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On characteristically simple conservative algebras

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Abstract. A classification of finite, characteristically simple, conservative algebras is given and several corollaries are derived. Among others it is shown that every nontrivial, at least three element, finite, conservative algebra with primitive automorphism group is functionally complete, and a nontrivial ρ-pattern algebra where ρ is a regular relation is functionally complete if and only if the intersection of the equivalence relations determining ρ is the equality relation.

1. Notions and notations

The notions collected below can be found in various texts and papers; we present them only for the convenience of the readers.

Let A be a nonempty set. The full symmetric group on A is denoted by S_A. If $n \geq 1$ then we put $\mathbf{n} = \{1, \ldots, n\}$, and we write S_n instead of $S_{\mathbf{n}}$. A permutation group $G \leq S_{\mathbf{n}}$ is *transitive* if for any $x, y \in \mathbf{n}$ there exists a $\pi \in G$ such that $x\pi = y$; G is *primitive* if $(\mathbf{n}; G)$ is a simple algebra and $|G| > 1$ (if $n = 2$). If G is a subgroup of S_A and H is a subgroup of S_m, $m \geq 1$, then $G \uparrow H$ denotes the wreath product of G and H, i.e., $G \uparrow H$ is the permutation group on A^m consisting of all permutations of the form

$$(\pi_1, \ldots, \pi_m, \tau) : A^m \to A^m, \quad (a_1, \ldots, a_m) \mapsto (a_{1\tau}\pi_1, \ldots, a_{m\tau}\pi_m),$$

where $\pi_1, \ldots, \pi_m \in G$ and $\tau \in H$.

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An operation f on a set A is *trivial* if it is a projection; f is a *conservative operation* if each nonempty subset of A is a subalgebra of $(A; f)$. A ternary operation f on A is a *majority operation* if for all $x, y \in A$ we have $f(x, x, y) = f(x, y, x) = f(y, x, x) = x$; f is a *Mal'cev operation* if $f(x, y, y) = f(y, y, x) = x$ for all $x, y \in A$. An n-ary operation t on A is said to be an i-th *semi-projection* ($n \geq 3$, $1 \leq i \leq n$) if for all $x_1, \ldots, x_n \in A$ we have $t(x_1, \ldots, x_n) = x_i$ whenever at least two elements among $x_1, \ldots, x_n$ are equal.

The clone of all term operations and the clone of all polynomial operations of an algebra $\mathbf{A}$ are denoted by $\operatorname{Clo}\mathbf{A}$ and $\operatorname{Pol}\mathbf{A}$, respectively. The algebra $\mathbf{A}$ is said to be *trivial* (*conservative*) if every fundamental operation of $\mathbf{A}$ is trivial (conservative).

Two algebras $\mathbf{A}$ and $\mathbf{B}$ are called *term equivalent* if they have a common base set and $\operatorname{Clo}\mathbf{A} = \operatorname{Clo}\mathbf{B}$; the algebras $\mathbf{A}$ and $\mathbf{B}$ are *equivalent* if $\mathbf{A}$ is isomorphic to an algebra term equivalent to $\mathbf{B}$. A finite algebra $\mathbf{A} = (A, F)$ is *functionally complete* if $\operatorname{Pol}\mathbf{A}$ is the set of all operations on A. The automorphism group of $\mathbf{A}$ is denoted by $\operatorname{Aut}\mathbf{A}$. The algebra $\mathbf{A}$ is *characteristically simple* if $(A; F \cup \operatorname{Aut}\mathbf{A})$ is simple.

By an *automorphism of a relation* ρ on A we mean a permutation π on A such that π and π^{-1} preserve ρ. The group of all automorphisms of ρ will be denoted by $\operatorname{Aut}\rho$.

If ρ is an h-ary relation on a set A, π is a mapping from A to a set B, and $m \geq 1$, then ρ^m, resp., ρ^π denote the h-ary relations on A^m, resp., on B defined as follows:

$$\rho^m = \{((x_{11}, \ldots, x_{1m}), \ldots, (x_{h1}, \ldots, x_{hm})) : \\ (x_{1i}, \ldots, x_{hi}) \in \rho,\ i = 1, \ldots, m\},$$

and

$$\rho^\pi = \{(x_1\pi, \ldots, x_h\pi) : (x_1, \ldots, x_h) \in \rho\}.$$

It is easy to check that $\operatorname{Aut}\rho^m = \operatorname{Aut}\rho \uparrow S_m$.

