Publ. Math. Debrecen
57 / 3-4 (2000), 435–444

$(J^2 = \pm 1)$-metric manifolds

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Abstract. Manifolds with a $(1,1)$ tensor field J such that $J^2 = \pm 1$, and with an adapted semi-Riemannian metric g verifying $g(JX, JY) = \pm g(X, Y)$ have been studied in the last years, developing four independent geometries. This paper is of survey character, showing that these geometries are strongly related among them and with the theory of 3-webs.

Introduction

Manifolds (M, J, g) where J is a $(1,1)$ tensor field such that $J^2 = \pm 1$ and g is a (pseudo)-Riemannian metric such that $g(JX, JY) = \pm g(X, Y)$ for all vector fields X, Y tangent to M, have been studied in the last decades, developing four different geometries:

(1) If $J^2 = 1$ and $g(JX, JY) = g(X, Y)$, then (M, J, g) is an *(indefinite) Riemannian almost product manifold.*

(2) If $J^2 = 1$ and $g(JX, JY) = -g(X, Y)$, then (M, J, g) is an *almost para-Hermitian manifold.*

(3) If $J^2 = -1$ and $g(JX, JY) = g(X, Y)$, then (M, J, g) is an *(indefinite) almost Hermitian manifold.*

(4) If $J^2 = -1$ and $g(JX, JY) = -g(X, Y)$, then (M, J, g) is a *Norden manifold.*

In the cases (2), (3) and (4) M is an even-dimensional orientable manifold, because: in cases (2) and (4) the metric g has signature (n, n); in

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Mathematics Subject Classification: 53C15.
Key words and phrases: Riemannian almost product, para-Hermitian, Hermitian, Norden, biparacomplex, 3-web.

cases (2) and (3), the *fundamental form* Ω given by $\Omega(X,Y) = g(JX,Y)$ is a nondegenerate 2-form, and then (M,Ω) is an almost symplectic manifold; in cases (3) and (4) (M,J) is an almost complex manifold. Then, one can expect that there exist some relations among manifolds of types (2), (3) and (4). (Obviously, (indefinite) Riemannian almost product manifolds may be odd-dimensional and non-orientable.)

As we have said, these four different geometries have independently grown. For example, there exist classifications of all of them: see [11] (resp. [9], [10], [8]) for the manifolds (1) (resp. (2), (3) and (4)) in the above list.

The main aim of the present paper is to show that there exist some relations among different ($J^2 = \pm 1$)-metric structures and with the geometry of 3-webs. The organization of the paper is as follows:

In Section 1, we show the main definitions and results about biparacomplex structures, which will be used later. A manifold endowed with an integrable biparacomplex structure is a manifold endowed with a 3-web.

In Section 2, we obtain almost para-Hermitian and biparacomplex structures on manifolds endowed with other geometries: Norden surfaces (Proposition 2.1 and Corollary 2.2) and almost complex manifolds endowed with a purely real distribution (Remark 2.3). As one can see, an almost complex structure is not enough to obtain an almost para-complex one.

In Section 3, we consider an almost para-Hermitian manifold. We prove that it does always admit: an almost Hermitian structure (Proposition 3.1), and a biparacomplex structure (Remark 3.4). We also study in Theorem 3.3 the case when the four geometries of the beginning of the paper appear: when M is a biparacomplex manifold with respect to both its almost para-Hermitian structures and the induced almost Hermitian one. Moreover, para-Kähler manifolds are symplectic, and then compact para-Kähler manifolds have some of the topological obstructions of compact Kähler ones: those which are given by the compact symplectic structure (see Proposition 3.5).

In Section 4 we study the geometry of the tangent bundle of a Riemannian manifold in the light of the results obtained in the above sections.

