Publ. Math. Debrecen
57 / 3-4 (2000), 487–498

Some sufficient conditions for a map to be harmonic

By HURIYE ARIKAN (Istanbul) DEMIR N. KUPELI (Ankara)

Abstract. Some integral sufficient conditions for a map to be harmonic are obtained. In achieving this result, the divergence and Laplacian of a vector field along a map are defined and a divergence theorem for a vector field along a map (the generalized divergence theorem) is used.

1. Introduction

It is claimed in ([1], p. 9) that a map from a compact Riemannian manifold to a Riemannian manifold is harmonic if the k^{th} covariant differential of its tension field vanishes. See ([5], Prop. 2.5) for the proof of this result when the (first) covariant differential of its tension field vanishes. In this paper, we generalize this result to integral inequalities involving divergence and Laplacian of the tension field which in turn also provides a proof of the above claim (see Theorem 3.1 and Remark 3.11). For this, first we define divergence of a vector field along a map. Then we give a divergence theorem for a vector field along a map, called the generalized divergence theorem (Theorem 2.2). In fact, this theorem plays the crucial role in obtaining the mentioned generalization of the result above. Also we use two complementary theorems in achieving this result (see Theorems 2.5 and 2.6). But these latter two theorems are the straightforward generalizations of the well-known results of Bochner on vector fields to vector fields along a map. Cf. ([4], p. 158) and ([3], p. 46). Finally, we

Mathematics Subject Classification: 58C35, 58E20.
Key words and phrases: harmonic map, divergence and Laplacian of a vector field along a map, divergence theorem for a vector field along a map (the generalized divergence theorem), closed geodesic.

make an application of our above mentioned result to closed geodesics on Riemannian manifolds.

The main result (Theorem 3.1) of this paper may also be considered as an application of the generalized divergence theorem to harmonic maps. Indeed, the generalized divergence theorem plays the central role in obtaining Theorem 3.1 about harmonicity of maps between Riemannian manifolds.

Throughout this paper, everything at hand is assumed to be smooth.

2. Preliminaries

Let (V_1, g_1) and (V_2, g_2) be real inner product spaces of dimensions n_1 and n_2 respectively, and let $T : (V_1, g_1) \to (V_2, g_2)$ be a linear transformation. The *adjoint* ${}^*T$ of T is defined to be the unique linear transformation ${}^*T : (V_2, g_2) \to (V_1, g_1)$, such that for all $x \in V_1$ and $y \in V_2$

$$g_1(x, {}^*Ty) = g_2(Tx, y).$$

The adjoint linear transformation enables us to define an inner product $\langle\ ,\ \rangle$ in the space $L(V_1; V_2)$ of linear transformations from V_1 to V_2 by

$$\langle T, S\rangle = \operatorname{trace} {}^*T \circ S.$$

Note that if $\{x_1, \ldots, x_{n_1}\}$ is an orthonormal basis for (V_1, g_1) then

$$\langle T, S\rangle = \sum_{i=1}^{n_1} g_2(Sx_i, Tx_i).$$

Also, let $\|\ \|^2$ be the square norm on $L(V_1; V_2)$ induced by $\langle\ ,\ \rangle$, that is,

$$\|T\|^2 = \langle T, T\rangle.$$

Now let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds of dimensions n_1 and n_2 with Levi–Civita connections $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, respectively. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map. We denote the set of vector fields on M_1 by ΓTM_1 and the set of vector fields along f by $\Gamma_f TM_2$. We also denote the pullback of $\overset{2}{\nabla}$ along f by $\overset{2}{\nabla}$. Recall that the map

$$\nabla f_* : \Gamma TM_1 \times \Gamma TM_1 \to \Gamma_f TM_2$$

defined by

$$(\nabla f_*)(X,Y) = \overset{2}{\nabla}_X f_* Y - f_*\Big(\overset{1}{\nabla}_X Y\Big)$$

is called the *second fundamental form* of f. The trace $\tau(f)$ of ∇f_* is called the *tension field* of f. That is,

$$\tau(f) = \operatorname{trace} \nabla f_* = \sum_{i=1}^{n_1} (\nabla f_*)(X_i, X_i),$$

where $\{X_1, \ldots, X_{n_1}\}$ is a local orthonormal frame for TM_1. If $\tau(f) = 0$ then f is called *harmonic*.

Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between Riemannian manifolds (M_1, g_1) and (M_2, g_2). For a given $Z \in \Gamma_f TM_2$, define a bundle homomorphism

$${}^*f_* \overset{2}{\nabla} Z : TM_1 \to TM_1$$

by

$$\Big({}^*f_* \overset{2}{\nabla} Z\Big)x = {}^*f_{*_{p_1}} \overset{2}{\nabla}_x Z,$$

where $x \in T_{p_1}M_1$ and ${}^*f_{*_{p_1}}$ is the adjoint of $f_{*_{p_1}}$.

Definition 2.1. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between Riemannian manifolds (M_1, g_1) and (M_2, g_2). Then the *divergence* of $Z \in \Gamma_f TM_2$ is defined by

$$\operatorname{div} Z = \operatorname{trace} {}^*f_* \overset{2}{\nabla} Z.$$

Note that if $\{X_1, \ldots, X_{n_1}\}$ is a local orthonormal frame for TM_1, then

$$\operatorname{div} Z = \operatorname{trace} {}^*f_* \overset{2}{\nabla} Z = \sum_{i=1}^{n_1} g_1\Big(\Big({}^*f_* \overset{2}{\nabla} Z\Big)X_i, X_i\Big) = \sum_{i=1}^{n_1} g_2\Big(\overset{2}{\nabla}_{X_i} Z, f_* X_i\Big).$$

A motivation for the definition of the divergence of a vector field along a map may be found in [2]. Also in [2], a generalization of the divergence theorem to vector fields along a map was obtained. Since this generalization is not well-known and plays a crucial role in the proof of the main theorem of this paper, we give this theorem with its proof here.

Theorem 2.2 (The Generalized Divergence Theorem)**.** *Let (M_1, g_1) be an oriented Riemannian manifold with boundary ∂M_1 (possibly $\partial M_1 = \emptyset$) and Riemannian volume form μ_1, and let (M_2, g_2) be a Riemannian manifold. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map and $Z \in \Gamma_f TM_2$ with compact support. Then*

$$\int_{M_1} (\operatorname{div} Z)\mu_1 + \int_{M_1} g_2(Z, \tau(f))\mu_1 = \int_{\partial M_1} g_2(Z, f_* N_1)\mu_{1_{\partial M_1}},$$

where N_1 is the unit outward normal vector field to ∂M_1 and $\mu_{1_{\partial M_1}}$ is the induced Riemannian volume on ∂M_1.

Proof. Let ${}^*f_* Z$ be a vector field on M_1 defined by

$$({}^*f_* Z)(p_1) = {}^*f_{*_{p_1}} Z(p_1)$$

at each $p_1 \in M_1$, where ${}^*f_{*_{p_1}}$ is the adjoint of $f_{*_{p_1}}$. Now, if $\{X_1, \ldots, X_{n_1}\}$ is an adapted moving frame for TM_1 near p_1, that is, $\{X_1, \ldots, X_{n_1}\}$ is a local orthonormal frame for TM_1 with $(\overset{1}{\nabla} X_i)(p_1) = 0$ for $i = 1, 2, \ldots, n_1$ (cf. [4], pp. 151–152), then we have at p_1,

$$\begin{aligned}
\operatorname{div} {}^*f_* Z &= \sum_{i=1}^{n_1} g_1\Big(\overset{1}{\nabla}_{X_i}({}^*f_* Z), X_i\Big) \\
&= \sum_{i=1}^{n_1} X_i g_1({}^*f_* Z, X_i) = \sum_{i=1}^{n_1} X_i g_2(Z, f_* X_i) \\
&= \sum_{i=1}^{n_1} g_2\Big(\overset{2}{\nabla}_{X_i} Z, f_* X_i\Big) + \sum_{i=1}^{n_1} g_2\Big(Z, \overset{2}{\nabla}_{X_i} f_* X_i\Big) \\
&= \sum_{i=1}^{n_1} g_2\Big(\overset{2}{\nabla}_{X_i} Z, f_* X_i\Big) + \sum_{i=1}^{n_1} g_2(Z, (\nabla f_*)(X_i, X_i)) \\
&= \operatorname{div} Z + g_2(Z, \tau(f)).
\end{aligned}$$