A binary reflexive and symmetric relation ρ on A is called *central* if $\rho \neq A^2$ and there exists a $c \in A$ such that $(a, b) \in \rho$ whenever $a = c$ or $b = c$. Then the set

$$C = \{c \in A : (c, x) \in \rho \text{ for all } x \in a\}$$

is called the *center* of ρ. Clearly $C \neq A$.

Let $k \geq 3$. A family $T = \{\Theta_1, \ldots, \Theta_m\}$ $(m \geq 1)$ of equivalence relations on A is called *k-regular* if each Θ_i $(1 \leq i \leq m)$ has exactly k blocks and $\Theta_T = \Theta_1 \cap \cdots \cap \Theta_m$ has exactly k^m blocks (i.e., the intersection $\bigcap_{i=1}^m B_i$ of arbitrary blocks B_i of Θ_i $(i = 1, \ldots, m)$ is nonempty). The relation determined by T is

$$\lambda_T = \{(a_1, \ldots, a_k) \in A^k : \text{for every } i\ (1 \leq i \leq m),\ a_1, \ldots, a_k \text{ are not pairwise incongruent modulo } \Theta_i\}.$$

Let $\mathbf{U} = (U; F)$ be a unary algebra and let $m, n \geq 1$. For given mappings $\sigma : \mathbf{m} \to \mathbf{m}$, $\mu : \mathbf{m} \to \mathbf{n}$ and unary term operations $g_1, \ldots, g_m$ of $\mathbf{U}$ let us define an n-ary operation $h_\mu^\sigma[g_1, \ldots, g_m]$ on U^m as follows: For $x_i = (x_i^1, \ldots, x_i^m) \in U^m$, $i = 1, \ldots, n$, set

$$h_\mu^\sigma[g_1, \ldots, g_m](x_1, \ldots, x_n) = (g_1(x_{1\mu}^{1\sigma}), \ldots, g_m(x_{m\mu}^{m\sigma})).$$

Now the *m'th matrix power* of $\mathbf{U}$, denoted by $\mathbf{U}^{[m]}$, is the algebra with universe U^m and with all the functions $h_\mu^\sigma[g_1, \ldots, g_m]$ described above as fundamental operations.

An algebra $\mathbf{A}$ is *semi-affine* with respect to an elementary Abelian p-group $\bar{\mathbf{A}}$ (p prime), if $\mathbf{A}$ and $\bar{\mathbf{A}}$ have a common base set A and the quaternary relation

$$\{(x, y, z, t) \in A^4 : x - y + z = t\}$$

is a compatible relation of $\mathbf{A}$; if, in addition, $x - y + z$ is a term operation of $\mathbf{A}$ then $\mathbf{A}$ is said to be *affine* with respect to $\bar{\mathbf{A}}$.

Consider an h-ary relation on a set A and let $n \geq 1$. We say that two n-tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in A^n$ are of the same ρ-pattern if for any $i_1, \ldots, i_n \in \mathbf{n}$, $(x_{i_1}, \ldots, x_{i_n}) \in \rho$ if and only if $(y_{i_1}, \ldots, y_{i_n}) \in \rho$. An n-ary operation f on A is called a *ρ-pattern operation* if for any $(x_1, \ldots, x_n) \in A^n$, $f(x_1, \ldots, x_n) = x_i$ for some i, where i depends on the ρ-pattern of $(x_1, \ldots, x_n)$ only. By a *ρ-pattern algebra* we mean an algebra on A whose fundamental operations are all ρ-pattern operations. If ρ is the equality relation then the ρ-pattern operations are the well-known pattern operations (see [2], [5]).

2. Results

Our aim is to give a classification of finite, characteristically simple, conservative algebras and to derive several corollaries. We need the following result from [8].

Theorem 2.1. *For a finite, surjective, characteristically simple algebra* $\mathbf{A}$ *one of the following conditions holds:*

(2.1.1) $\mathbf{A}$ *is isomorphic to* $\mathbf{B}^m$ $(m \geq 1)$ *where* $\mathbf{B}$ *is a functionally complete algebra and* $\operatorname{Aut}\mathbf{B}^m = \operatorname{Aut}\mathbf{B} \uparrow S_m$.

(2.1.2) $\mathbf{A}$ *is isomorphic to* $\mathbf{B}^m$ $(m \geq 1)$ *where* $\mathbf{B}$ *is a simple algebra that is affine with respect to an elementary Abelian* p*-group (*p *is prime).*

(2.1.3) $\mathbf{A}$ *is equivalent to* $\mathbf{U}^{[m]}$ $(m \geq 1)$ *for an at least two element unary algebra* $\mathbf{U} = (U; G)$ *where* $G \leq S_U$ *and either* $|G| = 1$ *or* G *is a primitive permutation group on* U *or* G *is a simple group acting regularly on* U.