1. Biparacomplex structures and 3-webs

Cruceanu introduced in [4] the notion of *biparacomplex manifold* in the following way: A *biparacomplex structure* on a manifold M is given by two anticommutative almost product structures F and P, i.e., two tensor fields F and P of type $(1,1)$ verifying $F^2 = P^2 = 1$, $F \circ P + P \circ F = 0$. Then, there are four equidimensional and supplementary distributions, defined by the eigenspaces associated with $+1$ and -1 of the automorphisms F and P (namely F^+, F^-, P^+, P^-). In particular, M has even dimension, F and P are almost paracomplex structures (because $\dim F^+ = \dim F^-$, $\dim P^+ = \dim P^-$) and F (resp. P) is an isomorphism between P^+ and P^- (resp. between F^+ and F^-).

Another equivalent introduction of this structure is also obtained in [4]; one can study manifolds endowed with three equidimensional supplementary distributions: for all $x \in M$, the tangent space of M at x is decomposed as $T_x M = V_1(x) \oplus V_2(x) = V_1(x) \oplus V_3(x) = V_2(x) \oplus V_3(x)$, V_1, V_2, V_3 being the distributions. If F is the almost product structure given by $F^+ = V_1, F^- = V_2$ and P is the almost product structure given by $P^+ = V_3, P^- = F(V_3) = V_4$, one easily can check that (M, F, P) is a biparacomplex manifold.

Now we can state

Proposition 1.1 [4]. *A manifold M is endowed with three supplementary distributions iff there exist two anticommuting almost product structures on the manifold.*

One can easily prove that the following conditions are equivalent: (1) the Nijenhuis tensor fields of F and P vanish; (2) the distributions V_i are involutive for all $i \in \{1,2,3\}$. In this case, M is said to be endowed with a *3-web*.

Moreover, if (M, F, P) is a biparacomplex manifold, then one can consider $J = P \circ F$, which is an *almost complex structure* on M. So, a biparacomplex manifold is an even-dimensional orientable manifold which has two almost product structures and one almost complex one.

One also can define an *almost tangent* structure K given by: $K(X) = P(X)$ if $X \in F^+$, and $K(X) = 0$ if $X \in F^-$.

Remark 1.2. Let (M, F, P) be a biparacomplex manifold, then one can consider $F_2 : F^+ \oplus F^- \to F^-$ the projection over F^-. The couple (F_2, P) of polynomial structures verifies: $F_2^2 = F_2$; $P^2 = 1$; $P \circ F_2 = (1 - F_2) \circ P$ and it is such that $F^+ = \ker F_2$; $F^- = \ker(1 - F_2)$; $P^+ = \ker(P - 1)$. In [14], a couple (P, B) of tensor fields of type $(1,1)$ satisfying the above conditions is called a (P,B)-*structure* on M associated with the three distributions F^+, F^- and P^+. It is easily seen that a (P,B)-structure on M defines three supplementary distributions.

Now we will study metrics adapted to a biparacomplex structure.

Definition 1.3 (see [13]). Let (M, F, P) be a biparacomplex manifold and let g be a pseudo-Riemannian metric on M. Then (M, F, P, g) is said to be an $(\varepsilon_1, \varepsilon_2)$ *pseudo-Riemannian biparacomplex manifold*, where $\varepsilon_1, \varepsilon_2 \in \{+, -\}$ according to the following relations:

$$g(FX, FY) = \varepsilon_1 g(X, Y); \qquad g(PX, PY) = \varepsilon_2 g(X, Y).$$

Example. In [2], Blažić introduces a quotient manifold $P_n(B) = S^{4n+3}_{2n+1}/S^3_1$, where S^i_j denotes the sphere of dimension i and pseudo-Riemannian metric of index j, and calls $P_n(B)$ the *paraquaternionic projective space* (in a sense different from [6]). Then he proves that $P_n(B)$ admits three tensor fields E, F, I of type $(1,1)$ verifying $E^2 = F^2 = -I^2 = 1$; $I \circ E = -E \circ I = F$ and $g(EX, EY) = g(FX, FY) = -g(X, Y)$ for all vectors fields X, Y on $P_n(B)$, where g is a pseudo-Riemannian metric of signature (n, n) on $P_n(B)$. The author also proves that the Levi–Civita connection ∇ of g parallelizes E and F. It is easy seen that (E, F) is a biparacomplex structure on $P_n(B)$.

