Publ. Math. Debrecen
42 / 1–2 (1993), 65–101

On the numbers of families of solutions of systems of decomposable form equations

By K. GYŐRY (Debrecen)

Dedicated to Professor Lajos Tamássy on his 70th birthday

§1. Introduction

The main purpose of this paper is to describe the structure of the sets of solutions of systems of decomposable form equations over finitely generated domains over $\mathbb{Z}$. Such a system of equations can be written in the form

$$F_i(x_1,\dots,x_m)=\beta_i \quad \text{in } x_1,\dots,x_m\in R,\quad i=1,\dots,k, \tag{1.1}$$

or, more generally, [1)]

$$F_i(x_1,\dots,x_m)\in \beta_i R^* \text{ in } x_1,\dots,x_m\in R,\quad i=1,\dots,k, \tag{1.2}$$

where R is a finitely generated extension ring of $\mathbb{Z}$ in a finitely generated extension field K of $\mathbb{Q}$, the β_i are elements of K^* and the F_i are decomposable forms with coefficients in K (i.e. homogeneous polynomials which factorize into linear factors over some finite extension of K). In (1.2), two solutions are identified if they differ only by a proportional factor from R^*. Decomposable form equations (case $k=1$) and systems of decomposable form equations are of fundamental importance in the theory of diophantine equations. Many problems in number theory can be reduced to equations or systems of equations of this type. Norm form equations form an important class of decomposable form equations. In the case that $K=\mathbb{Q}$, $R=\mathbb{Z}$, $k=1$ and F_1 is a norm form, Schmidt [29] proved that

---

[1)] For any integral domain R, R^* will denote the unit group of R. If in particular K is a field, then $K^*=K\setminus\{0\}$.

Research supported in part by Grant 1641 from the Hungarian National Foundation for Scientific Research.

all solutions of (1.1) belong to finitely many families of solutions. This was extended by Schlickewei [25] to the case of arbitrary finitely generated subrings R of $\mathbb{Q}$. Later, Laurent [18] generalized these results to norm form equations over finitely generated extension rings R of $\mathbb{Z}$ (general case), with a slightly weaker notion of family of solutions.

In our paper, we define families of solutions for systems of equations (1.1) and (1.2) and generalize the results of Schmidt, Schlickewei and Laurent for arbitrary systems of decomposable form equations over finitely generated domains over $\mathbb{Z}$. We prove that the set of solutions of (1.1) (resp. of (1.2)) is the union of finitely many families of solutions. Moreover, we derive upper bounds for the numbers of these families of solutions which are *independent* of the coefficients of the decomposable forms involved (cf. Theorems 1,2 in § 3). In fact, we consider more general systems of decomposable form equations (cf. §§ 2 and 3). As a consequence, we get (cf. Corollaries 1 and 1') a general and quantitative finiteness criterion for (1.2) to have only finitely many solutions for every finitely generated subring R of K and every β_i in K^*. This implies some earlier results of Evertse and Győry [6] and Evertse, Gaál and Győry [10] on decomposable form equations.

More precise and explicit versions of Theorem 1 are established (cf. Theorems 4, 4' in § 4) in the special case when K is an algebraic number field and R is the ring of S-integers of K (number field case). As a consequence, an explicit upper bound is given for the number of solutions, provided that this number is finite (cf. Corollary 2). Further, a quantitative finiteness criterion is established in the case when the ground ring is fixed, and only the constant terms β_i vary.

In § 5, our results are specialized to norm form equations. We obtain as a consequence quantitative versions (cf. Theorems 6 and 7) of the above mentioned finiteness results of Schmidt, Schlickewei and Laurent on families of solutions. As a further consequence, we establish in the number field case some finiteness criteria and, under the finiteness condition, we derive explicit upper bounds for the numbers of solutions (cf. Theorem 8 and Corollary 4). These are quantitative generalizations for the number field case of well-known qualitative finiteness results of Schmidt [28] and Schlickewei [25] concerning the numbers of solutions of norm form equations. It should be remarked that in the case $K=\mathbb{Q}$, $R=\mathbb{Z}$, Schmidt [32] has recently obtained a better bound for the number of solutions of norm form equations in terms of certain parameters.