(2.1.4) $\mathbf{A}$ *has a compatible binary central relation preserved by every automorphism.*

(2.1.5) $\mathbf{A}$ *is isomorphic to* $\mathbf{B}^m$ $(m \geq 1)$ *where* $\mathbf{B}$ *is a simple algebra that has a compatible bounded partial order* ρ *such that for every* $\pi \in \operatorname{Aut}\mathbf{B}$ *we have either* $\rho^\pi = \rho$ *or* $\rho^\pi = \rho^{-1}$. *Moreover,* $\operatorname{Aut}\mathbf{B}^m = \operatorname{Aut}\mathbf{B} \uparrow S_m$.

Using this theorem we can characterize the finite, conservative, characteristically simple algebras. First consider some lemmas:

Lemma 2.2. *Let* $\mathbf{A} = (A; F)$ *be an at least two element algebra and let* $k \geq 2$ *be a positive integer. If* $\mathbf{A}^k$ *is a conservative algebra then either* $\mathbf{A}$ *is trivial or* $k = 2$ *and* $\mathbf{A}$ *is term equivalent to the algebra* $(\{0,1\}; xy + xz + yz)$ *where* $(\{0,1\}; +, \cdot)$ *is the two element field.*

Proof. Let $\mathbf{A}$ be nontrivial algebra and suppose that $\mathbf{A}^k$ is conservative for some $k \geq 2$. First we establish some properties of the term operations of $\mathbf{A}$. Let $f \in \operatorname{Clo}\mathbf{A}$ be a nontrivial n-ary operation, $n \geq 1$.

Claim 1. *f cannot be binary.*

Indeed, if $n=2$ then since f is not a projection, there exist $a_i, b_i \in A$, $i=1,2$, such that $f(a_1,b_1) \neq a_1$, $f(a_2,b_2) \neq b_2$ and thus

$$\begin{aligned}&(f((a_1,a_2,\dots),(b_1,b_2,\dots))\\&\qquad\qquad = (f(a_1,b_1), f(a_2,b_2),\dots) \notin \{(a_1,a_2,\dots),(b_1,b_2,\dots)\}\end{aligned}$$

showing that f is not a conservative operation on A^k.

Claim 2. *f cannot be a semiprojection.*

Let f be a first semiprojection. Since f is not the first projection, $3 \leq n \leq |A|$ and for some pairwise different elements $a_1,\dots,a_n \in A$ we have $f(a_1,\dots,a_n) \neq a_1$. Without loss of generality we can suppose that $f(a_1,\dots,a_n) \neq a_2,\dots,a_{n-1}$. Then

$$\begin{aligned}&f((a_1,a_1,\dots),(a_2,a_1,\dots),(a_3,a_3,\dots),\dots,(a_n,a_n,\dots))\\&\quad = (f(a_1,a_2,a_3\dots,a_n), f(a_1,a_1,a_3,\dots,a_n)),\dots)\\&\quad = (f(a_1,a_2,a_3\dots,a_n),a_1,\dots)\end{aligned}$$

which shows that f is not a conservative operation on A^k.

Claim 3. *f cannot be a Mal'cev operation.*

Indeed, if f is a Mal'cev operation and $a,b \in A$, $a \neq b$, then

$$f((a,b,\dots),(b,b,\dots),(b,a,\dots)) = (f((a,b,b), f(b,b,a),\dots) = (a,a,\dots)$$

shows that f is not a conservative operation on A^k.

Claim 4. *If f is a majority operation then $k=2$ and $\mathbf{A}$ is term equivalent to the two element algebra $(\{0,1\}; xy+xz+yz)$ where $(\{0,1\};+,\cdot)$ is the two element field.*

Let f be a majority operation. If $k \geq 3$ and $a,b \in A$, $a \neq b$ then

$$\begin{aligned}&f((a,b,b,\dots),(b,a,b,\dots),(b,b,a,\dots))\\&\quad = (f((a,b,b), f(b,a,b),(b,b,a),\dots) = (b,b,b,\dots)\end{aligned}$$

shows that f is not a conservative operation on A^k. If $k=2$ and $|A| \geq 3$ then let $a,b,c \in A$ be pairwise distinct elements. Clearly, $f(a,b,c)$ is different from at least two of the elements a, b, c, say from a and b. Then

$$f((a,a),(b,a),(c,b)) = (f(a,b,c), f(a,a,b)) = (f(a,b,c),a)$$

shows that f is not a conservative operation on A^2. Hence $k=2$ and $|A|=2$. Finally, using Post's lattice [6], it is easy to check that $(\{0,1\}; xy+xz+yz)$ is the only two element conservative algebra having a majority term operation where $(\{0,1\};+,\cdot)$ is the two element field. This completes the proof of Claim 4.

