z][y,x,z,t]^{-1}[[t,z],[y,x]][[t,y],[z,x]]. \qquad \square\end{aligned}$$

We need a characterization of elements of $S^{(2)}(\mathfrak{N}_4)$. Every element from $S^{(n)}(\mathfrak{N}_4)$ has a form $x_1^a x_2^a \dots x_n^a \cdot c$, where c belongs to the commutator subgroup (see Lemma 4 of [2]). Moreover, we have

Lemma 3. *An element $w(x,y)$ from $F_2(\mathfrak{N}_4)$ belongs to $S^{(2)}(\mathfrak{N}_4)$ if and only if*

$$w(x,y) = x^a y^a [y,x]^b [y,x,x]^{c_1} [y,x,y]^{c_2} [y,x,x,x]^{d_1} [y,x,x,y]^{d_2} [y,x,y,y]^{d_3}$$

where

$$a^2 = 2b, \quad c_1 + c_2 = a\binom{a}{2}, \quad d_1 + d_3 = a\binom{a}{3}, \quad 2d_2 = \binom{a}{2}\binom{a}{2}.$$

Proof. We have to prove the equality

$$\begin{aligned} w(y,x) &= y^a x^a [x,y]^b [x,y,y]^{c_1} [x,y,x]^{c_2} [x,y,y,y]^{d_1} [x,y,y,x]^{d_2} [x,y,x,x]^{d_3} \\ &= x^a y^a [y^a, x^a][y,x]^{-b}[y,x,x]^{-c_2}[y,x,y]^{-c_1}[y,x,x,x]^{-d_3} \\ &\quad \times [y,x,x,y]^{-d_2}[y,x,y,y]^{-d_1} \\ &= x^a y^a [y,x]^{a^2-b}[y,x,x]^{a\cdot\binom{a}{2}-c_2}[y,x,y]^{a\cdot\binom{a}{2}-c_1}[y,x,x,x]^{a\cdot\binom{a}{3}-d_3} \\ &\quad \times [y,x,x,y]^{\binom{a}{2}\binom{a}{2}-d_2}[y,x,y,y]^{a\cdot\binom{a}{3}-d_1} = w(x,y). \end{aligned}$$

The lemma now follows from the fact that in the free nilpotent group a presentation of the element as a product of basic commutators is unique [7]. □

3. Proofs of main results

Now we are ready to prove our theorems.

Proof of Theorem 1. It follows from the Lemma 3 that every element of $S^{(2)}(\mathfrak{N}_4)$ has a form

$$\begin{aligned} &x^{4m}y^{4m}[y,x]^{8m^2}[y,x,x]^c[y,x,y]^{8m^2(4m-1)-c}[y,x,x,x]^d \\ &\times[y,x,x,y]^{2m^2(4m-1)^2}[y,x,y,y]^{\frac{1}{3}8m^2(4m-1)(4m-2)-d}, \end{aligned}$$

where m, c, d are arbitrary integers. So, the group $S^{(2)}(G)$ is generated by three elements

$$\begin{gathered} u_1 = [y,x,x][y,x,y]^{-1}, \qquad u_2 = [y,x,x,x][y,x,y,y]^{-1}, \\ u_3 = x^4y^4[y,x]^8[y,x,x]^{24}[y,x,x,x]^{16}[y,x,x,y]^{18}. \end{gathered}$$

We have $u_3u_1 \neq u_1u_3 = u_3u_1u_2^4$ and commutator of any two 2-symmetric words from $S^{(2)}(\mathfrak{N}_4)$ belongs to the centre, so the theorem is proved. □

Proof of Theorem 2. Every element of $S^{(3)}(\mathfrak{N}_4)$ has a form

$$v(x,y,z) = x^a y^a z^a [y,x]^b [z,x]^b [z,y]^b v_1(x,y) v_2(x,z) v_3(y,z) v_0(x,y,z)$$

where

$$v_i(x,y) = [y,x,x]^{c_{i,1}}[y,x,y]^{c_{i,2}}[y,x,x,x]^{d_{i,1}}[y,x,x,y]^{d_{i,2}}[y,x,y,y]^{d_{i,3}}$$