Thus

$$\operatorname{div} {}^*f_* Z = \operatorname{div} Z + g_2(Z, \tau(f)).$$

Now, by applying the usual divergence theorem to ${}^*f_* Z$, we obtain

$$\int_{M_1} (\operatorname{div} {}^*f_* Z)\mu_1 = \int_{\partial M_1} g_1({}^*f_* Z, N_1)\mu_{1_{\partial M_1}} = \int_{\partial M_1} g_2(Z, f_* N_1)\mu_{1_{\partial M_1}}.$$

Hence it follows that

$$\int_{M_1} (\operatorname{div} Z)\mu_1 + \int_{M_1} g_2(Z, \tau(f))\mu_1 = \int_{\partial M_1} g_2(Z, f_* N_1)\mu_{1_{\partial M_1}}. \qquad \square$$

Note that if $(M_1, g_1) = (M_2, g_2)$ and $f = \mathrm{id}$, then the generalized divergence theorem reduces to the usual divergence theorem.

Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between Riemannian manifolds (M_1, g_1) and (M_2, g_2), and let $Z \in \Gamma_f TM_2$. Recall that the map

$$\nabla \overset{2}{\nabla} Z : \Gamma TM_1 \times \Gamma TM_1 \to \Gamma_f TM_2$$

defined by

$$\left(\nabla \overset{2}{\nabla} Z\right)(X, Y) = \overset{2}{\nabla}_X \overset{2}{\nabla}_Y Z - \overset{2}{\nabla}_{\overset{1}{\nabla}_X Y} Z$$

is called the *second covariant differential* of Z.

Definition 2.3. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between Riemannian manifolds (M_1, g_1) and (M_2, g_2). Then the *Laplacian* of $Z \in \Gamma_f TM_2$ is defined by

$$\Delta Z = \operatorname{trace} \nabla \overset{2}{\nabla} Z.$$

Note that, if $\{X_1, \ldots, X_{n_1}\}$ is a local orthonormal frame for TM_1, then

$$\Delta Z = \operatorname{trace} \nabla \overset{2}{\nabla} Z = \sum_{i=1}^{n_1} \left(\nabla \overset{2}{\nabla} Z\right)(X_i, X_i).$$

Lemma 2.4. *Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between Riemannian manifolds (M_1, g_1) and (M_2, g_2). If $Z \in \Gamma_f TM_2$ then*

$$-\frac{1}{2} \overset{1}{\Delta} g_2(Z, Z) = g_2(\Delta Z, Z) + \left\| \overset{2}{\nabla} Z \right\|^2,$$

where $\overset{1}{\Delta}$ is the Laplacian on (M_1, g_1).

Proof (Following [4], p. 158.). Let $\overset{1}{\nabla} g_2(Z, Z)$ denote the gradient of $g_2(Z, Z)$ on (M_1, g_1). First note that, for any $X \in \Gamma TM_1$,

$$g_1\left(\overset{1}{\nabla} g_2(Z, Z), X\right) = X g_2(Z, Z) = 2 g_2\left(\overset{2}{\nabla}_X Z, Z\right).$$