In § 6, some applications are presented to generalized systems of unit equations and, in the number field case, an explicit version of a finiteness result of Laurent [18] is established (cf. Theorem 9).

In the general case, our proofs involve a general finiteness result of Evertse and Győry [8] on unit equations whose proof depends among other things on Schlickewei's p-adic generalization [24] of Schmidt's subspace theorem [30].Consequently, all results in this paper are ineffective.

It is a consequence of the non-explicit character of the utilized result of [8] that, in the general case, we are not able to make explicit our upper bounds in terms of each parameter. Over number fields, for the case of S-unit equations, the result in question of [8] has recently been made explicit by Schlickewei [26] by means of his p-adic generalization [27] of Schmidt's recent quantitative subspace theorem [31]. This explicit version enables us to make explicit all our bounds in the number field case.

Schmidt [29] and Schlickewei [25] deduced their finiteness results on the families of solutions of norm form equations from their subspace theorems. In [6], we gave with Evertse another proof for these theorems by using qualitative finiteness results (cf. [21], [4]) on the number of non-degenerate solutions (cf. § 6) of unit equations. Further, we pointed out that our general finiteness criteria obtained in [6] for decomposable form equations are in fact equivalent with these finiteness results concerning unit equations. The results of the present paper illustrate that there is also a close connection between the structure of the sets of solutions of systems of decomposable form equations and that of systems (or generalized systems) of unit equations. Apart from the form of the bounds, our Theorems 4 and 4' concerning systems of decomposable form equations and our Theorem 9 concerning generalized systems of unit equations are in fact equivalent.

Obviously, the main results of our paper can also be applied to other classes of decomposable form equations, for example to Thue equations, discriminant form equations and index form equations. We shall not, however, deal with these applications because effective results and better upper bounds are known for these equations (see e.g. [12], [13], [7] and the references given there).

Finally, we remark that, in the number field case, some qualitative finiteness results on families of solutions of decomposable form equations have been established independently by Evertse (private communication).

§2. Notation, terminology and preliminary remarks

For certain applications, it will be convenient to consider decomposable forms and systems of decomposable form equations in a more general context. Let K be a finitely generated extension field of $\mathbb{Q}$, and V a finite dimensional K-vector space. A *decomposable form* on V over K is a function $F : V \to K$ for which there are a field extension G/K and K-linear functions $\ell_1, \dots, \ell_f : V \to G$ such that

$$F(\mathbf{x}) = \ell_1(\mathbf{x}) \cdots \ell_f(\mathbf{x}) \text{ for all } \mathbf{x} \in V. \tag{2.1}$$

Then we say that F factorizes into linear factors over G and that F is of degree f. The degree is uniquely determined. F is called *reducible* over K if it is the product of two decomposable forms on V of lower degree over K and *irreducible* otherwise. The *rank* of F is defined as the dimension of the

G-vector space of K-linear functions generated by $\{\ell_1, \dots, \ell_f\}$. The rank is independent of the choice of $\ell_1, \dots, \ell_f$ and G and is at most $\dim_K V$. The form F is called of *maximal rank* over K if $\mathrm{rank} F = \dim_K V$. If in particular $V = K^n$ for some $n \geq 2$ and $\mathbf{e}_1 = (1, 0, \dots, 0)^T, \dots, \mathbf{e}_n = (0, \dots, 0, 1)^T$ is the standard basis of K^n, we identify $F(\mathbf{x})$ on K^n with the homogeneous polynomial

$$F(\mathbf{X}) = F(X_1\mathbf{e}_1 + \cdots + X_n\mathbf{e}_n) \in K[X_1, \dots, X_n].$$

This polynomial is also called a decomposable form. For these definitions and further general facts about decomposable forms, we refer to [9].