and v_0 is a product of basic commutators on exactly three letters. Simple calculation using transpositions of generators, shows that $v_1 = v_2 = v_3$. Since $v(x, y, 1)$ belongs to $S^{(2)}(\mathfrak{N}_4)$, we can apply Lemma 3. So we have

$$v_1(x, y) = [y, x, x]^{c_1}[y, x, y]^{c_2}[y, x, x, x]^{d_1}[y, x, x, y]^{d_2}[y, x, y, y]^{d_3}$$

and a, b, c_1, c_2, d_1, d_2, d_3 satisfy the conditions of Lemma 3. We put

$$\begin{aligned} v_0(x, y, z) = {} & [y, x, z]^{c_3}[z, x, y]^{c_4}[y, x, x, z]^{d_4}[y, x, y, z]^{d_5}[y, x, z, z]^{d_6} \\ & \times [z, x, x, y]^{d_7}[z, x, y, y]^{d_8}[z, x, y, z]^{d_9} \\ & \times [[z, x], [y, x]]^{e_1}[[z, y], [y, x]]^{e_2}[[z, y], [z, x]]^{e_3} \end{aligned}$$

and rewrite the element $v(y, x, z)$ as a product of basic commutators. We consider now only the basic commutators on three letters. By rewriting v_0 we obtain

$$\begin{aligned} v_0(y, x, z) = {} & [y, x, z]^{-c_3-c_4}[z, x, y]^{c_4}[y, x, x, z]^{-d_5-d_8}[y, x, y, z]^{-d_4-d_7} \\ & \times [y, x, z, z]^{-d_6-d_9}[z, x, x, y]^{d_8}[z, x, y, y]^{d_7}[z, x, y, z]^{d_9} \\ & \times [[z, x], [y, x]]^{-e_2+c_4+2d_8}[[z, y], [y, x]]^{-e_1+c_4+2d_7} \\ & \times [[z, y], [z, x]]^{-e_3+c_4} \end{aligned}$$

and from $v(y, x, z)(v_0(y, x, z))^{-1}$ we have

$$[y, x, z]^{a^3}[y, x, x, z]^{a^2\binom{a}{2}}[y, x, y, z]^{a^2\binom{a}{2}}[y, x, z, z]^{a^2\binom{a}{2}}[[z, y], [z, x]]^{b^2}.$$

The same calculation for $v(y, z, x)$ gives us

$$\begin{aligned} v_0(y, z, x) = {} & [y, x, z]^{-c_3-c_4}[z, x, y]^{c_3}[y, x, x, z]^{-d_6-d_9}[y, x, y, z]^{-d_4-d_7} \\ & \times [y, x, z, z]^{-d_5-d_8}[z, x, x, y]^{d_6}[z, x, y, y]^{d_4}[z, x, y, z]^{d_5} \\ & \times [[z, x], [y, x]]^{e_3+c_3-d_9+2d_6}[[z, y], [y, x]]^{e_1+c_3+2d_4} \\ & \times [[z, y], [z, x]]^{e_2+c_3+d_5} \end{aligned}$$

from rewriting $v_0(y, z, x)$ and from $v(y, z, x)(v_0(y, z, x))^{-1}$

$$\begin{aligned} & [y, x, z]^{a^3}[y, x, x, z]^{a^2\binom{a}{2}}[y, x, y, z]^{a^2\binom{a}{2}}[y, x, z, z]^{a^2\binom{a}{2}} \\ & \times[[z, x], [y, x]]^{-b^2}[[z, y], [y, x]]^{-b^2}[[z, y], [z, x]]^{-b^2}. \end{aligned}$$

Comparing the powers of basic commutators we obtain

$$a^2 = 2b,\ c_3 = c_4 = c,\ d_4 = d_6 = d_7 = d_8 = d,\ d_5 = d_9 = d_0,\ 3c = a^3,$$
$$2d + d_0 = a^2\binom{a}{2},\ 2d_0 = b^2,\ e_1 + e_2 = c + 2d,\ 2e_3 = c + b^2,$$
$$e_1 + b^2 + d_0 = e_3 + c + 2d,\ e_2 + b^2 = e_1 + c + 2d,\ e_3 + b^2 = e_2 + c + d_0.$$