Now we are in a position to complete the proof of the lemma. Since $\mathbf{A}$ is a nontrivial algebra therefore there is a nontrivial operation in Clo $\mathbf{A}$ which is either an idempotent binary operation or a majority operation or a Mal'cev operation or a semi-projection (see e.g. [4]). Taking into consideration Claims 1–4, from this our statement follows. $\square$

Lemma 2.3. *Up to equivalence* $(\{0,1\}; x+y+z)$ *is the only conservative affine algebra where* $(\{0,1\};+)$ *is the two element group.*

Proof. Let $\mathbf{A}=(A;F)$ be a conservative algebra that is affine with respect to an elementary Abelian p-group $(A;+)$ (p prime). If $|A|>2$ and $a,b\in A\setminus\{0\}$ are two distinct elements, then $a+b\neq a,b$ and $a+b=a-0+b$ belongs to the subalgebra generated by $\{a,b\}$ showing that $\mathbf{A}$ is not conservative. Hence $|A|=2$. Finally, taking into consideration Post's lattice [6] one can easily check that $(\{0,1\}; x+y+z)$ is the only conservative affine algebra on $\{0,1\}$. $\square$

Lemma 2.4. *If a matrix power* $\mathbf{U}^{[m]}$ *of an at least two element unary algebra* $\mathbf{U}$ *is conservative then* $\mathbf{U}$ *is trivial and* $m=1$.

Proof. It follows directly from the definition of matrix powers of unary algebras. $\square$

Theorem 2.5. *For a nontrivial, finite, conservative, characteristically simple algebra* $\mathbf{A}$ *one of the following conditions holds:*

(2.5.1) $\mathbf{A}$ *is functionally complete.*

(2.5.2) $\mathbf{A}$ *is equivalent to* $(\{0,1\}; x+y+z)$ *where* $(\{0,1\};+)$ *is the two element group.*

(2.5.3) $\mathbf{A}$ *is equivalent to* $(\{0,1\}; xy+xz+yz)^2$ *where* $(\{0,1\};+,\cdot)$ *is the two element field.*

(2.5.4) $\mathbf{A}$ *has a compatible binary central relation preserved by every automorphism.*

(2.5.5) $\mathbf{A}$ *is a simple algebra that has a compatible bounded partial order* ρ *such that for every* $\pi \in \operatorname{Aut}\mathbf{A}$ *we have either* $\rho^\pi = \rho$ *or* $\rho^\pi = \rho^{-1}$.

Proof. Let $\mathbf{A}$ be a nontrivial, finite, conservative, characteristically simple algebra and apply Theorem 2.1 for $\mathbf{A}$. According to Lemma 2.4, case (2.1.3) cannot occur, and (2.1.4) is the same as (2.5.4). In case (2.1.2), taking into consideration Lemma 2.3, we have (2.5.2). In case (2.1.1), by Lemma 2.2, we have that $m = 1$, i.e., (2.5.1) holds for $\mathbf{A}$. Finally in case (2.1.5), again by Lemma 2.2, we have (2.5.3) or (2.5.5). □

Corollary 2.6. *For a nontrivial, finite, conservative, characteristically simple algebra* $\mathbf{A}$ *with transitive automorphism group one of the following conditions holds:*

(2.6.1) $\mathbf{A}$ *is functionally complete.*

(2.6.2) $\mathbf{A}$ *is equivalent to* $(\{0,1\}; x+y+z)$ *where* $(\{0,1\};+)$ *is the two element group.*

(2.6.3) $\mathbf{A}$ *is equivalent to* $(\{0,1\}; xy+xz+yz)^k$ *where* $k \leq 2$ *and* $(\{0,1\};+,\cdot)$ *is the two element field.*

Proof. Let $\mathbf{A} = (A; F)$ be a nontrivial, finite, conservative, characteristically simple algebra with transitive automorphism group and apply Theorem 2.5 for $\mathbf{A}$. Cases (2.5.1) and (2.5.2) are the same as (2.6.1) and (2.6.2). Case (2.5.3) is the same as (2.6.3) with $k = 2$. It is known and easy to see that if a permutation preserves a central relation then it preserves the centre of the relation (see [7]). Therefore, by the transitivity of $\operatorname{Aut}\mathbf{A}$, the case (2.5.4) cannot occur.

