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On certain tensor fields on contact metric manifolds

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Dedicated to Professor Lajos Tamássy on his 70th birthday

In Sasakian manifolds, Matsumoto and Chūman [3] defined a contact Bochner curvature tensor (see also Yano [6]). This is invariable with respect to a D-homothetic deformation (see Tanno [5] about a D-homothetic deformation). In this paper we define a new tensor field which is invariable with respect to the D-homothetic deformation in a contact metric manifold and call it an E-tensor. Then we shall study the contact metric manifold with vanishing E-tensor.

1. Preliminaries

Let M be a $(2n+1)$-dimensional contact metric manifold with the structure tensors (ϕ, ξ, η, g), where ϕ is a linear mapping $TM \to TM$ (TM is the tangent bundle over M), ξ is a vector field, η a l-form, and g a Riemannian metric on M, such that

$$\phi\xi = 0, \quad \eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(X) = g(\xi, X)$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(\phi X, Y) = d\eta(X, Y)$$

for any vector fields X and Y on M. Now we define an operator h by $h = -\frac{1}{2}\mathcal{L}_\xi\phi$ where $\mathcal{L}$ denotes Lie differentiation. Then the vector field ξ is Killing if and only if h vanishes. It is well known that h and ϕh are symmetric operators, h anti-commutes with ϕ (i.e., $\phi h + h\phi = 0$), $h\xi = 0$, $\eta \circ h = 0$, $\operatorname{Tr} h = 0$ and $\operatorname{Tr} \phi h = 0$ (see [1]). Moreover the following general formulas for a contact metric manifold M were obtained

(1.1) $$\nabla_X \xi = \phi X + \phi h X$$

(1.2) $$g(Q\xi, \xi) = 2n - \operatorname{Tr} h^2$$

$$(1.3)\qquad \sum_{i=1}^{2n+1} (\nabla_{E_i}\phi)E_i = -2n\xi \quad (\{E_i\} \text{ is an orthonormal frame})$$

$$(1.4)\quad (\nabla_{\phi X}\phi)\phi Y + (\nabla_X\phi)Y = -2g(X,Y)\xi + \eta(Y)\big(X + hX + \eta(X)\xi\big)$$
$$\Big((\nabla_X\phi)\xi = -\eta(X)\xi + X + hX\Big)$$

$$(1.5)\qquad \phi(\nabla_\xi h)X = X - \eta(X)\xi - h^2X - R(X,\xi)\xi,$$

where ∇ is the covariant differentiation with respect to g, Q is the Ricci operator of M, R is the curvature tensor field of M and $\operatorname{Tr} h$ denotes the trace of h (see [1] [2] and [4]). Moreover, by means of $\phi h\xi = 0$, we get

$$(1.6)\qquad \phi(\nabla_Y h)\xi = -hY - h^2Y.$$

If ξ is Killing in a contact metric manifold M, M is said to be a K-contact Riemannian manifold. If a contact metric manifold M is normal (i.e., $N + d\eta\otimes\xi = 0$, where N denotes the Nijenhuis tensor formed with ϕ), M is called a Sasakian manifold. A Sasakian manifold is a K-contact Riemannian manifold. In a Sasakian manifold with structure tensors (ϕ, ξ, η, g) we have

$$\nabla_X\xi = \phi X, \quad (\nabla_X\phi)Y = R(X,\xi)Y = -g(X,Y)\xi + \eta(Y)X$$
$$\phi Q = Q\phi, \qquad Q\xi = 2n\xi \quad \text{(see e.g., [7])}.$$

2. D-homothetic deformations

Let M be an $(m+1)$-dimensional $(m = 2n)$ contact metric manifold. Now we define the tensor field $\tilde{B}$ in M by

$$(2.1)\qquad \tilde{B}(X,Y) = R(X,Y) + \frac{1}{m+4}\Big(QY\wedge X - QX\wedge Y$$
$$+ Q\phi Y\wedge\phi X - Q\phi X\wedge\phi Y + g(Q\phi X, Y)\phi - g(Q\phi Y, X)\phi$$
$$+ 2g(\phi X, Y)Q\phi + \eta(Y)QX\wedge\xi + \eta(X)\xi\wedge QY\Big)$$
$$-\frac{k+m}{m+4}\Big(\phi Y\wedge\phi X + 2g(\phi X, Y)\phi\Big) - \frac{k-4}{m+4}Y\wedge X$$
$$+\frac{k}{m+4}\Big(\eta(Y)\xi\wedge X + \eta(X)Y\wedge\xi\Big),$$

where $k = \dfrac{S+m}{m+2}$ (S is the scalar curvature of M) and $(X\wedge Y)Z = g(Y,Z)X - g(X,Z)Y$ (cf., [3]). If M is a Sasakian manifold, $\tilde{B}$ coincides with the contact Bochner curvature tensor by Matsumoto and Chūman [3] and the following are satisfied

$$\tilde{B}(\xi,Y)Z=\tilde{B}(X,Y)\xi=0, \qquad \eta(\tilde{B}(X,Y)Z)=0, \tag{2.2}$$
$$\tilde{B}(\phi X,\phi Y)Z=\tilde{B}(X,Y)Z,$$

where we have used $R(\phi X,Y)Z-R(\phi Y,X)Z=g(\phi Z,X)Y-g(\phi Z,Y)X-g(Z,X)\phi Y+g(Z,Y)\phi X$ in a Sasakian manifold.