Now let $\{X_1, \dots, X_{n_1}\}$ be an adapted moving frame for TM_1 near $p_1 \in M_1$. Then at p_1,

$$\begin{aligned}
-\frac{1}{2}\overset{1}{\Delta} g_2(Z,Z) &= \frac{1}{2}\sum_{i=1}^{n_1} g_1\Big(\overset{1}{\nabla}_{X_i}\overset{1}{\nabla} g_2(Z,Z), X_i\Big) \\
&= \frac{1}{2}\sum_{i=1}^{n_1} X_i g_1\Big(\overset{1}{\nabla} g_2(Z,Z), X_i\Big) = \sum_{i=1}^{n_1} X_i g_2\Big(\overset{2}{\nabla}_{X_i} Z, Z\Big) \\
&= \sum_{i=1}^{n_1} g_2\Big(\overset{2}{\nabla}_{X_i}\overset{2}{\nabla}_{X_i} Z, Z\Big) + \sum_{i=1}^{n_1} g_2\Big(\overset{2}{\nabla}_{X_i} Z, \overset{2}{\nabla}_{X_i} Z\Big) \\
&= g_2(\Delta Z, Z) + \Big\|\overset{2}{\nabla} Z\Big\|^2.
\end{aligned}$$

Thus

$$-\frac{1}{2}\overset{1}{\Delta} g_2(Z,Z) = g_2(\Delta Z, Z) + \Big\|\overset{2}{\nabla} Z\Big\|^2. \qquad \square$$

Theorem 2.5. *Let (M_1, g_1) be an oriented compact Riemannian manifold with Riemannian volume form μ_1 and let (M_2, g_2) be a Riemannian manifold. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map and $Z \in \Gamma_f TM_2$. If $\int_{M_1} g_2(\Delta Z, Z)\mu_1 \geq 0$ then $\overset{2}{\nabla} Z = 0$, that is, Z is parallel.*

Proof. Since

$$\int_{M_1} \Big(\overset{1}{\Delta} g_2(Z,Z)\Big)\mu_1 = 0,$$

it follows from Lemma 2.4 that

$$\int_{M_1} g_2(\Delta Z, Z)\mu_1 + \int_{M_1} \Big\|\overset{2}{\nabla} Z\Big\|^2 \mu_1 = 0.$$

Hence, since $\int_{M_1} g_2(\Delta Z, Z)\mu_1 \geq 0$, it follows that $\|\overset{2}{\nabla} Z\|^2 = 0$, that is, $\overset{2}{\nabla} Z = 0$. $\square$

Let (M_1, g_1) be an oriented compact Riemannian manifold with Riemannian volume form μ_1 and let (M_2, g_2) be a Riemannian manifold. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map. Considering $\Gamma_f TM_2$ as a real vector space, introduce on $\Gamma_f TM_2$ the inner product

$$(Y, Z) = \int_{M_1} g_2(Y, Z)\mu_1,$$

where $Y, Z \in \Gamma_f TM_2$. Then note that $(\Gamma_f TM_2, (\ ,\ ))$ is an inner product space and the Laplacian $\Delta : \Gamma_f TM_2 \to \Gamma_f TM_2$ is a linear operator. Furthermore we have the following properties of Δ in $(\Gamma_f TM_2, (\ ,\ ))$:

Theorem 2.6. *Let (M_1, g_1) be an oriented compact Riemannian manifold with Riemannian volume form μ_1 and let (M_2, g_2) be a Riemannian manifold. Let $f : (M_1, g_1) \to (M_2, g_2)$ be a map. Then the Laplacian $\Delta : \Gamma_f TM_2 \to \Gamma_f TM_2$ is a self-adjoint, negative semi-definite operator with respect to $(\ ,\ )$.*

Proof (Following [3], p. 46.). Let $Y, Z \in \Gamma_f TM_2$ and $\{X_1, \ldots, X_{n_1}\}$ be an oriented adapted moving frame for TM_1 near $p_1 \in M_1$. Then at p_1, we have

$$\begin{aligned}\sum_{i=1}^{n_1} X_i g_2\Big(\overset{2}{\nabla}_{X_i} Y, Z\Big) &= \sum_{i=1}^{n_1} g_2\Big(\overset{2}{\nabla}_{X_i}\overset{2}{\nabla}_{X_i} Y, Z\Big) + \sum_{i=1}^{n_1} g_2\Big(\overset{2}{\nabla}_{X_i} Y, \overset{2}{\nabla}_{X_i} Z\Big)\\ &= g_2(\Delta Y, Z) + \Big\langle \overset{2}{\nabla} Y, \overset{2}{\nabla} Z\Big\rangle.\end{aligned}$$