A $(-, -)$ pseudo-Riemannian manifold is called a *paraquaternionic Hermitian manifold* in [2].

2. Neutral metrics

Let M be a manifold endowed with a *neutral metric*, i.e., with a pseudo-Riemannian metric g of signature (n, n). Then one can easily prove the following

Proposition 2.1. *If (M, g) is a neutral metric with* $\dim M = 2$, *then M is an almost para-Hermitian manifold.*

Proof. One can consider the maximal isotropic distributions V_1 and V_2, which verify the following properties for all $x \in M$:

(1) $T_x M = (V_1)_x \oplus (V_2)_x$

(2) $\dim(V_1)_x = \dim(V_2)_x = 1$

(3) $g\,|_{(V_1)_x} = g\,|_{(V_2)_x} = 0$.

Then one can define an almost product structure F such that $(V_1)_x$ (resp. $(V_2)_x$) is the eigenspace associated with $+1$ (resp. -1), for each $x \in M$. Then, by a straightforward computation, one concludes that (M, F, g) is an almost para-Hermitian manifold. □

The above proposition cannot be generalized to dim : $M > 2$ because one cannot choose two isotropic maximal distributions in a canonical way.

Corollary 2.2. *Every Norden surface admits an almost para-Hermitian structure and a $(-,+)$ pseudo-Riemannian biparacomplex structure.*

Proof. Taking into account the above proposition, if (M, J, g) is a Norden surface, then M admits a para-Hermitian structure (F, g). One easily checks that both distributions V_1 and $J(V_1)$ satisfy conditions (1), (2) and (3) in the proof of Proposition 2.1. Then $V_2 = J(V_1)$, and consequently J is an isomorphism between both distributions which are defined by the eigenspaces of F. Now it is an easy exercise to show that $J \circ F + F \circ J = 0$, and then $(M, F, P = J \circ F)$ is a biparacomplex manifold. Finally, taking vector fields X and Y on M, one has $g(FX, FY) = -g(X, Y)$ by the above proposition, and $g(PX, PY) = g(JFX, JFY) = -g(FX, FY) = g(X, Y)$, thus proving that (M, F, P, g) is a $(-,+)$ pseudo-Riemannian biparacomplex manifold. □

Remark 2.3. One can obtain a para-Hermitian structure on an almost complex $2n$-dimensional manifold (M, J) if one has a *purely real* n-dimensional distribution (i.e. a distribution D such that $J(D) \cap D = \{0\}$). Moreover, in this case M admits a biparacomplex structure, because one can follow the proof of the above corollary (observe that we have obtained in the corollary a purely real distribution by means of the neutral metric). This result has been obtained by Bejan in [1, p. 18].

3. Almost para-Hermitian manifolds

Let (M, F, g) be an almost para-Hermitian manifold. Then the metric g has signature (n, n) and the dimension of the manifold M is even. One can define an almost symplectic structure Ω given by $\Omega(X, Y) = g(FX, Y)$ for all vector fields X, Y on M, and then M is orientable. Moreover, naming the distributions generated by the eigenspaces of F as F^+ and F^-, one has that $TM = F^+ \oplus F^-$, and then (M, Ω) is an *almost bilagrangian manifold*, in the sense that it is an almost symplectic manifold with two transverse Lagrangian distributions.

The authors do not know whether more considerations about other structures related to this one of an almost para-Hermitian manifold have been published. Now we will present our results. First of all, we will show that every almost para-Hermitian manifold admits an almost Hermitian structure:

Proposition 3.1. *Let (M, F, g) be an almost para-Hermitian manifold with fundamental form Ω. Then M admits an almost Hermitian structure (J, G), where G is a Riemannian metric, such that its fundamental form (or Kähler form) coincides with Ω.*

Proof. As is well known (see, e.g., [12, Theorem 8.13], if a manifold M admits a symplectic 2-form Ω then it admits an almost Hermitian structure (J, G) such that $\Omega(X, Y) = G(JX, Y)$. The proof finishes when one takes Ω as the fundamental form of (M, F, g). □

Remark 3.2. An *almost para-Kähler manifold* is an almost para-Hermitian manifold (M, F, g) whose fundamental 2-form Ω is closed. Then (M, J, G) is an almost Hermitian manifold, J being the almost complex structure induced by F, because Ω is the fundamental form of both structures J and F.