We consider a D-homothetic deformation $g^\star=\alpha g+\alpha(\alpha-1)\eta\otimes\eta$, $\phi^\star=\phi$, $\xi^\star=\alpha^{-1}\xi$, $\eta^\star=\alpha\eta$ on a contact metric manifold M, where α is a positive constant, and for a D-homothetic deformation we say that $M(\phi,\xi,\eta,g)$ is D-homothetic to $M(\phi^\star,\xi^\star,\eta^\star,g^\star)$. It is well known that if a contact metric manifold $M(\phi,\xi,\eta,g)$ is D-homothetic to $M(\phi^\star,\xi^\star,\eta^\star,g^\star)$, $M(\phi^\star,\xi^\star,\eta^\star,g^\star)$ is a contact metric manifold. Moreover if $M(\phi,\xi,\eta,g)$ is a K-contact Riemannian manifold (resp. Sasakian manifold), then $M(\phi^\star,\xi^\star,\eta^\star,g^\star)$ is also a K-contact Riemannian manifold (resp. Sasakian manifold) (see [5]). Denoting by $W^i{}_{jk}$ the difference $\Gamma^\star{}_{jk}{}^i-\Gamma_{jk}{}^i$ of Christoffel symbols, by (1.1), we have in a contact metric manifold

$$\begin{aligned}W(X,Y)&=(\alpha-1)\big(\eta(Y)\phi X+\eta(X)\phi Y\big)\\&\qquad+\frac{\alpha-1}{2\alpha}\big((\nabla_X\eta)(Y)+(\nabla_Y\eta)(X)\big)\xi\quad\text{(see [5])}\\&=(\alpha-1)\big(\eta(Y)\phi X+\eta(X)\phi Y\big)+\frac{\alpha-1}{\alpha}g(\phi hX,Y)\xi.\end{aligned}$$

Putting this into

$$\begin{aligned}R^\star(X,Y)Z=R(X,Y)Z&+(\nabla_XW)(Z,Y)-(\nabla_YW)(Z,X)\\&+W(W(Z,Y),X)-W(W(Z,X),Y)\end{aligned}$$

and using (1.1), we have

$$\begin{aligned}R^\star(X,Y)Z&=R(X,Y)Z+(\alpha-1)\\&\times\Big(2g(\phi X,Y)\phi Z+g(\phi Z,Y)\phi X-g(\phi Z,X)\phi Y+\eta(Y)(\nabla_X\phi)Z\\&\qquad-\eta(X)(\nabla_Y\phi)Z+\eta(Z)(\nabla_X\phi)Y-\eta(Z)(\nabla_Y\phi)X\Big)\\&\qquad-(\alpha-1)^2\Big(\eta(Z)\eta(X)Y-\eta(Z)\eta(Y)X\Big)\\&-\frac{\alpha-1}{\alpha}\Big(g\big(X,(\nabla_Y\phi)hZ\big)\xi-g\big(Y,(\nabla_X\phi)hZ\big)\xi+g\big(X,\phi(\nabla_Yh)Z\big)\xi\\&\qquad-g\big(Y,\phi(\nabla_Xh)Z\big)\xi+g(X,\phi hZ)\phi hY-g(Y,\phi hZ)\phi hX\Big)\end{aligned}\tag{2.3}$$

$$-\frac{(\alpha-1)^2}{\alpha}\Big(\eta(X)g(hZ,Y)\xi-\eta(Y)g(hZ,X)\xi\Big).$$

Here, choosing a ϕ-basis and using (1.3) and (1.5), we get

$$\begin{aligned}(2.4)\qquad \mathrm{Ric}^\star(X,Y)&=\mathrm{Ric}(X,Y)\\ &+(\alpha-1)\big(-2g(X,Y)+2(2n+1)\eta(X)\eta(Y)\big)\\ &+2n(\alpha-1)^2\eta(X)\eta(Y)-\frac{\alpha-1}{\alpha}\Big(-g(X,Y)\\ &+\eta(X)\eta(Y)-2g(hX,Y)+g(hX,hY)+g\big(R(X,\xi)\xi,Y\big)\Big),\end{aligned}$$

where Ric is the Ricci curvature of M. From (2.4) we find

$$\begin{aligned}(2.5)\qquad Q^\star X&=\frac{1}{\alpha}QX+\frac{\alpha-1}{\alpha}\big(-2X+2(2n+1)\eta(X)\xi\big)\\ &-\frac{\alpha-1}{\alpha^2}g(X,Q\xi)\xi-2n\left(\frac{\alpha-1}{\alpha}\right)^2\eta(X)\xi\\ &-\frac{\alpha-1}{\alpha^2}\big(-X+\eta(X)\xi-2hX+h^2X+R(X,\xi)\xi\big),\end{aligned}$$

where we have used $Q^\star\xi=\frac{1}{\alpha}Q\xi-\dfrac{\alpha-1}{\alpha^2}g(Q\xi,\xi)\xi+2n\dfrac{\alpha^2-1}{\alpha^2}\xi$.
By virtue of (1.2) we have

$$(2.6)\qquad S^\star=\frac{1}{\alpha}S-2n\frac{\alpha-1}{\alpha}+\frac{\alpha-1}{\alpha^2}\operatorname{Tr}h^2.$$