If we now define on M_1 a 1-form ω by setting

$$\omega(X) = g_2\Big(\overset{2}{\nabla}_X Y, Z\Big),$$

then it is not difficult to show that the above equation tells us

$$d * \omega = \Big(g_2(\Delta Y, Z) + \Big\langle \overset{2}{\nabla} Y, \overset{2}{\nabla} Z\Big\rangle\Big)\mu_1,$$

where $*$ is the Hodge star operator. Integrating by using Stokes' theorem, since $\partial M_1 = \emptyset$, we get

$$\int_{M_1} g_2(\Delta Y, Z)\mu_1 = -\int_{M_1} \Big\langle \overset{2}{\nabla} Y, \overset{2}{\nabla} Z\Big\rangle \mu_1.$$

Hence the result now follows immediately. $\square$

3. Sufficient conditions for harmonicity

Let Δ^k denote the k th power of the Laplacian $\Delta : \Gamma_f TM_2 \to \Gamma_f TM_2$ and define $\Delta^0 = \mathrm{id}$, that is, $\Delta^0 = \mathrm{id}$ and $\Delta^k = \Delta \cdots \Delta$ ($k \geq 1$ times).

Now we are ready to state the main theorem of this paper.

Theorem 3.1. *Let (M_1, g_1) be an oriented compact Riemannian manifold with Riemannian volume form μ_1 and let(M_2, g_2) be a Riemannian manifold. A map $f : (M_1, g_1) \to (M_2, g_2)$ is harmonic if it satisfies one of the conditions below for some integer $k \geq 0$.*

a) $(-1)^k \int_{M_1} (\operatorname{div} \Delta^k \tau(f))\mu_1 \geq 0$

b) $\int_{M_1} g_2(\Delta^{k+1}\tau(f), \Delta^k \tau(f))\mu_1 \geq 0$.

We prove this theorem by induction. In the lemmas below, first we show that Theorem 3.1 is true for $k = 0, 1, 2, 3$.

Throughout the lemmas, let $f : (M_1, g_1) \to (M_2, g_2)$ be a map between Riemannian manifolds (M_1, g_1) and (M_2, g_2), where (M_1, g_1) is oriented and compact with Riemannian volume form μ_1.

Lemma 3.2. *If $\int_{M_1} (\operatorname{div} \tau(f))\mu_1 \geq 0$ then f is harmonic.*

Proof. Since $\partial M_1 = \emptyset$, by Theorem 2.2,

$$\int_{M_1} (\operatorname{div} \tau(f))\mu_1 + \int_{M_1} g_2(\tau(f), \tau(f))\mu_1 = 0.$$

Hence by $\int_{M_1} (\operatorname{div} \tau(f))\mu_1 \geq 0$ it follows that $g_2(\tau(f), \tau(f)) = 0$, that is, $\tau(f) = 0$ □

Lemma 3.3. *If $\int_{M_1} g_2(\Delta\tau(f), \tau(f))\mu_1 \geq 0$ then f is harmonic.*

Proof. By Theorem 2.5, $\overset{2}{\nabla}\tau(f) = 0$. Thus $\operatorname{div} \tau(f) = 0$ and it follows from Lemma 3.2 that f is harmonic. □

Lemma 3.4. *If $\int_{M_1} (\operatorname{div} \Delta\tau(f))\mu_1 \leq 0$ then f is harmonic.*

Proof. Since $\partial M_1 = \emptyset$, by Theorem 2.2,

$$\int_{M_1} (\operatorname{div} \Delta\tau(f))\mu_1 + \int_{M_1} g_2(\Delta\tau(f), \tau(f))\mu_1 = 0.$$

Hence by $\int_{M_1} (\operatorname{div} \Delta\tau(f))\mu_1 \leq 0$, we have $\int_{M_1} g_2(\Delta\tau(f), \tau(f))\mu_1 \geq 0$ and it follows from Lemma 3.3 that f is harmonic. □

