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The homogeneous lift to the tangent bundle of a Finsler metric

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Abstract. Notice that the Sasaki–Matsumoto lift $\overset{0}{\mathbb{G}}$ from (1.4) of a Finsler metric tensor g is not homogeneous on the fibers of the tangent bundle. We correct this inconvenient by introducing a new kind of lift $\mathbb{G}$ of g, given by (2.1), which is 0-homogeneous. Some properties of the Riemannian space $(\widetilde{TM}, \mathbb{G})$ are studied. The almost complex structure $\mathbb{F}$, from (3.1) is introduced. It has the property of homogeneity and $(\mathbb{G}, \mathbb{F})$ is an almost Hermitian structure. We prove that in fact $(\mathbb{G}, \mathbb{F})$ is a conformal almost Kählerian structure. It depends only on the fundamental function of the Finsler space considered.

Introduction

The Sasaki–Matsumoto lift $\overset{0}{\mathbb{G}}$ [6], [11] to the manifold $\widetilde{TM} = TM \setminus \{O\}$ of a Finsler metric tensor g is extremely important in the study of the geometry of a Finsler space $F^n = (M, F(x,y))$. $\overset{0}{\mathbb{G}}$ determines a Riemannian structure on $\widetilde{TM}$, which depends only on the fundamental function F. It is not difficult to see that $\overset{0}{\mathbb{G}}$ does not have a Finslerian meaning. More precisely, $\overset{0}{\mathbb{G}}$ is not homogeneous with respect to the vertical variables y^i. Consequently, we cannot study global properties – as the Gauss–Bonnet Theorem – for the Finsler space F^n by means of this lift [4], [5]. Also, since the two terms of the metric $\overset{0}{\mathbb{G}}$ do not have the same physical dimensions, it does not satisfy the principles of the Post-Newtonian Calculus and so it is not convenient for a gauge theory.

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In the present paper, using the same ideas as in the Riemannian case [10] we define a new lift $\mathbb{G}$, (2.1), to $\widetilde{TM}$, which depends only on the fundamental function F of the Finsler space F^n and which is 0-homogeneous on the fibers of the tangent bundle TM.

Some geometrical properties of the space $(\widetilde{TM}, \mathbb{G})$ are studied by means of the Cartan nonlinear connection of the space F^n: the canonical metrical N-connection, the Levi–Civita connection of $\mathbb{G}$ etc.

We introduce the natural almost complex structure $\mathbb{F}$ by the formulae (3.1). It has the property of homogeneity and depends only on F. The main result is as follows: The space $(\widetilde{TM}, \mathbb{G}, \mathbb{F})$ is almost Hermitian and its associated almost symplectic structure θ is such that $d\theta = 0$ (modulo θ). We prove that this space is in fact conformal almost Kählerian. It represents the geometrical model of the Finsler space F^n with respect to the homogeneous lift $\mathbb{G}$.

1. Preliminaries

Let $F^n = (M, F)$ be a Finsler space, M being a real n-dimensional differentiable manifold and $F : TM \to \mathbb{R}$ its fundamental function. F is of C^∞-class on $\widetilde{TM} = TM \setminus \{O\}$ and continuous on the null section of the natural projection $\pi : TM \to M$. The fundamental tensor field of F^n is

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}, \quad \forall (x, y) \in \widetilde{TM}. \tag{1.1}$$

The regular Lagrangian $F^2(x, y)$ determines the canonical spray $S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}$ with the coefficients $G^i = \frac{1}{2}\gamma^i_{jk}(x, y)y^j y^k$, where $\gamma^i_{jk}(x, y)$ are the Christoffel symbols of the metric tensor $g_{ij}(x, y)$. The Cartan nonlinear connection N of the space F^n has the coefficients $N^i_j = \frac{\partial G^i}{\partial y^j}$.