From (2.1), (2.3), (2.4), (2.5) and (2.6), after some lengthy computation we obtain

$$\begin{aligned}(2.7)\qquad \tilde{B}^\star(X,Y)Z&=\tilde{B}(X,Y)Z+(\alpha-1)\Big(\eta(Y)g(X,Z)\xi\\ &-\eta(X)g(Y,Z)\xi+2\eta(X)\eta(Z)Y-2\eta(Y)\eta(Z)X+\eta(Y)(\nabla_X\phi)Z\\ &+\eta(Z)(\nabla_X\phi)Y-\eta(X)(\nabla_Y\phi)Z-\eta(Z)(\nabla_Y\phi)X\Big)\\ &-\frac{\alpha-1}{\alpha}\Big(g\big(X,(\nabla_Y\phi)hZ\big)\xi-g\big(Y,(\nabla_X\phi)hZ\big)\xi+g\big(X,\phi(\nabla_Yh)Z\big)\xi\\ &-g\big(Y,\phi(\nabla_Xh)Z\big)\xi+g(X,\phi hZ)\phi hY-g(Y,\phi hZ)\phi hX\Big)\\ &-\frac{(\alpha-1)^2}{\alpha}\Big(\eta(X)g(hZ,Y)\xi-\eta(Y)g(hZ,X)\xi\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2n+4}\Big\{\frac{\alpha-1}{\alpha}\Big(\eta(X)\eta(Z)g(Y,Q\xi)\xi-\eta(Y)\eta(Z)g(X,Q\xi)\xi\\
&+g(Y,Z)g(X,Q\xi)\xi-g(X,Z)g(Y,Q\xi)\xi+g(\phi Y,Z)g(\phi X,Q\xi)\xi\\
&-g(\phi X,Z)g(\phi Y,Q\xi)\xi-2g(\phi X,Y)g(\phi Z,Q\xi)\xi\Big)\\
&+2n(\alpha-1)\Big(\eta(Y)g(X,Z)\xi-\eta(X)g(Y,Z)\xi\Big)\\
&-\big(g(X,Z)-\eta(X)\eta(Z)\big)\Big(2n\frac{(\alpha-1)^2}{\alpha}\eta(Y)\xi\\
&+\frac{\alpha-1}{\alpha}\big(-Y+\eta(Y)\xi-2hY+h^2Y+R(Y,\xi)\xi\big)\Big)\\
&+\frac{\alpha-1}{\alpha}\Big(-g(Y,Z)+\eta(Y)\eta(Z)-2g(hY,Z)+g(hY,hZ)\\
&+g\big(R(Y,\xi)\xi,Z\big)\Big)X+\big(g(Y,Z)-\eta(Y)\eta(Z)\big)\\
&\times\Big(2n\frac{(\alpha-1)^2}{\alpha}\eta(X)\xi+\frac{\alpha-1}{\alpha}\big(-X+\eta(X)\xi-2hX+h^2X+R(X,\xi)\xi\big)\Big)\\
&-\frac{\alpha-1}{\alpha}\Big(-g(X,Z)+\eta(X)\eta(Z)-2g(hX,Z)+g(hX,hZ)+g\big(R(X,\xi)\xi,Z\big)\Big)Y\\
&-\frac{\alpha-1}{\alpha}g(\phi X,Z)\Big(-\phi Y-2h\phi Y+h^2\phi Y+R(\phi Y,\xi)\xi\Big)\\
&+\frac{\alpha-1}{\alpha}\Big(-g(\phi Y,Z)-2g(h\phi Y,Z)+g(h\phi Y,hZ)+g\big(R(\phi Y,\xi)\xi,Z\big)\Big)\phi X\\
&+\frac{\alpha-1}{\alpha}g(\phi Y,Z)\Big(-\phi X-2h\phi X+h^2\phi X+R(\phi X,\xi)\xi\Big)\\
&-\frac{\alpha-1}{\alpha}\Big(-g(\phi X,Z)-2g(h\phi X,Z)+g(h\phi X,hZ)+g\big(R(\phi X,\xi)\xi,Z\big)\Big)\phi Y\\
&-\frac{\alpha-1}{\alpha}\Big(-2g(\phi X,Y)-2g(\phi hX,hY)+g\big(R(\phi X,\xi)\xi,Y\big)-g\big(R(\phi Y,\xi)\xi,X\big)\Big)\phi Z\\
&-2\frac{\alpha-1}{\alpha}g(\phi X,Y)\Big(-\phi Z-2h\phi Z+h^2\phi Z+R(\phi Z,\xi)\xi\Big)\\
&+\frac{\alpha-1}{\alpha}\Big(-g(X,Z)+\eta(X)\eta(Z)+2g(hX,Z)+g(hX,hZ)\\
&+g\big(R(X,\xi)\xi,Z\big)\Big)\eta(Y)\xi-\frac{\alpha-1}{\alpha}\Big(-g(Y,Z)+\eta(Y)\eta(Z)
\end{aligned}
$$

$$-2g(hY,Z)+g(hY,hZ)+g\big(R(Y,\xi)\xi,Z\big)\Big)\eta(X)\xi\Big\}$$

$$-\frac{1}{(2n+4)(2n+2)}\frac{\alpha-1}{\alpha}\operatorname{Tr}h^2\Big(g(\phi X,Z)\phi Y-g(\phi Y,Z)\phi X$$

$$+2g(\phi X,Y)\phi Z+g(X,Z)Y-g(Y,Z)X-\eta(Y)g(X,Z)\xi$$

$$+\eta(X)g(Y,Z)\xi+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y\Big)$$