Lemma 3.5. *If $\int_{M_1} g_2(\Delta^2\tau(f), \Delta\tau(f))\mu_1 \geq 0$ then f is harmonic.*

Proof. By Theorem 2.5, $\overset{2}{\nabla}\Delta\tau(f) = 0$. Thus $\operatorname{div} \Delta\tau(f) = 0$ and it follows from Lemma 3.4 that f is harmonic. □

Lemma 3.6. *If* $\int_{M_1}(\operatorname{div}\Delta^2\tau(f))\mu_1 \geq 0$ *then* f *is harmonic.*

Proof. Since $\partial M_1 = \emptyset$, by Theorem 2.2,

$$\int_{M_1}(\operatorname{div}\Delta^2\tau(f))\mu_1 + \int_{M_1} g_2(\Delta^2\tau(f), \tau(f))\mu_1 = 0.$$

But by Theorem 2.6,

$$\int_{M_1} g_2(\Delta^2\tau(f), \tau(f))\mu_1 = \int_{M_1} g_2(\Delta\tau(f), \Delta\tau(f))\mu_1.$$

Hence by $\int_{M_1}(\operatorname{div}\Delta^2\tau(f))\mu_1 \geq 0$, it follows that $g_2(\Delta\tau(f), \Delta\tau(f)) = 0$, that is, $\Delta\tau(f) = 0$. Thus either of Lemmas 3.4 or 3.5 implies that f is harmonic. □

Lemma 3.7. *If* $\int_{M_1} g_2(\Delta^3\tau(f), \Delta^2\tau(f))\mu_1 \geq 0$ *then* f *is harmonic.*

Proof. By Theorem 2.5, $\overset{2}{\nabla}\Delta^2\tau(f) = 0$. Thus $\operatorname{div}\Delta^2\tau(f) = 0$ and it follows from Lemma 3.6 that f is harmonic. □

Lemma 3.8. *If* $\int_{M_1}(\operatorname{div}\Delta^3\tau(f))\mu_1 \leq 0$ *then* f *is harmonic.*

Proof. Since $\partial M_1 = \emptyset$, by Theorem 2.2,

$$\int_{M_1}(\operatorname{div}\Delta^3\tau(f))\mu_1 + \int_{M_1} g_2(\Delta^3\tau(f), \tau(f))\mu_1 = 0.$$

But by Theorem 2.6,

$$\int_{M_1} g_2(\Delta^3\tau(f), \tau(f))\mu_1 = \int_{M_1} g_2(\Delta^2\tau(f), \Delta\tau(f))\mu_1.$$

Hence by $\int_{M_1}(\operatorname{div}\Delta^3\tau(f))\mu_1 \leq 0$, we have $\int_{M_1} g_2(\Delta^2\tau(f), \Delta\tau(f)) \geq 0$, and it follows from Lemma 3.5 that f is harmonic. □

Lemma 3.9. *If* $\int_{M_1} g_2(\Delta^4\tau(f), \Delta^3\tau(f))\mu_1 \geq 0$ *then* f *is harmonic.*

Proof. By Theorem 2.5, $\overset{2}{\nabla}\Delta^3\tau(f) = 0$. Thus $\operatorname{div}\Delta^3\tau(f) = 0$ and it follows from Lemma 3.8 that f is harmonic. □

Proof of Theorem 3.1. The above lemmas show that Theorem 3.1 is true for $k = 0, 1, 2, 3$. Now suppose the theorem is true for $k = 0, 1, 2, 3, \ldots, 2m, 2m+1$, where $m \geq 1$. We show that the theorem is true for $k = 2m+2$ and $k = 2m+3$.

