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On cosymplectic quasi-Sasakian manifolds with quasi-Reeb vector field

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Abstract. A cosymplectic quasi-Sasakian manifold M (see [O]) with quasi-Reeb vector field is considered. We study some distinguished vector fields on M: skew symmetric Killing vector fields [MRV] and vector fields which define strong automorphisms of the symplectic structure. Some foliations on M are obtained.

Let $M(\phi,\Omega,\eta,\xi,g)$ be a $(2m+1)$-dimensional cosymplectic quasi-Sasakian manifold (abbr. CQS) in the sense of [O], i.e. the structure tensors satisfy:

$$\phi^2 = -I + \eta\otimes\xi, \quad d\Omega = 0, \quad d\eta = 0, \quad \xi(\eta) = 1. \tag{0.1}$$

If J means the anti-invariant operator of square $+1$ [R3], then [BR] have initiated the case when the covariant differential of the structure vector ξ satisfies:

$$\nabla\xi = c(J\circ\phi)dp, \tag{0.2}$$

where c is a non vanishing constant (called the essential constant) and dp the soldering form of M. Such a manifold is called a CQS manifold with quasi-Reeb vector field ξ (abbr. CQSQR).

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Clearly the distribution $\{Z \in \Gamma TM; \eta(X) = 0\}$ is a horizontal involutive distribution.

In the present paper, we study some properties of skew symmetric Killing vector fields [R1] (abbr. SSK) and of vector fields which define strong automorphisms of the $(1 \times Sp(2m, \mathbb{R}))$-structure considered, i.e. $\mathcal{L}_Z\Omega = 0$, $\mathcal{L}_Z\eta = 0$, where $\mathcal{L}_Z$ is the Lie derivative with respect to Z.

In Section 2 it is shown that the existence of an SSK vector field X is assured by an exterior differential system in involution (in the sense of [C]) and the following properties are proved:

(i) M is foliated by surfaces M_X of constant Ricci curvature, tangent to X and its generative $\mathcal{T}$.

(ii) $\|X\|^2$ is an isoparametric function [W], where $\|X\|^2 = g(X, X)$.

(iii) the conditions:

a) $\|X\|^2$ is an eigenfunction of Δ;

b) X is an affine vector field,

are mutually equivalent.

In Section 3 we obtain a necessary and sufficient condition for a strong automorphism of the $(1 \times Sp(2m, \mathbb{R}))$-structure to be a Killing vector field.

In Section 4 one considers on the horizontal hypersurface M_ξ defined by $\eta = 0$ two associated principal vector fields W and $\overline{W}$ in the sense of [Ph]. Then if W and $\overline{W}$ are SSK vector fields having ξ as generative, this implies that both define strong automorphisms of the $(1 \times Sp(2m, \mathbb{R}))$-structure under consideration.

1. Preliminaries

Let (M, g) be a $(2m+1)$-dimensional oriented C^∞-manifold with Riemannian metric g. Let ΓTM be the set of sections of the tangent bundle and ∇ be the covariant derivative operator defined by g. Assume that M carries the quadruple of structure tensors $(\phi, \Omega, \eta, \xi)$, where ϕ is a $(1, 1)$ tensor field, Ω is a closed 2-form of rank $2m$, η a closed Pfaffian and $\xi = \eta^\sharp$ the structure vector field (one may also call ξ the quasi-Reeb vector field (abbr. QR)). Then, if these tensor fields satisfy:

$$(1.1) \qquad \begin{cases} \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \\ g(\phi Z, \phi Z') = g(Z, Z') - \eta(Z)\eta(Z'), \quad \eta(Z) = g(\xi, Z), \\ d\Omega = 0, \quad \Omega(Z, Z') = g(Z, \phi Z'), \quad \Omega^m \wedge \eta \neq 0, \end{cases}$$

and

$$d\eta = 0, \tag{1.2}$$

one says [O] that M is a quasi-Sasakian manifold endowed with a cosymplectic structure $(1 \times Sp(2m, \mathbb{R}))$, and the distribution $D_\eta = \{Z \in \Gamma TM; \eta(Z) = 0\}$, which is called the horizontal distribution, is always involutive.