Now we shall introduce in M an E-tensor by

$$E(X,Y)Z=\tilde{B}(X,Y)Z-\eta(X)\tilde{B}(\xi,Y)Z-\eta(Y)\tilde{B}(X,\xi)Z \tag{2.8}$$

$$-\eta(Z)\tilde{B}(X,Y)\xi-\eta\big(\tilde{B}(X,Y)Z\big)\xi-\tilde{B}(\phi X,\phi Y)Z+\eta(Z)\tilde{B}(\phi X,\phi Y)\xi$$

$$+\eta\big(\tilde{B}(\phi X,\phi Y)Z\big)\xi-\phi\tilde{B}(X,\phi Y)Z-\phi\tilde{B}(\phi X,Y)Z+\eta(X)\phi\tilde{B}(\xi,\phi Y)Z$$

$$+\eta(Y)\phi\tilde{B}(\phi X,\xi)Z+\eta(Z)\phi\tilde{B}(X,\phi Y)\xi+\eta(Z)\phi\tilde{B}(\phi X,Y)\xi$$

$$+\eta(X)\eta\big(\tilde{B}(\xi,Y)Z\big)\xi+\eta(Y)\eta\big(\tilde{B}(X,\xi)Z\big)\xi.$$

In particular if M is Sasakian, $E=0$ from (2.2).

Theorem 2.1. *The E-tensor is invariant with respect to the D-homothetic deformation $M(\phi,\xi,\eta,g)\to M(\phi^\star,\xi^\star,\eta^\star,g^\star)$ on a contact metric manifold M.*

Proof. Using (1.4), (1.5), (1.6) and (2.7), we find

$$-\eta^\star(X)\tilde{B}^\star(\xi^\star,Y)Z=-\eta(X)\tilde{B}^\star(\xi,Y)Z=-\eta(X)\tilde{B}(\xi,Y)Z \tag{2.9}$$

$$+(\alpha-1)\Big(-\eta(X)\eta(Z)Y+\eta(X)\eta(Z)hY+\eta(X)(\nabla_Y\phi)Z\Big)$$

$$+\frac{\alpha-1}{\alpha}\Big(\eta(X)\eta(Y)\eta(Z)\xi+\eta(X)g\big(Y,R(Z,\xi)\xi\big)\xi\Big)$$

$$+\frac{(\alpha-1)^2}{\alpha}\eta(X)g(Y,Z)\xi+\frac{(\alpha-1)(\alpha-2)}{\alpha}\eta(X)g(Y,hZ)\xi$$

$$-\frac{1}{2n+4}\frac{\alpha-1}{\alpha}\Big(g(Y,Z)-\eta(Y)\eta(Z)\Big)\Big(g(Q\xi,\xi)-2n\Big)\eta(X)\xi$$

$$-\eta^\star(Y)\tilde{B}^\star(X,\xi^\star)Z=-\eta(Y)\tilde{B}(X,\xi)Z \tag{2.10}$$

$$+(\alpha-1)\Big(\eta(Y)\eta(Z)X-\eta(Y)\eta(Z)hX-\eta(Y)(\nabla_X\phi)Z\Big)$$

$$-\frac{\alpha-1}{\alpha}\Big(\eta(X)\eta(Y)\eta(Z)\xi+\eta(Y)g\big(X,R(Z,\xi)\xi\big)\xi\Big)$$

$$-\frac{(\alpha-1)^2}{\alpha}\eta(Y)g(Z,X)\xi-\frac{(\alpha-1)(\alpha-2)}{\alpha}\eta(Y)g(hZ,X)\xi$$
$$+\frac{1}{2n+4}\frac{\alpha-1}{\alpha}\Big(g(X,Z)-\eta(X)\eta(Z)\Big)\Big(g(Q\xi,\xi)-2n\Big)\eta(Y)\xi$$

$$-\eta^\star(Z)\tilde{B}^\star(X,Y)\xi^\star=-\eta(Z)\tilde{B}(X,Y)\xi \tag{2.11}$$
$$+(\alpha-1)\Big(-\eta(X)\eta(Z)Y+\eta(Y)\eta(Z)X-\eta(Y)\eta(Z)hX$$
$$+\eta(X)\eta(Z)hY-\eta(Z)(\nabla_X\phi)Y+\eta(Z)(\nabla_Y\phi)X\Big)$$