Finally suppose that (2.5.5) holds for $\mathbf{A}$ with the bounded partial order ρ and let 0 and 1 be the least and greatest element of ρ, respectively. For any $\pi \in \operatorname{Aut}\mathbf{A}$ we have either $\rho^\pi = \rho$ or $\rho^\pi = \rho^{-1}$ implying that $\{0,1\}\pi = \{0,1\}$. Since $\operatorname{Aut}\mathbf{A}$ is transitive, it follows that $|A| = 2$. Using Post's lattice [6], it is easy to check that $(\{0,1\}; xy+xz+yz)$ is the only two element conservative algebra on $\{0,1\}$ with transitive automorphism group having the property (2.5.5), i.e., we have (2.6.3) with $k = 1$. □

The next corollary extends classical results on the functional completeness of finite discriminator algebras ([3], [12]), and more generally, those concerning pattern algebras ([2]).

Corollary 2.7. *Every at least three element nontrivial, finite, conservative algebra with primitive automorphism group is functionally complete.*

Proof. Since $\operatorname{Aut}(\{0,1\}; xy + xz + yz)^2 = S_2 \uparrow S_2$ is not primitive, our statement follows from Corollary 2.6. $\square$

The next corollaries supplement the results of [9], [10] and [11] where certain types of ρ-pattern algebras are investigated in the cases when ρ is a permutation, a binary central relation or a bounded partial order.

Corollary 2.8. *Let A be an at least three element finite set and let ρ be a relation on A such that $\operatorname{Aut}\rho$ is a primitive permutation group on A of composite order. Then for any $k \geq 1$, every nontrivial ρ^m-pattern algebra on A^m is functionally complete.*

Proof. Let ρ be a relation on an at least three element set A such that $\operatorname{Aut}\rho$ is a primitive permutation group on A of composite order, and consider a nontrivial ρ^m-pattern algebra $\mathbf{A}$ on A^m, $m \geq 1$. First observe that $\operatorname{Aut}\rho \uparrow S_m \subseteq \operatorname{Aut}\rho^m$. On the other hand, since $\operatorname{Aut}\rho$ is a primitive group of composite order, $\operatorname{Aut}\rho \uparrow S_m$ is primitive (see [1]). It is also easy to check that $\operatorname{Aut}\rho^m \subseteq \operatorname{Aut}\mathbf{A}$, and therefore $\operatorname{Aut}\mathbf{A}$ is primitive. Finally apply Corollary 2.8. $\square$

Corollary 2.9. *Let A be an at least three element finite set and let*

$$\iota_h = \{(x_1, \dots, x_h) \in A : |\{x_1, \dots, x_h\}| \leq h - 1\}, \quad 3 \leq h \leq |A|.$$

Then every nontrivial ι_h^m-pattern algebra on A^m is functionally complete, $m \geq 1$.

Proof. Since $\operatorname{Aut}\iota_h = S_A$, our statement is a special case of Corollary 2.8. $\square$

Corollary 2.10. *Let λ_T be k-regular relation on an at least three element finite set A determined by a k-regular family T of equivalence relations. Then a nontrivial λ_T-pattern algebra is functionally complete if and only if $\bigcap T$ is the equality relation.*

Proof. Let λ_T be k-regular relation on an at least three element finite set A determined by a k regular family $T = \{\Theta_1, \dots \Theta_m\}$ of equivalence relations and consider a nontrivial λ_T-pattern algebra $\mathbf{A}$. It is easy to check that $\bigcap T$ is a congruence relation of $\mathbf{A}$. Therefore if $\mathbf{A}$ is functionally complete then $\bigcap T$ is the equality relation.

Now suppose that $\bigcap T$ is the equality relation. Then there is a bijection $\pi : A \to \mathbf{k}^m$ such that $\rho^\pi = \iota_k^m$ (see [7]). Therefore $\mathbf{A}$ is equivalent to an ι_k^m-pattern algebra $\mathbf{B}$ on $\mathbf{k}^m$. By Corollary 2.9, $\mathbf{B}$ is functionally complete implying that $\mathbf{A}$ is also functionally complete. □

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