We also recall that $\flat : TM \to T^*M$, $\sharp : T^*M \to TM$ mean the musical isomorphisms defined by g, and

$$\Omega^\flat : TM \to T^*M, \quad Z \mapsto -i_Z\Omega = {}^\flat Z, \quad Z \in \Gamma TM \tag{1.3}$$

denotes the symplectic isomorphism, where i_Z is the interior product operator with respect to Z.

Further, if we set $A^q(M, TM) = \mathrm{Hom}(\Lambda^q TM, TM)$ (elements of $A^q(M, TM)$ are vector valued q-forms), then following [P], $d^\nabla : A^q(M, TM) \to A^{q+1}(M, TM)$ denotes the exterior covariant operator with respect to ∇.

It should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$. If $p \in M$, then the vector valued 1-form $dp \in A^1(M, TM)$ is the canonical vector valued 1-form of M and is called the soldering form [Di]. A (non-parallel) vector field X on a Riemannian (or pseudo-Riemannian) manifold is, following [R2], is said to be exterior concurrent (abbr. EC) if

$$d^\nabla(\nabla X) = \nabla^2 X = r \wedge dp \tag{1.4}$$

for some 1-form r, called the concurrence form associated with X. The above formula is equivalent to

$$\nabla^2 X = -\frac{1}{n-1}\,\mathrm{Ric}(X) X^\flat \wedge dp, \tag{1.5}$$

where $\mathrm{Ric}(X)$ denotes the Ricci curvature of M with respect to X and $n = \dim M$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is isoparametric [W] if $\|\nabla f\|^2$ and $\mathrm{div}(\nabla f)$ are functions of f ($\nabla f = \mathrm{grad}\, f$).

Let $\mathcal{O} = \mathrm{vect}\{e_A, A = 1, \dots, n\}$ be a local field of adapted vectorial frames over M, and let $\mathcal{O}^* = \mathrm{covect}\{\omega^A\}$ be its associated coframe. Then the soldering form dp is expressed by

$$dp = \omega^A \otimes e_A, \tag{1.6}$$

and E. Cartan's structure equations written in the indexless manner are:

$$\nabla e = \theta \otimes e, \tag{1.7}$$

$$d\omega = -\theta \wedge \omega, \tag{1.8}$$

$$d\theta = -\theta \wedge \theta + \Theta. \tag{1.9}$$

In the above equations, θ (resp. Θ) are the local connection forms in the tangent bundle TM (resp. the curvature 2-forms of M).

On a $(2m+1)$-dimensional manifold carrying the structure tensors ϕ and Ω one sets generally

$$\Omega = \omega^i \wedge \omega^{i^*}, \quad i \in \{1, \dots, m\},\ i^* = i + m, \tag{1.10}$$

and the $(1,1)$ tensor field ϕ induces the Kaehlerian relations for the horizontal connection forms

$$\theta^i_j = \theta^{i^*}_{j^*}, \quad \theta^{i^*}_j = \theta^{j^*}_i. \tag{1.11}$$

Further, following [R3] (see also [VR]) the anti-invariant operator with respect to ϕ is defined by

$$Je_i = e_{i^*}, \quad Je_{i^*} = e_i, \quad J^2 = I, \tag{1.12}$$

and one has

$$J \circ \phi + \phi \circ J = 0, \quad J\xi = 0. \tag{1.13}$$

In order to simplify, we set $\mathcal{A} = J \circ \phi$ and agree to call $\mathcal{A}$ the mixed anti-invariant operator (abbr. MA). By (1.7) we write:

$$\nabla \xi = c(J \circ \phi)dp, \quad c = \text{const.}, \tag{1.14}$$

and it is easily seen that equations (1.1) and (1.2) are satisfied.

Such a quasi-Sasakian manifold is defined as a cosymplectic quasi-Sasakian manifold with $J\phi$-structure vector field ξ. We agree to call it a quasi-Reeb vector field. One may write (1.14) as

$$\nabla \xi = c(\omega^i \otimes e_i - \omega^{i^*} \otimes e_{i^*}), \quad c \neq 0, \quad c = \text{const.}, \tag{1.15}$$

and the constant c will be called the essential constant. By (1.15), we notice that a short calculation gives

$$\operatorname{div} \xi = 0. \tag{1.16}$$

2. Skew symmetric Killing vector fields on a CQSQR-manifold

In this section we study some properties of skew symmetric Killing vector fields X on a CQSQR manifold $M(\phi,\Omega,\eta,\xi,J,g)$ defined by (0.1) and (0.2). Following [R1] such a vector field is defined by

$$\nabla X = X\wedge \mathcal{T} = \tau\otimes X - X^\flat\otimes \mathcal{T}, \tag{2.1}$$

where $\tau=\mathcal{T}^\flat$ and the vector field $\mathcal{T}$ is called the generative of X (see also [MRV]), and as in [R1] we assume that $\mathcal{T}$ is a closed torse forming (abbr. TF) [Y].