Following the notation in the above proposition one has an almost para-Hermitian manifold (M, F, g) and its associated almost Hermitian structure (M, J, G). Now one can look for a biparacomplex structure on M and one can ask whether (M, J, g) (resp. (M, F, G)) is a Norden (resp. Riemannian almost product) manifold. The three problems are related:

Theorem 3.3. *Let (M, F, g) be an almost para-Hermitian manifold and let (M, J, G) be its associated almost Hermitian structure. The following conditions are equivalent:*

(1) *$(M, F, P = J \circ F)$ is a biparacomplex manifold;*

(2) *(M, J, g) is a Norden manifold;*

(3) *(M, F, G) is a Riemannian almost product manifold.*

Moreover, in this case (M, F, P, g) is a $(-, +)$ pseudo-Riemannian biparacomplex manifold and (M, F, P, G) is a $(+, +)$ Riemannian biparacomplex manifold.

Proof. One easily can check that condition (1) is equivalent to $F \circ J + J \circ F = 0$, and G being Riemannian this is equivalent to $G((F \circ J + J \circ F)X, Y) = 0$, for all vector fields X, Y on M. Let Ω be the fundamental form of both F and J. Then one has:

$$G((J \circ F)X, Y) = \Omega(FX, Y) = g(F^2X, Y) = g(X, Y).$$

Then, condition (1) is equivalent to $G((F \circ J)X, Y) = -g(X, Y)$. Now we can prove the result:

(1)⇔(2): $G((F \circ J)X, Y) = G(JFJX, JY) = \Omega(FJX, JY) = g(F^2JX, JY) = g(JX, JY)$, thus proving the result.

(3)⇒(1): $G((F \circ J)X, Y) = G(F^2JX, FY) = G(JX, FY) = \Omega(X, FY) = g(FX, FY) = -g(X, Y)$.

(1)⇒(3): $G(FX, FY) = G(JFX, JFY) = G(FJX, FJY) = -g(X, FJY) = g(FX, JY) = \Omega(X, JY) = G(JX, JY) = G(X, Y)$.

The last part of the proof is trivial, because P is an isometry for both metrics g and G:

$$g(PX, PY) = g(JFX, JFY) = -g(FX, FY) = g(X, Y)$$

$$G(PX, PY) = G(JFX, JFY) = G(FX, FY) = G(X, Y)$$

thus finishing the proof. □

Remark 3.4. One always can obtain a biparacomplex structure on an almost para-Hermitian $2n$-dimensional manifold (M, F, g): taking into account Proposition 3.1, one has an almost Hermitian structure (J, G) on M and a purely real n-dimensional distribution given by F^+ (which is purely real because F^+ and JF^+ are G-orthogonal). Then, by Remark 2.3 the result follows. Observe that if $JF^+ = F^-$ then the biparacomplex structure coincides with that given in the above Theorem 3.3.

A *para-Kähler manifold* is an almost para-Hermitian manifold (M, F, g) such that $\nabla F = 0$, where ∇ is the Levi–Civita connection of g. As is well known, this condition is equivalent to the following: the 2-form Ω is closed and the Nijenhuis tensor of F vanishes, i.e., $d\Omega = 0$ and $N_F = 0$. Thus, a para-Kähler manifold is symplectic and one obtains

Proposition 3.5. *If (M, F, g) is a compact para-Kähler manifold of dimension $2n$, then all the even-dimensional Betti numbers of M are nonzero, all the odd-dimensional Betti numbers of M are even, and $b_r(M) \leq b_{r+2}(M)$, $0 \leq r < n$, with the corresponding inequalities for $r \geq n$.*

Proof. This is a direct consequence of the similar result for compact symplectic manifolds (see, e.g., [12, Corollary 8.41]). □