Let $\int_{M_1}(\operatorname{div}\Delta^{2m+2}\tau(f))\mu_1 \geq 0$. Then, since $\partial M_1 = \emptyset$, by Theorem 2.2,

$$\int_{M_1}(\operatorname{div}\Delta^{2m+2}\tau(f))\mu_1 + \int_{M_1} g_2(\Delta^{2m+2}\tau(f), \tau(f))\mu_1 = 0.$$

But by Theorem 2.6,

$$\int_{M_1} g_2(\Delta^{2m+2}\tau(f), \tau(f))\mu_1 = \int_{M_1} g_2(\Delta^{m+1}\tau(f), \Delta^{m+1}\tau(f))\mu_1.$$

Hence by $\int_{M_1}(\Delta^{2m+2}\tau(f))\mu_1 \geq 0$, it follows that $g_2(\Delta^{m+1}\tau(f), \Delta^{m+1}\tau(f)) = 0$, that is, $\Delta^{m+1}\tau(f) = 0$. Thus, by the induction hypothesis, either of (a) or (b) implies that f is harmonic.

Now let $\int_{M_1} g_2(\Delta^{2m+3}\tau(f), \Delta^{2m+2}\tau(f))\mu_1 \geq 0$. Then by Theorem 2.5, $\overset{2}{\nabla}\Delta^{2m+2}\tau(f) = 0$. Thus $\operatorname{div}\Delta^{2m+2}\tau(f) = 0$ and it follows from the above case that f is harmonic. Consequently we showed that the theorem is true for $k = 2m+2$. Now we show that the Theorem is true for $k = 2m+3$.

Let $\int_{M_1}(\operatorname{div}\Delta^{2m+3}\tau(f))\mu_1 \leq 0$. Then, since $\partial M_1 = \emptyset$, by Theorem 2.2,

$$\int_{M_1}(\operatorname{div}\Delta^{2m+3}\tau(f))\mu_1 + \int_{M_1} g_2(\Delta^{2m+3}\tau(f), \tau(f))\mu_1 = 0.$$

But by Theorem 2.6,

$$\int_{M_1} g_2(\Delta^{2m+3}\tau(f), \tau(f))\mu_1 = \int_{M_1} g_2(\Delta^{m+2}\tau(f), \Delta^{m+1}\tau(f))\mu_1.$$

Hence by $\int_{M_1}(\operatorname{div}\Delta^{2m+3}\tau(f))\mu_1 \leq 0$, we have $\int_{M_1} g_2(\Delta^{m+2}\tau(f), \Delta^{m+1}\tau(f))\mu_1 \geq 0$, and it follows by the induction hypothesis that (b) implies f is harmonic.

Now let $\int_{M_1} g_2(\Delta^{2m+4}\tau(f), \Delta^{2m+3}\tau(f))\mu_1 \geq 0$. Then, by Theorem 2.5, $\overset{2}{\nabla}\Delta^{2m+3}\tau(f) = 0$. Thus $\operatorname{div}\Delta^{2m+3}\tau(f) = 0$ and it follows from the above case that f is harmonic. This completes the proof of the theorem. $\square$

Finally we make an application of Theorem 3.1 to closed geodesics on Riemannian manifolds. Let $(M_1, g_1) = (S^1, d\theta^2)$, where θ is the polar coordinate on S^1, and orient S^1 by $\left[\frac{\partial}{\partial\theta}\right]$. Then note that $d\theta$ is the Riemannian volume form of $(S^1, d\theta^2)$. Also, let $(M_2, g_2) = (M, g)$ be a Riemannian manifold with Levi–Civita connection ∇. Let $\gamma : (S^1, d\theta^2) \to (M, g)$ be a (curve) map. Define the velocity vector field of γ by $\dot\gamma = \gamma_* \frac{\partial}{\partial\theta}$. Now, if we set $\nabla^k_{\frac{\partial}{\partial\theta}} = \nabla_{\frac{\partial}{\partial\theta}} \cdots \nabla_{\frac{\partial}{\partial\theta}}$ ($k \geq 1$ times), then it can be easily seen that $\tau(\gamma) = \nabla_{\frac{\partial}{\partial\theta}} \dot\gamma$, $\operatorname{div} \Delta^k \tau(\gamma) = g(\nabla^{2k+2}_{\frac{\partial}{\partial\theta}} \dot\gamma, \dot\gamma)$ and $g(\Delta^{k+1}\tau(\gamma), \Delta^k\tau(\gamma)) = g(\nabla^{2k+3}_{\frac{\partial}{\partial\theta}} \dot\gamma, \nabla^{2k+1}_{\frac{\partial}{\partial\theta}} \dot\gamma)$, where $k \geq 0$. Thus by Theorem 3.1, if either