This implies $3c+4d-6d_0 = 0$ and for some integer k the equalities $a = 6k$, $b = 18k^2$, $c = 12k^2$, $d_0 = 3^4 \cdot 2k^4$ and $d = 2\cdot 3^3 k^3(6k-1) - 3^4 \cdot k^4$. But then we obtain $k^2(1 - 6k) = 0$ and consequently $k = 0$ and $e_1 = e_2 = e_3 = 0$, which finishes the proof. □

Proof of Theorem 3. Let $w = w(x, y, z, t)$ belong to $S^{(4)}(\mathfrak{N}_4)$ and let

$$w_2 = x^a y^a z^a t^a [y, x]^b [z, x]^b [t, x]^b [z, y]^b [t, y]^b [t, z]^b.$$

Since the words $w(x, y, z, 1)$, $w(x, y, 1, t)$, $w(x, 1, z, t)$, $w(1, y, z, t)$ are both in $S^{(3)}(\mathfrak{N}_4)$ we have

$$w(x, y, z, t) = w_2 w_1(x, y, z) w_1(x, y, t) w_1(x, z, t) w_1(y, z, t) w_0 = w_1' \cdot w_0,$$

where w_1' is a product of commutators which contain exactly 3 letters and

$$w_0 = [y, x, z, t]^{f_1} [z, x, y, t]^{f_2} [t, x, y, z]^{f_3} [[t, x], [z, y]]^{f_4} \times [[t, y], [z, x]]^{f_5} [[t, z][y, x]]^{f_6}$$

is the product of all basic commutators on exactly 4 letters and w_2 is trivial because $a = b = 0$.

Using Lemmas 1, 2 we rewrite $w(y, x, z, t)$ as a product of basic commutators. Then we obtain

$$[y, x, z, t]^{-f_1-f_2-f_3} [z, x, y, t]^{f_2} [t, x, y, z]^{f_3} [[t, x], [z, y]]^{f_5} \times [[t, y], [z, x]]^{f_4} [[t, z], [y, x]]^{-f_6}.$$

So we deduce that

$$2f_1 + f_2 + f_3 = 0, \quad f_4 = f_5, \quad f_5 = f_4, \quad 2f_6 = f_3.$$

The similar calculations for the element $w(y,z,t,x)$ give

$$[y,x,z,t]^{-f_1-f_2-f_3}[z,x,y,t]^{f_1}[t,x,y,z]^{f_2}[[t,x],[z,y]]^{-f_1-f_6} \times[[t,y],[z,x]]^{f_2+f_5}[[t,z],[y,x]]^{f_2+f_4}.$$

It follows that $f_1=f_2=f_3=f_4=f_5=f_6=0$ and Theorem 3 is proved. □

References

[1] C. K. Gupta and W. Hołubowski, On 2-symmetric words in groups, *Arch. der Math.* (*to appear*).

[2] W. Hołubowski, Symmetric words in metabelian groups, *Comm. Algebra* **23** (14) (1995), 5161–5167.

[3] W. Hołubowski, Symmetric words in a free nilpotent group of class 5, Groups St Andrews 1997 in Bath, vol. I, *London Math. Soc. Lect. Notes Ser.* **260** (1999), 363–367.

[4] S. Krstič, On symmetric words in nilpotent groups, *Publ. Inst. Math. (Beograd) (N.S)* **27** (41) (1980), 139–142.

[5] O. Macedońska, On symmetric words in nilpotent groups, *Fund. Math.* **120** (1984), 119–125.

[6] O. Macedońska and D. Solitar, On binary σ-invariants words in a group, *Contemp. Math.* **169** (1994), 431–449.

[7] H. Neumann, Varieties of groups, *Springer V.*, *Berlin–Heidelberg–New York*, 1967.

[8] E. Płonka, Symmetric operations in groups, *Colloq. Math.* **21** (1970), 179–186.

[9] E. Płonka, On symmetric words in free nilpotent groups, *Bull. Acad. Polon. Sci.* **18** (1970), 427–429.

[10] E. Płonka, Symmetric words in nilpotent groups of class ≤ 3, *Fund. Math.* **97** (1977), 95–103.

WALDEMAR HOŁUBOWSKI
INSTITUTE OF MATHEMATICS
SILESIAN TECHNICAL UNIVERSITY
UL.KASZUBSKA 23
44–101 GLIWICE
POLAND

E-mail: wholub@zeus.polsl.gliwice.pl

(Received January 18, 1999; revised June 15, 1999)