4. The tangent bundle of a Riemannian manifold

Let M be an n-dimensional manifold endowed with a metric g and let ∇ be the Levi–Civita connection of g. In [5] the following almost complex structure on TM is introduced: $JX^V = -X^H$, $JX^H = X^V$, where X is a vector field on M and V (resp. H) denotes the vertical (resp. the horizontal) lift to the tangent bundle (see [15]). One can also introduce the almost para-complex structure on TM defined by $FX^V = -X^V$, $FX^H = X^H$. (The opposite of this structure has been introduced in [3].) Let g^D (resp. g^H) be the diagonal (resp. horizontal) lift of g and ∇^H the horizontal lift of ∇. Then we have

Proposition 4.1 ([15]). *(TM, J, g^D) is an almost Kähler manifold and $\nabla^H g^D = 0$.*

One can also prove that (TM, F, g^H) is an almost para-Hermitian manifold and $\nabla^H g^H = 0$.

Let Ω be the almost symplectic structure of (TM, F, g^H) and $\bar{\Omega}$ the fundamental form of (TM, J, g^D). One can obtain:

$$\Omega(X^V,Y^V)=0=\Omega(X^H,Y^H), \quad \Omega(X^V,Y^H)=-(g(X,Y))^V=-\Omega(X^H,Y^V),$$

$$\bar{\Omega}(X^V,Y^V)=0=\bar{\Omega}(X^H,Y^H), \quad \bar{\Omega}(X^V,Y^H)=-(g(X,Y))^V=-\bar{\Omega}(X^H,Y^V).$$

Then, the almost Hermitian structure (TM, J, g^D) is associated with the almost para-Hermitian structure (TM, F, g^H) in the sense of Proposition 3.1.

Let P be the almost para-complex structure on TM given by $P = J \circ F$. One can check that (TM, F, P) is a biparacomplex manifold and then, by Theorem 3.3, (TM, J, g^H) is a Norden manifold and (TM, F, g^D) is a Riemannian almost product manifold. The connection ∇^H is *well-adapted* to the geometry of TM, because one has

Theorem 4.2 ([7, Theorem 1]). *∇^H is the unique connection on TM such that*

$$\nabla^H J = 0, \quad \nabla^H F = 0, \quad \nabla^H g^D = 0, \quad \nabla^H g^H = 0.$$

It is easily seen that $\nabla^H P = 0$.

Taking into account that ∇ is symmetric, one has

$$\mathrm{Tor}_{\nabla^H}(X^V, Y^H) = (\mathrm{Tor}_\nabla(X,Y))^V = 0,$$

for any every vector fields X, Y on M, where Tor_{∇^H} is the torsion tensor field of ∇^H, thus proving that ∇^H is the canonical connection of the biparacomplex structure (F,P). This connection is also called the *Chern connection* of the 3-web defined by the distributions (F^+, F^-, P^+).

If ∇ is locally flat, i.e., $\mathrm{Tor}_\nabla = 0$ and $R_\nabla = 0$, where R_∇ denotes the curvature tensor of ∇, then ∇^H is also locally flat, taking into account the following

Proposition 4.3 ([15, Proposition 7.3, 7.4]). *Let ∇ be a symmetric connection. Then the connection ∇^H is symmetric if and only if $R_\nabla = 0$. In this case, one also has $R_{\nabla^H} = 0$.*

Let us assume that (M,g) is locally flat, and that ∇ is the Levi–Civita connection of g. Then, ∇^H is the Levi–Civita connection of g^D and g^H. Taking into account that ∇^H is symmetric and that $\nabla^H J = \nabla^H F = \nabla^H P = 0$, we can conclude that $N_J = N_F = N_P = 0$, where N_K denotes the Nijenhuis tensor field of the $(1,1)$-tensor field K.

We see that (TM, J, g^D) is a Kähler manifold and (TM, F, g^H) a para-Kähler manifold. Moreover, (TM, F, P, g^H) is a $(-,+)$ Kähler biparacomplex manifold and (TM, F, P, g^D) a $(+,+)$ Kähler biparacomplex manifold.

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(Received April 2, 1999; revised September 22, 1999)