N determines a distribution on $\widetilde{TM}$, which is supplementary to the vertical distribution V. We have the following direct sum of linear spaces:

$$T_u(\widetilde{TM}) = N_u \oplus V_u, \quad \forall u \in \widetilde{TM}. \tag{1.2}$$

An adapted basis to N_u and V_u is given by $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, $(i = 1, \dots, n)$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i(x, y)\frac{\partial}{\partial y^j}. \tag{1.3}$$

The dual basis is $(dx^i, \delta y^i)$ with

$$\delta y^i = dy^i + N^i_j(x,y)dx^j. \tag{1.3'}$$

M. Matsumoto [6] extended to Finsler spaces F^n the notion of Sasaki lift, considering the tensor field

$$\overset{0}{\mathbb{G}}(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j, \quad \forall (x,y) \in \widetilde{TM}. \tag{1.4}$$

It easily follows that $\overset{0}{\mathbb{G}}$ is a Riemannian metric globally defined on $\widetilde{TM}$ and depending only on the fundamental function F of the Finsler space F^n.

Let us consider the homothety $h_t : (x,y) \to (x,ty)$, $t \in \mathbb{R}^*$ on the fibers of the tangent bundle TM. Then $\overset{0}{\mathbb{G}}$ is transformed as follows:

$$\overset{0}{\mathbb{G}} \circ h_t(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + t^2 g_{ij}(x,y)\delta y^i \otimes \delta y^j, \quad \forall t \in \mathbb{R}^*.$$

We see that the Sasaki–Matsumoto lift $\overset{0}{\mathbb{G}}$ is not homogeneous on the fibers of the tangent bundle TM.

Next we consider the $\mathcal{F}(\widetilde{TM})$-linear mapping $\mathbb{F} : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, defined by

$$\overset{0}{\mathbb{F}}\left(\frac{\delta}{\delta x^i}\right) = -\frac{\partial}{\partial y^i}, \quad \overset{0}{\mathbb{F}}\left(\frac{\partial}{\partial y^i}\right) = \frac{\delta}{\delta x^i}, \qquad (i = 1, \dots, n). \tag{1.5}$$

As $\overset{0}{\mathbb{F}}$ maps the 1-homogeneous vector field $\frac{\delta}{\delta x^i}$ onto 0-homogeneous vector fields $\frac{\partial}{\partial y^i}$, $(i = 1, \dots, n)$, it does not preserve the property of homogeneity of the vector fields on $\widetilde{TM}$.

It is known that $\overset{0}{\mathbb{F}}$ is an almost complex structure on $\widetilde{TM}$ depending only on the fundamental function F which becomes a complex structure on $\widetilde{TM}$ if and only if the horizontal distribution N is integrable.

Now let us consider the Cartan–Poincaré forms

$$\overset{\circ}{\omega} = \frac{1}{2}\frac{\partial F^2}{\partial y^i}dx^i, \quad \overset{\circ}{\theta} = g_{ij}(x,y)\delta y^i \wedge dx^j. \tag{1.6}$$

Evidently, $\overset{\circ}{\omega}$ and $\overset{\circ}{\theta}$ are globally defined on $\widetilde{TM}$ and $\overset{\circ}{\theta}$ is an almost symplectic structure on $\widetilde{TM}$.

As is known, between $\overset{\circ}{\omega}$ and $\overset{\circ}{\theta}$ there is the relation

$$d\overset{\circ}{\omega} = \overset{\circ}{\theta}, \tag{1.7}$$

d being the exterior differential operator.

It follows that $\overset{\circ}{\theta}$ is a closed 2-form. In other words, $\overset{\circ}{\theta}$ is a symplectic structure. Remarking that the pair $(\overset{0}{\mathbb{G}}, \overset{0}{\mathbb{F}})$ is an almost Hermitian structure having $\overset{\circ}{\theta}$ as its associated symplectic structure, we recall the known result that $H^{2n} = (\widetilde{TM}, \overset{0}{\mathbb{G}}, \overset{0}{\mathbb{F}})$ is an almost Kählerian space.

In the terminology of the book [7], H^{2n} is the almost Kählerian model on $\widetilde{TM}$ of the Finsler space F^n considered. This is important for the geometry of the Finsler space F^n.