If $Z\in\Gamma TM$ is any vector field on M, then by reference to (1.7) and (1.15) its covariant differential is expressed by

$$\begin{aligned}\nabla Z = (dZ^i + Z^a\theta_a^i + cZ^0\omega^i)\otimes e_i + (dZ^{i^*} + Z^a\theta_a^{i^*} - cZ^o\omega^{i^*})\otimes e_{i^*}\\ +\big(dZ^0 - c(Z^i\omega^i - Z^{i^*}\omega^{i^*})\big)\otimes\xi,\end{aligned}\tag{2.2}$$

where $a\in\{1,\dots,2m\}$.

If X coincides with the SSK vector field, then one derives by (2.1)

$$dX^\flat = 2\tau\wedge X^\flat, \tag{2.3}$$

and so one refinds Rosca's lemma for SSK vector fields [R1], i.e. $X^\flat$ is an exterior recurrent [D] form, having τ as recurrence form. In addition, if $\mathcal{T}$ is a closed TF, then one has

$$\nabla\mathcal{T} = f dp - \tau\otimes\mathcal{T},\quad f\in C^\infty M, \tag{2.4}$$

and it is easily seen that

$$d\tau = 0. \tag{2.5}$$

Setting $s=g(X,\mathcal{T})$, one quickly derives from (2.1) that

$$ds\wedge X^\flat = 0, \tag{2.6}$$

and so, we may set

$$s = s_0 = \text{const}. \tag{2.7}$$

Further, from (2.1) and (2.3) a short calculation gives

$$ds = \big(f - \|\mathcal{T}\|^2\big)X^\flat \Rightarrow f = \|\mathcal{T}\|^2\,, \tag{2.8}$$

and under these conditions one has

$$[X, \mathcal{T}] = 0, \tag{2.9}$$

which shows that X and $\mathcal{T}$ commute. Moreover, considering $\langle \mathcal{T}, \mathcal{T}\rangle$ and taking account of (2.5), it follows from (2.8) that $d\|\mathcal{T}\|^2 = 0$, and so by (2.8) one may write

$$f = \|\mathcal{T}\|^2 = \text{const}. \tag{2.10}$$

Operating now on (2.1) and (2.4) by d^∇, one quickly derives by (2.10)

$$\begin{cases} \nabla^2 X = f X^\flat \wedge dp \\ \nabla^2 \mathcal{T} = f \mathcal{T}^\flat \wedge dp. \end{cases} \tag{2.11}$$

This proves the significant fact that both X and $\mathcal{T}$ are exterior concurrent vector fields with the constant conformal factor f. Hence, following [MRV], one may write:

$$f = -\frac{1}{2m}\,\text{Ric}(X) = -\frac{1}{2m}\,\text{Ric}(\mathcal{T}).$$

Clearly, by (2.3) the distribution $D_X = \{X, \mathcal{T}\}$ is involutive, and since the property of exterior concurrency is preserved by linearity, one may say that D_X is an autoparallel exterior concurrent distribution whose leaves are surfaces of constant Ricci curvature.

On the other hand, one derives from (2.1):

$$\nabla\,\|X\|^2 = c\,\|X\|^2\,\mathcal{T} - 2s_0 X, \quad s_0 = \text{const.}, \tag{2.12}$$

and one may write

$$\big\|\nabla\|X\|^2\big\|^2 = 8\|X\|^4 f + 2s_0^2\|X\|^2, \tag{2.13}$$

and one also infers from (2.12):

$$\text{div}\big(\nabla\|X\|^2\big) = 2(2m+1)f\,\|X\|^2 - 2s_0. \tag{2.14}$$

Hence, since $\left\|\nabla\|X\|^2\right\|^2$ and $\operatorname{div}(\nabla\|X\|^2)$ are functions of $\|X\|^2$, we conclude that $\|X\|^2 : \mathbb{R}^{2m+1} \to \mathbb{R}$ is an isoparametric function (see 1).