$$-\eta^\star\big(\tilde{B}^\star(X,Y)Z\big)\xi^\star=-\eta\big(\tilde{B}^\star(X,Y)Z\big)\xi \tag{2.12}$$
$$=-\eta\big(\tilde{B}(X,Y)Z\big)\xi-\frac{\alpha-1}{\alpha}\Big(\eta(X)g(Z,hY)\xi-\eta(Y)g(Z,hX)\xi$$
$$-g\big(X,(\nabla_Y\phi)hZ\big)\xi+g\big(Y,(\nabla_X\phi)hZ\big)\xi-g\big(X,\phi(\nabla_Yh)Z\big)\xi$$
$$+g\big(Y,\phi(\nabla_Xh)Z\big)\xi\Big)+\frac{1}{2n+4}\Big\{\frac{\alpha-1}{\alpha}\Big(-\eta(X)\eta(Z)g(Y,Q\xi)\xi$$
$$+\eta(Y)\eta(Z)g(X,Q\xi)\xi-g(Y,Z)g(X,Q\xi)\xi+g(X,Z)g(Y,Q\xi)\xi$$
$$-g(\phi Y,Z)g(\phi X,Q\xi)\xi+g(\phi X,Z)g(\phi Y,Q\xi)\xi+2g(\phi X,Y)g(\phi Z,Q\xi)\xi\Big)$$
$$-2n(\alpha-1)\Big(g(X,Z)\eta(Y)\xi-g(Y,Z)\eta(X)\xi\Big)$$
$$+2n\frac{(\alpha-1)^2}{\alpha}\Big(g(X,Z)-\eta(X)\eta(Z)\Big)\eta(Y)\xi$$
$$+\frac{\alpha-1}{\alpha}\Big(g(Y,Z)-\eta(Y)\eta(Z)+2g(hY,Z)-g(hY,hZ)$$
$$-g\big(R(Y,\xi)\xi,Z\big)\Big)\eta(X)\xi-2n\frac{(\alpha-1)^2}{\alpha}\Big(g(Y,Z)-\eta(Y)\eta(Z)\Big)\eta(X)\xi$$
$$-\frac{\alpha-1}{\alpha}\Big(g(X,Z)-\eta(X)\eta(Z)+2g(hX,Z)-g(hX,hZ)$$
$$-g\big(R(X,\xi)\xi,Z\big)\Big)\eta(Y)\xi+\frac{\alpha-1}{\alpha}\Big(g(X,Z)-\eta(X)\eta(Z)-2g(hX,Z)$$
$$-g(hX,hZ)-g\big(R(X,\xi)\xi,Z\big)\Big)\eta(Y)\xi-\frac{\alpha-1}{\alpha}\Big(g(Y,Z)-\eta(Y)\eta(Z)$$

$$+2g(hY,Z)-g(hY,hZ)-g\big(R(Y,\xi)\xi,Z\big)\Big)\eta(X)\xi\Big\}.$$

From (2.7), (2.11) and (2.12), we get

$$\begin{aligned}
(2.13)\qquad & -\tilde{B}^\star(\phi^\star X,\phi^\star Y)Z=-\tilde{B}(\phi X,\phi Y)Z\\
&+(\alpha-1)\Big(-\eta(Z)(\nabla_{\phi X}\phi)\phi Y+\eta(Z)(\nabla_{\phi Y}\phi)\phi X\Big)\\
&+\frac{\alpha-1}{\alpha}\Big(g\big(\phi X,(\nabla_{\phi Y}\phi)hZ\big)\xi-g\big(\phi Y,(\nabla_{\phi X}\phi)hZ\big)\xi\\
&+g\big(\phi X,\phi(\nabla_{\phi Y}h)Z\big)\xi-g\big(\phi Y,\phi(\nabla_{\phi X}h)Z\big)\xi+g(X,hZ)hY-g(Y,hZ)hX\Big)\\
&+\frac{1}{2n+4}\Big\{\frac{\alpha-1}{\alpha}\Big(-g(\phi Y,Z)g(\phi X,Q\xi)\xi+g(\phi X,Z)g(\phi Y,Q\xi)\xi\\
&+2g(\phi X,Y)g(\phi Z,Q\xi)\xi-g(Y,Z)g(X,Q\xi)\xi+g(X,Z)g(Y,Q\xi)\xi\\
&-g(X,Z)\eta(Y)g(Q\xi,\xi)\xi+g(Y,Z)\eta(X)g(Q\xi,\xi)\xi\\
&-g(Y,Q\xi)\eta(X)\eta(Z)\xi+g(X,Q\xi)\eta(Y)\eta(Z)\xi\Big)\\
&+\frac{\alpha-1}{\alpha}g(\phi X,Z)\Big(-\phi Y-2h\phi Y+h^2\phi Y+R(\phi Y,\xi)\xi\Big)\\
&-\frac{\alpha-1}{\alpha}\Big(-g(\phi Y,Z)-2g(h\phi Y,Z)+g(h\phi Y,hZ)+g\big(R(\phi Y,\xi)\xi,Z\big)\Big)\phi X\\
&-\frac{\alpha-1}{\alpha}g(\phi Y,Z)\Big(-\phi X-2h\phi X+h^2\phi X+R(\phi X,\xi)\xi\Big)\\
&+\frac{\alpha-1}{\alpha}\Big(-g(\phi X,Z)-2g(h\phi X,Z)+g(h\phi X,hZ)+g\big(R(\phi X,\xi)\xi,Z\big)\Big)\phi Y\\
&+\frac{\alpha-1}{\alpha}\Big(g(X,Z)-\eta(X)\eta(Z)\Big)\Big(-Y+\eta(Y)\xi-hY+h^2Y+R(Y,\xi)\xi\Big)\\
&-\frac{\alpha-1}{\alpha}\Big(-g(Y,Z)X+\eta(Y)\eta(Z)X-2g(hY,Z)X+g(hY,hZ)X\\
&+g\big(R(Y,\xi)\xi,Z\big)X+g(Y,Z)\eta(X)\xi-\eta(X)\eta(Y)\eta(Z)\xi\\
&+2g(hY,Z)\eta(X)\xi-g(hY,hZ)\eta(X)\xi-g\big(R(Y,\xi)\xi,Z\big)\eta(X)\xi\Big)\\
&-\frac{\alpha-1}{\alpha}\Big(g(Y,Z)-\eta(Y)\eta(Z)\Big)\Big(-X+\eta(X)\xi-2hX+h^2X+R(Y,\xi)\xi\Big)\\
&+\frac{\alpha-1}{\alpha}\Big(-g(X,Z)Y+\eta(X)\eta(Z)Y-2g(hX,Z)Y+g(hX,hZ)Y
\end{aligned}$$