2. The homogeneous lift to $\widetilde{TM}$ of a Finsler metric

We define a new lift $\mathbb{G}$ on $\widetilde{TM}$ of the fundamental tensor field g_{ij} of a Finsler space $F^n = (M, F)$ which satisfies the following conditions:

1° $\mathbb{G}$ is 0-homogeneous with respect to y^i;

2° It depends only on the fundamental function F;

3° In the mechanical interpretation the terms of $\mathbb{G}$ have the same physical dimensions.

Definition 2.1. By the homogeneous lift to $\widetilde{TM}$ of the fundamental tensor field g_{ij} of a Finsler space F^n we mean the following tensor field on $\widetilde{TM}$:

$$\begin{gathered}\mathbb{G}(x,y) = g_{ij}(x,y)dx^i \otimes dx^j + \frac{a^2}{\|y\|^2} g_{ij}(x,y)\delta y^i \otimes \delta y^j, \\ \forall (x,y) \in \widetilde{TM},\end{gathered} \tag{2.1}$$

where $a > 0$ is a constant and

$$\|y\|^2 = g_{ij}(x,y)y^i y^j = F^2(x,y). \tag{2.2}$$

The constant a is required by the applications, in order that the physical dimension of the terms of G be the same.

We get without difficulty the

Theorem 2.1. *The pair $(\widetilde{TM}, \mathbb{G})$ is a Riemannian space. $\mathbb{G}$ is 0-homogeneous on the fibers of TM and it depends only on the fundamental function $F(x,y)$ of the Finsler space F^n.*

We consider $\mathbb{G}$ as a (h,v)-metric, that is,

$$\mathbb{G} = \mathbb{G}^H + \mathbb{G}^V,\ \mathbb{G}^H = g_{ij}(x,y)dx^i \otimes dx^j,\ G^V = h_{ij}(x,y)\delta y^i \otimes \delta y^j \tag{2.3}$$

$$h_{ij} = \frac{a^2}{\|y\|^2} g_{ij}(x,y). \tag{2.4}$$

Consequently, we can apply the theory of the (h,v)-Riemannian metric on $\widetilde{TM}$ investigated by R. Miron and M. Anastasiei in the books [7] and [8].

The equation $F(x_0,y) = a$ determines the so called indicatrix of the Finsler space F^n in the point $x_0 \in M$, [6].

Therefore, we have the

Proposition 2.1. *The homogeneous lift $\mathbb{G}$ of the metric tensor $g_{ij}(x,y)$ coincides with the Sasaki–Matsumoto lift of $g_{ij}(x,y)$ on the indicatrix $F(x_0,y) = a$ for every point $x_0 \in M$.*

A linear connection D on $\widetilde{TM}$ is called a metrical N-connection with respect to $\mathbb{G}$, if $D\mathbb{G} = 0$ and D preserves by parallelism the horizontal distribution N.

As we know [7], [8], there exist the metrical N-connections on $\widetilde{TM}$. We represent a linear connection D on $\widetilde{TM}$ in the adapted basis in the following form:

$$\begin{aligned} D_{\frac{\delta}{\delta x^k}} \frac{\delta}{\delta x^j} &= \overset{H}{L^i_{jk}} \frac{\delta}{\delta x^i} + \widetilde{L^i_{jk}} \frac{\partial}{\partial y^i}, \quad D_{\frac{\delta}{\delta x^k}} \frac{\partial}{\partial y^j} = \widetilde{\widetilde{L^i_{jk}}} \frac{\delta}{\delta x^i} + \overset{V}{L^i_{jk}} \frac{\partial}{\partial y^i}, \\ D_{\frac{\partial}{\partial y^k}} \frac{\delta}{\delta x^j} &= \overset{H}{C^i_{jk}} \frac{\delta}{\delta x^i} + \widetilde{C^i_{jk}} \frac{\partial}{\partial y^i}, \quad D_{\frac{\partial}{\partial y^k}} \frac{\partial}{\partial y^j} = \widetilde{\widetilde{C^i_{jk}}} \frac{\delta}{\delta x^i} + \overset{V}{C^i_{jk}} \frac{\partial}{\partial y^i}. \end{aligned} \tag{2.5}$$

where $(\overset{H}{L^i_{jk}}, \widetilde{L^i_{jk}}, \widetilde{\widetilde{L^i_{jk}}}, \overset{V}{L^i_{jk}}, \overset{H}{C^i_{jk}}, \widetilde{C^i_{jk}}, \widetilde{\widetilde{C^i_{jk}}}, \overset{V}{C^i_{jk}})$ are the coefficients of D.