Further, by the well known formula $\Delta\mu = -\operatorname{div}\nabla\mu, \mu \in C^\infty M$, it follows from (2.14) that

$$\Delta \|X\|^2 = -2(2m+1)f \|X\|^2 + 2s_0. \tag{2.15}$$

This equation affirms that the necessary and sufficient condition in order that $\|X\|^2$ be an eigenfunction of Δ is that the constant s_0 vanishes. In this case, since the constant $f = \|\mathcal{T}\|^2$ is positive definite, it follows by a known Proposition that the manifold M under consideration cannot be compact (see also [BR]).

In another order of ideas, remember that a vector field Z is affine if $\mathcal{L}_Z\nabla Z = 0$.

Then, coming back to the case under discussion, one finds by (2.9) and (2.10):

$$\mathcal{L}_X\nabla X = s_0 X^\flat \otimes \mathcal{T}, \tag{2.16}$$

and so by (2.15) and (2.16) we may assert that the conditions

(i) $\|X\|^2$ is an eigenfunction of Δ;

(ii) X is an affine vector field

are equivalent.

Finally, denote by Σ the exterior differential system which determines the vector field X. Then by (2.3) and (2.5) it is seen that the characteristic numbers (or E. Cartan's numbers) of Σ are $r = 2$, $s_0 = 0$, $s_1 = 2$. Since $r = s_0 + s_1$, it follows that Σ is in involution and by E. Cartan's test [C], we conclude that the existence of X is determined by an arbitrary function of one argument.

Summing up, we state the

Theorem 2.1. *Let $M(\phi, \Omega, \eta, \xi, J)$ be the CQSQR manifold of dimension $2m+1$ under consideration. The existence of an SSK vector field X having a TF vector field $\mathcal{T}$ as generative is assured by an exterior differential system in involution and the following properties hold:*

(i) *M is foliated by surfaces M_X of constant Ricci curvature, tangent to X and $\mathcal{T}$;*

(ii) *$\|X\|^2$ is an isoparametric function;*

(iii) *the conditions $\|X\|^2$ is an eigenfunction of Δ and X is an affine vector field are equivalent.*

3. Strong automorphisms

Let Y be any vector field on a cosymplectic quasi-Sasakian manifold M and let Ω (resp. η) be the structure 2-form (resp. the structure 1-form) which defines the cosymplectic structure $(1 \times Sp(2m, \mathbb{R}))$ of M.

Following a known definition, if Y defines an infinitesimal automorphism of both Ω and η, i.e.

$$\mathcal{L}_Y \Omega = 0, \quad \mathcal{L}_Y \eta = 0, \tag{3.1}$$

one says that Y is a strong automorphism of $(1 \times Sp(2m, \mathbb{R}))$.

Assume that M is a CQSQR manifold and set

$$Y = Y^a e_a + Y^0 \xi, \quad a \in \{1, \dots, 2m\}. \tag{3.2}$$

Since $d\Omega = 0$ and $\mathcal{L}_Y = di_Y + i_Y d$, one may write

$$\mathcal{L}_Y \Omega = 0 \Longleftrightarrow d^\flat Y = 0 \Longleftrightarrow d(\phi Y)^\flat = 0, \tag{3.3}$$

where ${}^\flat Y$ is the symplectic isomorphism.

In addition, since $d\eta = 0$, it is seen that $X\eta(Y) = 0$ (i.e. $Y^0 = \text{const.}$).

One finds after some caculations

$$(\phi Y)^\flat = \Sigma(Y^i \omega^{i^*} - Y^{i^*} \omega^i), \tag{3.4}$$

then from (1.8), (1.11) and (3.4), the equation (3.3) is expressed by

$$\begin{cases} dY^i + Y^a \theta_a^i - cY^i \eta = \lambda \omega^i, \\ dY^{i^*} + Y^a \theta_a^{i^*} + cY^{i^*} \eta = -\lambda \omega^{i^*}, \end{cases} \tag{3.5}$$

where λ is a certain scalar field.

