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Note on the superstability of the Cauchy functional equation

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Abstract. We give the characterization of function spaces for which the phenomenon of superstability of the Cauchy functional equation occurs.

In the stability theory of functional equations we have two kinds of results (see [2]). In the first case is the classical Hyers theorem where we have a rich family of unbounded solutions of the inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

which are not the solutions of the respective functional equation

$$f(x+y) = f(x) + f(y).$$

The second type of results is, for example, the result obtained by J. A. Baker [1] where every unbounded solution of the inequality

$$|f(x+y) - f(x)f(y)| \leq \varepsilon$$

satisfies the respective functional equation

$$f(x+y) = f(x)f(y).$$

Then we say that the functional equation is *superstable*.

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In the additive case for the Cauchy functional equation

$$f(x+y) = f(x) + f(y)$$

the phenomenon of superstability we can find in the papers of J. Tabor [4] and Zs. Páles [3]. The authors studied the stability problem of this equation in some function spaces. In the note we give the characterization of such function spaces for which this superstability phenomenon occurs.

First we recall the results of J. Tabor and Zs. Páles.

Theorem (J. Tabor). *Let $(S,+)$ be a group with a measure μ such that $\mu(S)=\infty$ and let $(H,+,d)$ be a metric Abelian group. Then every mapping $f : S \to H$ satisfying*

$$S \times S \ni (x,y) \to f(x+y) - f(x) - f(y) \in \mathcal{L}_p(S\times S)$$

for a certain $p > 0$, is additive.

Theorem (Zs. Páles). *Let $\varepsilon \geq 0$. Then every function $f : (0,\infty) \to \mathbb{R}$ fulfilling the inequality*

$$|f(x+y) - f(x) - f(y)| \leq \varepsilon \cdot e^{-(x+y)}, \quad x,y > 0$$

is additive.

Let $(S,+)$ be a semigroup and let $(H,+)$ be a group. For a function $f : S \to H$ by $Cf : S \times S \to H$ we denote the Cauchy difference of f, i.e.

$$Cf(x,y) = f(x+y) - f(x) - f(y), \quad x,y \in S.$$

By H^S we mean the set of all functions defined on S with values in H, and let $\mathcal{C}$ be the group of all constant functions from H^S

Assume that $\mathcal{F} \subset H^S$ is a group invariant with respect to left [right] translations, i.e.

$$x \in S,\ f \in \mathcal{F} \Rightarrow\ {}_xf(\cdot) = f(x+\cdot) \in \mathcal{F}$$
$$[x \in S,\ f \in \mathcal{F} \Rightarrow\ f_x(\cdot) = f(\cdot + x) \in \mathcal{F}].$$

We say that a subgroup Φ of $H^{S\times S}$ has the *left* [*right*] *$\mathcal{F}$-Fubini property* if and only if for every function $\phi \in \Phi$ the function of the second [first] variable $\phi(x,\cdot)$ [$\phi(\cdot,x)$] belongs to $\mathcal{F}$, where $x \in S$.

Under these assumptions we have the following

Theorem. *If $\mathcal{F} \cap \mathcal{C} = \{0_S\}$ and $\Phi \subset H^{S\times S}$ has the left [right] $\mathcal{F}$-Fubini property, then every mapping $f : S \to H$ satisfying $Cf \in \Phi$ is additive.*

Proof. Let $x, y \in S$ are fixed. Then since the group $\mathcal{F}$ is invariant with respect to left translations and Φ has the left $\mathcal{F}$-Fubini property it is evident that the function

$$S \ni z \to Cf(x, y+z) + Cf(y, z) - Cf(x+y, z) \in H \tag{1}$$

belongs to $\mathcal{F}$.

Moreover, the Cauchy difference of every function satisfies the cocycle functional equation, i.e.

$$Cf(x+y, z) + Cf(x, y) = Cf(x, y+z) + Cf(y, z),$$

for all $x, y, z \in S$. Therefore

$$Cf(x, y) = Cf(x, y+z) + Cf(y, z) - Cf(x+y, z), \tag{2}$$

which means that function (1) is constant and belongs to $\mathcal{F}$. The assumption $\mathcal{F} \cap \mathcal{C} = \{0_S\}$ implies that (1) is the zero function, so f is an additive mapping.

The proof of the "right" version is analogous. □

The Páles theorem we obtain by taking for $\mathcal{F}$ the family of all functions $f : (0, \infty) \to \mathbb{R}$ for which the mapping

$$(0, \infty) \ni x \mapsto |f(x)| \cdot e^{-x} \in \mathbb{R}$$

is bounded.

Moreover, using our theorem, the result of Zs. Páles can be generalized in the following way:

Corollary. *Let $(S, +)$ be a semigroup, $(Y, \|\cdot\|)$ be a normed space and let $p, q : S \to \langle 0, \infty)$. Assume that*

$$\inf\left\{\sum_{k=1}^{n} q(z_k + y) : y \in S\right\} = 0, \tag{3}$$

for all $z_k \in S$, $k = 1, 2, \ldots, n$ *and* $n \in \mathbb{N}$. *Then every function satisfying the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le p(x)q(y), \quad x, y \in S$$

is additive.

Condition (3) is fulfilled by every function $q : S \to \langle 0, \infty)$ satisfying

$$q(x+y) \le q(x)q(y), \quad x, y \in S$$

and

$$\inf\{q(x) : x \in S\} = 0.$$

References

[1] J. A. Baker, The stability of the cosine equation, *Proc. Amer. Math. Soc.* **80** (1980), 411–416.

[2] D. H. Hyers and Th. M. Rassias, Approximate homomorphisms, *Aequationes Math.* **44** (1992), 125–153.

[3] Zs. Páles, Remark 27, *Report of Meeting, Aequationes Math.* **53** (1997), 200–201.

[4] J. Tabor, Stability of the Cauchy type equation in $\mathcal{L}_p$ norms, *Result. Math.* **32** (1997), 145–158.

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