By a direct calculation we obtain

Theorem 2.2. *There exist the metrical N-connections D on $\widetilde{TM}$ with respect to $\mathbb{G}$, which depend only on the fundamental function $F(x,y)$ of the Finsler space F^n. One of them has the following coefficients:*

$$
(2.6) \qquad \begin{cases}
\widetilde{L^i_{jk}} = \widetilde{\widetilde{L^i_{jk}}} = \widetilde{C^i_{jk}} = \widetilde{\widetilde{C^i_{jk}}} = 0 \\
\overset{H}{L}{}^i_{jk} = \dfrac{1}{2} g^{is} \left(\dfrac{\delta g_{sk}}{\delta x^j} + \dfrac{\delta g_{js}}{\delta x^k} - \dfrac{\delta g_{jk}}{\delta x^s} \right) \\
\overset{V}{L}{}^i_{jk} = \dfrac{1}{2} h^{is} \left(\dfrac{\delta h_{sk}}{\delta x^j} + \dfrac{\delta h_{js}}{\delta x^k} - \dfrac{\delta h_{jk}}{\delta x^s} \right) \\
\overset{H}{C}{}^i_{jk} = \dfrac{1}{2} g^{is} \left(\dfrac{\partial g_{sk}}{\partial y^j} + \dfrac{\partial g_{js}}{\partial y^k} - \dfrac{\partial g_{jk}}{\partial y^s} \right) \\
\overset{V}{C}{}^i_{jk} = \dfrac{1}{2} h^{is} \left(\dfrac{\partial h_{sk}}{\partial y^j} + \dfrac{\partial h_{js}}{\partial y^k} - \dfrac{\partial h_{jk}}{\partial y^s} \right).
\end{cases}
$$

Of course, the structure equations of the previous connection can be written as in the books [7], [8].

For us it is important to express the coefficients $\left(\overset{H}{L}{}^i_{jk}, \widetilde{L^i_{jk}}, \widetilde{\widetilde{L^i_{jk}}}, \overset{V}{L}{}^i_{jk}, \overset{H}{C}{}^i_{jk}, \widetilde{C^i_{jk}}, \widetilde{\widetilde{C^i_{jk}}}, \overset{V}{C}{}^i_{jk}\right)$ of the Levi–Civita connection of the metric $\mathbb{G}$.

To this aim, expressing in the adapted basis the conditions

$$
X\mathbb{G}(Y,Z) - \mathbb{G}(D_X Y, Z) - \mathbb{G}(Y, D_X Z) = 0
$$

$$
D_X Y - D_Y X - [X,Y] = 0
$$

and using the torsions $R^i{}_{jk}$ and $P^i{}_{jk}$ of the Cartan connection $C\Gamma(N)$, we get:

Theorem 2.3. *The Levi–Civita connection of the Riemannian metric $\mathbb{G}$, in the adapted basis, has the following coefficients:*

$$
(2.7) \qquad \begin{cases}
\overset{H}{L}{}^i_{jk},\ \overset{V}{C}{}^i_{jk},\ \ \overset{H}{C}{}^i_{jk} = \widetilde{\widetilde{L^i_{jk}}} = C^i_{jk} + \frac{1}{2} g^{is} h_{mj} R^m{}_{sk}, \\
\overset{V}{L}{}^i_{jk} = F^i_{jk} - \frac{1}{2}(\delta^i_s \delta^m_j - g_{sj} g^{im}) P^s{}_{km}, \quad \widetilde{C^i_{jk}} = \overset{V}{L}{}^i_{jk} - B^i_{jk}, \\
\widetilde{L^i_{jk}} = -h^{is} C_{skj} + \frac{1}{2} R^i{}_{jk}, \quad \widetilde{\widetilde{C^i_{jk}}} = -h_{js} g^{im} \widetilde{C^s_{mk}}
\end{cases}
$$

where $\overset{H}{L}{}^i_{jk}$, $\overset{V}{C}{}^i_{jk}$ *are from* (2.6) *and* (F^i_{jk}, C^i_{jk}) *are the coefficients of the Cartan metrical connection of the Finsler space* F^n.