Now, using (2.2) and carrying out the calculations one derives:

$$\nabla Y = \mathcal{A}\big((\lambda + cY^0) dp + c(Y \wedge \xi)\big) - c(Y^i \omega^i - Y^{i^*} \omega^{i^*}) \otimes \xi, \tag{3.6}$$

where $\mathcal{A} = \phi \circ J$ is the mixed anti-invariant operator.

From (3.6) we quickly find

$$g(\nabla_Z Y, Z') + g(\nabla_{Z'}, Z) = 2(\lambda + cY^0) g(Z, \mathcal{A}Z'), \quad Z, Z' \in \Gamma TM,$$

which says that in order that Y be a Killing vector, the necessary and sufficient condition is that the conformal scalar associated with Y satisfies

$$\lambda + cY^0 = 0.$$

Theorem 3.1. *Let Y be a strong automorphism in the CQSQR manifold defined in Section 2, $Y^0 = \eta(Y)$ the constant vertical component of Y and λ the associated conformal scalar of Y. Then the necessary and sufficient condition in order that Y be a Killing vector is that*

$$\lambda + cY^0 = 0$$

holds good.

4. Principal vector fields

Let M_ξ be a hypersurface defined by $\eta = 0$, which foliates the manifold $M(\phi, \Omega, \eta, \xi, \mathcal{A})$ under consideration and let

$$L : TM_\xi \to TM_\xi, \quad LV = \nabla_V \xi$$

be the Weingarten map.

One finds from (1.15)

$$\begin{cases} L(JV + \phi V) = -c(JV + \phi V), \\ L(JV - \phi V) = c(JV - \phi V), \end{cases} \tag{4.1}$$

where J is the anti-invariant operator on M_ξ and V denotes any horizontal vector field.

The vector fields

$$W = JV + \phi V, \quad \overline{W} = JV - \phi V, \quad \eta(V) = 0,$$

have been defined in [BR] as the principal vector fields of M_ξ (see also [Ph]).

Taking into account (1.7) and the operators J and ϕ, one finds

$$\nabla W = dW^i \otimes e_{i^*} + W^i(\theta^a_{i^*} \otimes e_a + c\omega^{i^*} \otimes \xi), \tag{4.3}$$

and expressing that W is an SSK vector field having ξ as generative, one refinds Rosca's lemma

$$dW^\flat = 2\eta \wedge W^\flat, \tag{4.4}$$

and in addition

$$c = -1, \tag{4.5}$$

(4.6) $$\begin{cases} dW^i + W^j\theta^{i^*}_{j^*} = W^i\eta, \\ W^i\theta^{i^*}_j = 0. \end{cases}$$

In these conditions one finds

(4.7) $$(\phi W)^\flat = -W^i\omega^i = -i_W\Omega,$$

and making use of (1.1) and $\mathcal{L}_W = di_W + i_W d$, one infers

(4.8) $$d(\phi W)^\flat = 0 \Leftrightarrow \mathcal{L}_W\Omega = 0.$$

Also, we find that W is a horizontal vector field, i.e. $\eta(W) = 0$, if and only if $\mathcal{L}_W\eta = 0$. Thus W defines a strong automorphism of the cosymplectic structure $(1 \times Sp(2m, \mathbb{R}))$ of M.

Proceeding in a similar manner for the associated principal vector field $\overline{W}$ of W, one finds that the essential scalar c is equated by $+1$ and like W, the vector field $\overline{W}$ defines a strong automorphism of the $(1 \times Sp(2m, \mathbb{R}))$-structure considered.

On the other hand, it is easily seen that one has $d\|W\|^2 = 2\|W\|^2\eta$ and $d\|\overline{W}\|^2 = 2\|\overline{W}\|^2\eta$ and similarly as for $\|X\|^2$, we may prove that $\|W\|^2$ and $\|\overline{W}\|^2$ are isoparametric functions.

Theorem 4.1. *Let M_ξ be the hypersurface defined by $\eta = 0$ and let W and $\overline{W}$ be the principal vector fields defined by the Weingarten map L. If W and $\overline{W}$ are SSK vector fields having $\xi = \eta^\sharp$ as generative, then both W and $\overline{W}$ define a strong automorphism of the* $(1 \times Sp(2m, \mathbb{R}))$*-structure carried by the manifold M (CQSQR) under consideration.*

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