$$(-1)^{k+1} \int_0^{2\pi} g(\nabla^{2k}_{\frac{\partial}{\partial\theta}} \dot\gamma, \dot\gamma) d\theta \geq 0 \quad \text{or} \quad \int_0^{2\pi} g(\nabla^{2k+1}_{\frac{\partial}{\partial\theta}} \dot\gamma, \nabla^{2k-1}_{\frac{\partial}{\partial\theta}} \dot\gamma) d\theta \geq 0$$

for some integer $k \geq 1$, then γ is harmonic and hence a geodesic of (M, g), that is $\nabla_{\frac{\partial}{\partial\theta}} \dot\gamma = 0$.

Remark 3.10. It is easy to observe that Theorem 3.1 remains valid if (M_1, g_1) is not orientable. In this case, by passing to the Riemannian orientation covering $(\widetilde{M}_1, \widetilde{g}_1)$ of (M_1, g_1), since the Riemannian covering map $\chi : (\widetilde{M}_1, \widetilde{g}_1) \to (M_1, g_1)$ is a local isometry, the integral inequalities in the statement of Theorem 3.1 hold on (M_1, g_1) if and only if the corresponding integral inequalities hold on $(\widetilde{M}_1, \widetilde{g}_1)$ for the lift of f to $(\widetilde{M}_1, \widetilde{g}_1)$, that is $f \circ \chi$. (In fact the mentioned integrals on $(\widetilde{M}_1, \widetilde{g}_1)$ are the twice of the corresponding ones on (M_1, g_1).) Thus the same conclusion of Theorem 3.1 follows from the fact that $f \circ \chi$ is harmonic if and only if f is harmonic, since χ is a local isometry. (See [1], p. 15.)

Remark 3.11. Note that if $f : (M_1, g_1) \to (M_2, g_2)$ is a map between Riemannian manifolds (M_1, g_1) and (M_2, g_2), where (M_1, g_1) is compact, then the vanishing k^{th} covariant differential of $\tau(f)$ implies either (a) or (b) of Theorem 3.1. In fact, the vanishing odd powers of the covariant differential of $\tau(f)$ implies (a) of Theorem 3.1 and the vanishing even powers of the covariant differential of $\tau(f)$ implies (b) of Theorem 3.1. Hence this also proves the claim in ([1], p. 9) mentioned in the Introduction.

Acknowledgement. The Authors are grateful to the Department of Geometry and Topology of the University of Santiago de Compostela for their kind invitation and support during preparation of this paper.

References

[1] J. Eells and L. Lemaire, Two reports on harmonic maps, *World Scientific*, *Singapore*, 1995.

[2] E. García-Río and D. N. Kupeli, On affine Riemannian maps, *Arch. Math. (Basel)* **71** (1998), 71–79.

[3] H. B. Lawson, Lectures on minimal submanifolds, *Publish or Perish*, *Berkeley*, 1980.

[4] W. A. Poor, Differential geometric structures, *McGraw-Hill*, *New York*, 1981.

[5] K. Yano and S. Ishihara, Harmonic and relatively affine mappings, *J. Diff. Geom.* **10** (1975), 501–509.

HURIYE ARIKAN
CENTRE FOR INDIVIDUAL AND ACADEMIC DEVELOPMENT
SABANCI UNIVERSITY
TUZLA, 81474 ISTANBUL
TURKEY

E-mail: huriye@sabanciuniv.edu

DEMIR N. KUPELI
BOSNA–HERSEK CADDESI, #35/B–10
EMEK MAHALLESI
06510 ANKARA
TURKEY

E-mail: dnkupeli@superonline.com

(Received June 7, 1999; file arrived September 8, 1999)