Using the previous Levi–Civita connection we can study the main geometrical properties of the space $(\widetilde{TM}, \mathbb{G})$.

3. The almost Hermitian structure $(\mathbb{G}, \mathbb{F})$

The almost complex structure $\overset{0}{\mathbb{F}}$ defined by (1.5) does not preserve the property of homogeneity of the vector fields. It applies the 1-homogeneous vector fields $\frac{\delta}{\delta x^i}$, $(i = 1, \dots, n)$ onto the 0-homogeneous vector fields $\frac{\partial}{\partial y^i}$, $(i = 1, \dots, n)$.

We can eliminate this inconvenient by defining a new kind of almost complex structure $\mathbb{F} : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, setting

$$\mathbb{F}\left(\frac{\delta}{\delta x^i}\right) = -\frac{\|y\|}{a}\frac{\partial}{\partial y^i}, \quad \mathbb{F}\left(\frac{\partial}{\partial y^i}\right) = \frac{a}{\|y\|}\frac{\delta}{\delta x^i}, \qquad (i = 1, \dots, n). \tag{3.1}$$

Taking into account that the norm of the Liouville vector field, $\|y\|$, and the Cartan nonlinear connection N are defined on $\widetilde{TM}$, it is not difficult to prove

Theorem 3.1. *The following properties hold:*

1° $\mathbb{F}$ *is a tensor field of type* (1.1) *on* $\widetilde{TM}$.

2° $\mathbb{F} \circ \mathbb{F} = -I$.

3° $\mathbb{F}$ *depends only on the fundamental function* F *of the Finsler space* F^n.

4° *The* $\mathcal{F}(\widetilde{TM})$*-linear mapping* $\mathbb{F} : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$ *preserves the property of homogeneity of the vector fields from* $\chi(\widetilde{TM})$.

It is important to know when is $\mathbb{F}$ a complex structure.

Theorem 3.2. $\mathbb{F}$ *is a complex structure on* $\widetilde{TM}$ *if and only if the Finsler space* F^n *has the following property:*

$$R^h{}_{ij} = \frac{1}{a^2}(y_i \delta^h_j - y_j \delta^h_i). \tag{3.2}$$

Proof. The Nijenhuis tensor $\mathcal{N}_{\mathbb{F}}$:

$$\mathcal{N}_{\mathbb{F}}(X,Y)=\mathbb{F}^2[X,Y]+[\mathbb{F}X,\mathbb{F}Y]-\mathbb{F}[\mathbb{F}X,Y]-\mathbb{F}[X,\mathbb{F}Y]$$

vanishes if and only if the previous equations hold. □

Remark. If F^n is reducible to a Riemannian space, then the equation (3.2) says that it is of constant sectional curvature.

The pair $(\mathbb{G},\mathbb{F})$ has remarkable properties:

Theorem 3.3. *We have:*

1° $(\mathbb{G},\mathbb{F})$ *is an almost Hermitian structure on* $\widetilde{TM}$ *and depends only on the fundamental function* F *of the Finsler space* F^n.

2° *The associated almost symplectic structure* θ *has the expression*

$$\theta=\frac{a}{\|y\|}\overset{\circ}{\theta} \tag{3.3}$$

where $\overset{\circ}{\theta}$ *is the symplectic structure* (1.6).

3° *The following formula holds:*

$$d\theta=-\frac{a}{\|y\|}d\|y\|\wedge\theta. \tag{3.3'}$$

4° *Consequently,* $(\mathbb{G},\mathbb{F})$ *is a conformal almost Kählerian structure and we have*

$$d\theta=0 \text{ (modulo } \theta).$$

The conformal almost Kählerian space $(\widetilde{TM},\mathbb{G},\mathbb{F})$ is the geometrical model of the Finsler space F^n with respect to the homogeneous lift $\mathbb{G}$.

The previous considerations are important for the study of Finslerian gauge theory, [1]–[3], and in general in the Geometry of the Finsler space F^n.

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