$$+g\big(R(X,\xi)\xi,Z\big)Y+g(X,Z)\eta(Y)\xi-\eta(X)\eta(Y)\eta(Z)\xi$$

$$+2g(hX,Z)\eta(Y)\xi-g(hX,hZ)\eta(Y)\xi-g\big(R(X,\xi)\xi,Z\big)\eta(Y)\xi\Big)$$

$$+\frac{\alpha-1}{\alpha}\Big(2g(X,\phi Y)-2g(hX,h\phi Y)-g\big(R(X,\xi)\xi,\phi Y\big)+g\big(R(Y,\xi)\xi,\phi X\big)\Big)\phi Z$$

$$+2\frac{\alpha-1}{\alpha}g(\phi X,Y)\Big(-\phi Z-2h\phi Z+h^2\phi Z+R(\phi Z,\xi)\xi\Big)\Big\}$$

$$+\frac{1}{(2n+4)(2n+2)}\frac{\alpha-1}{\alpha}\operatorname{Tr}h^2\Big(g(X,Z)Y-g(Y,Z)X$$

$$+g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(Y)\eta(Z)X$$

$$-\eta(X)\eta(Z)Y+g(\phi X,Z)\phi Y-g(\phi Y,Z)\phi X-2g(X,\phi Y)\phi Z\Big)$$

$$\eta^\star(Z)\tilde{B}^\star(\phi^\star X,\phi^\star Y)\xi^\star=\eta(Z)\tilde{B}(\phi X,\phi Y)\xi \tag{2.14}$$

$$+(\alpha-1)\Big(\eta(Z)(\nabla_{\phi X}\phi)\phi Y-\eta(Z)(\nabla_{\phi Y}\phi)\phi X\Big)$$

$$\eta^\star\big(\tilde{B}^\star(\phi^\star X,\phi^\star Y)Z\big)\xi^\star=\eta\big(\tilde{B}(\phi X,\phi Y)Z\big)\xi \tag{2.15}$$

$$-\frac{\alpha-1}{\alpha}\Big(g\big(\phi X,(\nabla_{\phi Y}\phi)hZ\big)\xi-g\big(\phi Y,(\nabla_{\phi X}\phi)hZ\big)\xi$$

$$+g\big(\phi X,\phi(\nabla_{\phi Y}h)Z\big)\xi-g\big(\phi Y,\phi(\nabla_{\phi X}h)Z\big)\xi\Big)$$

$$+\frac{1}{2n+4}\frac{\alpha-1}{\alpha}\Big(g(\phi Y,Z)g(\phi X,Q\xi)\xi-g(\phi X,Z)g(\phi Y,Q\xi)\xi$$

$$-2g(\phi X,Y)g(\phi Z,Q\xi)\xi+g(Y,Z)g(X,Q\xi)\xi-g(X,Z)g(Y,Q\xi)\xi$$

$$+g(X,Z)\eta(Y)g(Q\xi,\xi)\xi-g(Y,Z)\eta(X)g(Q\xi,\xi)\xi$$

$$+g(Y,Q\xi)\eta(X)\eta(Z)\xi-g(X,Q\xi)\eta(Y)\eta(Z)\xi\Big).$$

Using (1.4), (1.5) and (2.12), we have

$$\eta^\star(X)\eta^\star\big(\tilde{B}^\star(\xi^\star,Y)Z\big)\xi^\star+\eta^\star(Y)\eta^\star\big(\tilde{B}^\star(X,\xi^\star)Z\big)\xi^\star \tag{2.16}$$

$$=\frac{\alpha-1}{\alpha}\Big(2\eta(X)g(Y,hZ)\xi-2\eta(Y)g(X,hZ)\xi+\eta(X)g(Y,Z)\xi$$

$$-\eta(Y)g(X,Z)\xi-\eta(X)g\big(Y,R(Z,\xi)\xi\big)\xi+\eta(Y)g\big(X,R(Z,\xi)\xi\big)\xi\Big)$$

$$+\frac{1}{2n+4}\frac{\alpha-1}{\alpha}\Big(g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\Big)\Big(g(Q\xi,\xi)-2n\Big)\xi.$$

Thus using (2.7), (2.9), (2.10), (2.11), (2.12), (2.13), (2.14), (2.15) and (2.16) we get

$$\begin{aligned}
(2.17)\qquad & \tilde{B}^\star(X,Y)Z-\eta^\star(X)\tilde{B}^\star(\xi^\star,Y)Z-\eta^\star(Y)\tilde{B}^\star(X,\xi^\star)Z\\
&-\eta^\star(Z)\tilde{B}^\star(X,Y)\xi^\star-\eta^\star\big(\tilde{B}^\star(X,Y)Z\big)\xi^\star-\tilde{B}^\star(\phi^\star X,\phi^\star Y)Z\\
&+\eta^\star(Z)\tilde{B}^\star(\phi^\star X,\phi^\star Y)\xi^\star+\eta^\star\big(\tilde{B}^\star(\phi^\star X,\phi^\star Y)Z\big)\xi^\star\\
&+\eta^\star(X)\eta^\star\big(\tilde{B}^\star(\xi^\star,Y)Z\big)\xi^\star+\eta^\star(Y)\eta^\star\big(\tilde{B}^\star(X,\xi^\star)Z\big)\xi^\star\\
&=\tilde{B}(X,Y)Z-\eta(X)\tilde{B}(\xi,Y)Z-\eta(Y)\tilde{B}(X,\xi)Z-\eta(Z)\tilde{B}(X,Y)\xi\\
&-\eta\big(\tilde{B}(X,Y)Z\big)\xi-\tilde{B}(\phi X,\phi Y)Z+\eta(Z)\tilde{B}(\phi X,\phi Y)\xi\\
&+\eta\big(\tilde{B}(\phi X,\phi Y)Z\big)\xi+\eta(X)\eta\big(\tilde{B}(\xi,Y)Z\big)\xi\\
&+\eta(Y)\eta\big(\tilde{B}(X,\xi)Z\big)\xi-2(\alpha-1)\Big(\eta(Y)\eta(Z)hX-\eta(X)\eta(Z)hY\Big)\\
&-\frac{\alpha-1}{\alpha}\Big(g(Y,hZ)hX-g(X,hZ)hY+g(X,\phi hZ)\phi hY-g(Y,\phi hZ)\phi hX\Big).
\end{aligned}$$

Here, substituting $\phi^\star Y$ for Y in (2.17) and transforming (2.17) by $\phi^\star$, we have

$$\begin{aligned}
(2.18)\ & -\phi^\star\tilde{B}^\star(X,\phi^\star Y)-\phi^\star\tilde{B}^\star(\phi^\star X,Y)Z+\eta^\star(X)\phi^\star\tilde{B}^\star(\xi^\star,\phi^\star Y)Z\\
&-\eta^\star(Y)\phi^\star\tilde{B}^\star(\phi^\star X,\xi^\star)Z-\eta^\star(Z)\phi^\star\tilde{B}^\star(X,\phi^\star Y)\xi^\star\\
&-\eta^\star(Z)\phi^\star\tilde{B}^\star(\phi^\star X,Y)\xi^\star-\eta^\star(Y)\eta^\star(Z)\phi^\star\tilde{B}^\star(\phi^\star X,\xi^\star)\xi^\star\\
&=-\phi\tilde{B}(X,\phi Y)-\phi\tilde{B}(\phi X,Y)Z+\eta(X)\phi\tilde{B}(\xi,\phi Y)Z\\
&-\eta(Y)\phi\tilde{B}(\phi X,\xi)Z-\eta(Z)\phi\tilde{B}(X,\phi Y)\xi-\eta(Z)\phi\tilde{B}(\phi X,Y)\xi\\
&-\eta(Y)\eta(Z)\phi\tilde{B}(\phi X,\xi)\xi-2(\alpha-1)\eta(X)\eta(Z)hY\\
&-\frac{\alpha-1}{\alpha}\Big(g(X,\phi hZ)h\phi Y-g(Y,hZ)hX+g(Y,\phi hZ)\phi hX+g(X,hZ)hY\Big).
\end{aligned}$$

On the other hand we find by virtue of (1.4)

$$\begin{aligned}
(2.19)\quad & -\eta^\star(Y)\eta^\star(Z)\phi^\star\tilde{B}^\star(\phi^\star X,\xi^\star)\xi^\star\\
&=-\eta(Y)\eta(Z)\phi\tilde{B}(\phi X,\xi)\xi-2(\alpha-1)\eta(Y)\eta(Z)hX\,.
\end{aligned}$$

Substituting (2.19) into (2.18) and combining (2.17) and (2.18), we obtain our result.

3. Contact metric manifolds with vanishing E-tensor

We define

$$s^{\#} = \sum_{i,j=1}^{2n+1} g(R(E_i, E_j)\phi E_j, \phi E_i), \tag{3.1}$$

where $\{E_i\}$ is an orthonormal frame.

Lemma 3.1. ([4]). *For any $(2n+1)$-dimensional contact metric manifold M we have*

$$s^{\#} - S + 4n^2 = \operatorname{Tr} h^2 + \frac{1}{2}\{\|\nabla\phi\|^2 - 4n\} \geq 0.$$

Moreover M is Sasakian if and only if $\|\nabla\phi\|^2 - 4n = 0$ or equivalently

$$s^{\#} - S + 4n^2 = 0.$$

Proposition 3.1. *Let M be a $(2n+1)$-dimensional contact metric manifold with vanishing E-tensor. Then we have*

$$2 \operatorname{Tr} h^2 = \|\nabla\phi\|^2 - 4n.$$

Proof. Since the E-tensor of M vanishes, we have

$$\begin{aligned}
g\big(\tilde{B}(X,Y)Z,W\big) &= \eta(X)g\big(\tilde{B}(\xi,Y)Z,W\big) \\
&+\eta(Y)g\big(\tilde{B}(X,\xi)Z,W\big) + \eta(Z)g\big(\tilde{B}(X,Y)\xi,W\big) + \eta(W)\eta\big(\tilde{B}(X,Y)Z\big) \\
&+g\big(\tilde{B}(\phi X,\phi Y)Z,W\big) - \eta(Z)g\big(\tilde{B}(\phi X,\phi Y)\xi,W\big) - \eta(W)\eta\big(\tilde{B}(\phi X,\phi Y)Z\big) \\
&-\eta(X)\eta(W)\eta\big(\tilde{B}(\xi,Y)Z\big) - \eta(Y)\eta(W)\eta\big(\tilde{B}(X,\xi)Z\big) + g\big(\phi\tilde{B}(X,\phi Y)Z,W\big) \\
&+g\big(\phi\tilde{B}(\phi X,Y)Z,W\big) - \eta(X)g\big(\phi\tilde{B}(\xi,\phi Y)Z,W\big) - \eta(Y)g\big(\phi\tilde{B}(\phi X,\xi)Z,W\big) \\
&-\eta(Z)g\big(\phi\tilde{B}(X,\phi Y)\xi,W\big) - \eta(Z)g\big(\phi\tilde{B}(\phi X,Y)\xi,W\big)
\end{aligned} \tag{3.2}$$

Taking $X = E_i$, $Y = E_j$, $Z = \phi E_j$, $W = \phi E_i$, ($\{E_i\}$ is a ϕ-basis) into each member of (3.2) and summing over i and j, we have

$$\sum_{i,j=1}^{2n+1} g\big(\tilde{B}(E_i,E_j)\phi E_j,\phi E_i\big) = s^{\#} - S - \frac{2(n+1)}{n+2}\operatorname{Tr} h^2 + 4n^2$$

$$\begin{aligned}
\sum_{i,j=1}^{2n+1} g\big(\tilde{B}(\phi E_i,\phi E_j)\phi E_j,\phi E_i\big) &= S - 2g(Q\xi,\xi) \\
&+ \frac{2(n+1)}{n+2}\big(g(Q\xi,\xi) - S\big) + \frac{3n(k+2n)}{n+2} + \frac{n(2n-1)(k-4)}{n+2}
\end{aligned}$$

$$\sum_{i,j=1}^{2n+1} g\bigl(\phi\tilde{B}(E_i,\phi E_j)\phi E_j,\phi E_i\bigr) = S - 2g(Q\xi,\xi) + \frac{2(n+1)}{n+2}\bigl(g(Q\xi,\xi) - S\bigr) + \frac{3n(k+2n)}{n+2} + \frac{n(2n-1)(k-4)}{n+2}$$

$$\sum_{i,j=1}^{2n+1} g\bigl(\phi\tilde{B}(\phi E_i,E_j)\phi E_j,\phi E_i\bigr) = -s^{\#} - \frac{2(n+1)}{n+2}\bigl(g(Q\xi,\xi) - S\bigr) - \frac{n(2n+1)}{n+2}(k+2n) - \frac{n(k-4)}{n+2}$$

$$\sum_{i,j=1}^{2n+1} \eta(E_i) g\bigl(\tilde{B}(\xi,E_j)\phi E_j,\phi E_i\bigr) = 0$$

$$\sum_{i,j=1}^{2n+1} \eta(E_j) g\bigl(\tilde{B}(E_i,\xi)\phi E_j,\phi E_i\bigr) = 0$$

$$\sum_{i,j=1}^{2n+1} \eta(E_i) g\bigl(\phi\tilde{B}(\xi,\phi E_j)\phi E_j,\phi E_i\bigr) = 0$$

$$\sum_{i,j=1}^{2n+1} \eta(E_j) g\bigl(\phi\tilde{B}(\phi E_j,\xi)\phi E_j,\phi E_i\bigr) = 0,$$

where we have used

$$\begin{aligned}
&\sum_{i,j=1}^{2n+1} g\bigl(R(\phi E_i,E_j)\phi E_j,E_i\bigr) = -\sum_{i,j=1}^{2n+1} g\bigl(\phi E_j, R(\phi E_i,E_j)E_i\bigr) \\
&= \sum_{i,j=1}^{2n+1} \Bigl(g\bigl(\phi E_j, R(E_j,E_i)\phi E_i\bigr) + g\bigl(\phi E_j, R(E_i,\phi E_i)E_j\bigr)\Bigr) \\
&= \sum_{i,j=1}^{2n+1} \Bigl(g\bigl(\phi E_j, R(E_j,E_i)\phi E_i\bigr) - g\bigl(\phi E_j, R(\phi E_i,E_j)E_i\bigr) \\
&\qquad - g\bigl(\phi E_j, R(E_j,E_i)\phi E_i\bigr)\Bigr) \\
&= \sum_{i,j=1}^{2n+1} \Bigl(g\bigl(\phi E_j, R(E_j,E_i)\phi E_i\bigr) + g\bigl(\phi E_j, R(E_j,\phi E_i)E_i\bigr) \\
&\qquad - g\bigl(\phi E_j, R(E_j,E_i)\phi E_i\bigr)\Bigr) = -s^{\#}
\end{aligned}$$

Therefore, using each of the above equations, (1.2) and $k = \dfrac{S+2n}{2n+2}$, we get

$$s^{\#} - S + 4n^2 = 2\,\mathrm{Tr}\,h^2.$$

From Lemma 3.1, we have our result.

By Proposition 3.1, we immediately get the following

Theorem 3.1. *Let M be a K-contact Riemannian manifold. Then M is a Sasakian manifold if and only if the E-tensor vanishes.*

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