Let R be a subring of K with 1 which is finitely generated over $\mathbb{Z}$ and which has K as its quotient field. By a theorem of Nagata [19], the integral closure of R in K is also finitely generated over $\mathbb{Z}$. Hence, for later convenience, we asssume that R itself is already integrally closed in K. As is known (see e.g. [17]), R^* is finitely generated. Let $\mathfrak{M}$ be an R-lattice, i.e. a finitely generated R-submodule of some K-vektor space. Put $K\mathfrak{M} := \{\lambda\mathbf{x} : \lambda \in K,\ \mathbf{x} \in \mathfrak{M}\}$ and assume that $n := \dim_K K\mathfrak{M} \geq 2$. Consider $k \geq 1$ decomposable forms $F_1, \dots, F_k$ on $K\mathfrak{M}$ over K with respective degrees $f_1, \dots, f_k$ which factorize into linear factors over G. The product $F_1 \dots F_k$ is also a decomposable form on $K\mathfrak{M}$ over K. We assume that it is of maximal rank over K. Further, we may assume that G is the splitting field of $F_1, \dots, F_k$, i.e. the smallest extension of K over which $F_1, \dots, F_k$ factorize into linear factors. Let $\beta_1, \dots, \beta_k$ be non-zero elements of R. We shall deal with the *system of decomposable form equations*

$$F_i(\mathbf{x}) \in \beta_i R^* \text{ in } \mathbf{x} \in \mathfrak{M}, \quad i = 1, \dots, k. \tag{2.2}$$

This is obviously a generalization of (1.2).

Denote by $\mathrm{Gal}(G/K)$ the Galois group of G/K. The form $F_1 \cdots F_k$ can be factorized as

$$F_1(\mathbf{x}) \cdots F_k(\mathbf{x}) = \alpha \prod_{j=1}^{f} \ell_j(\mathbf{x}) \text{ for all } \mathbf{x} \in K\mathfrak{M}, \tag{2.3}$$

where $\alpha \in K^*$ and $\ell_1, \dots, \ell_f : K\mathfrak{M} \to G$ are linear functions with the following properties: $\ell_i = \ell_j$ if ℓ_i and ℓ_j are linearly dependent over G and

$$\sigma(\ell_j) = \ell_{\sigma(j)} \text{ for } j = 1, \dots, f \text{ and for all } \sigma \in \mathrm{Gal}(G/K), \tag{2.4}$$

where $\{\sigma(1), \dots, \sigma(f)\}$ is a permutation of $\{1, \dots, f\}$. Moreover, putting $\mathcal{J} = \{1, \dots, f\}$, there is a partition $\{\mathcal{J}_1, \dots, \mathcal{J}_k\}$ of $\mathcal{J}$ such that for $i = 1, \dots, k$, there exists an $\alpha_i \in K^*$ for which

$$F_i(\mathbf{x}) = \alpha_i \prod_{j \in \mathcal{J}_i} \ell_j(\mathbf{x}) \text{ for } \mathbf{x} \in K\mathfrak{M},\ \sigma(\mathcal{J}_i) = \mathcal{J}_i \text{ for } \sigma \in \mathrm{Gal}(G/K). \tag{2.5}$$

Partition $\mathcal{J}$ into $\mathrm{Gal}(G/K)$-orbits $\mathcal{C}_1, \dots, \mathcal{C}_v$ sucht that j_1 and j_2 belong to the same orbit if and only if $\ell_{j_2} = \sigma(\ell_{j_1})$ for some $\sigma \in \mathrm{Gal}(G/K)$. For given w with $1 \leq w \leq v$, ℓ_j occurs with the same multiplicity, say e_w, in (2.3) for each $j \in \mathcal{C}_w$. For $k = 1$, $v = 1$, and $e_1 = 1$, $F_1(\mathbf{x})$ is in fact a *norm form* on $K\mathfrak{M}$ over K, and (2.2) is a *system of norm form equations.*

We introduce now the concept of family of solutions. Denote by $\mathbf{M}$ the set of tuples $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_f) \in G^f$ for which

$$\text{(2.6)} \qquad \begin{cases} \lambda_j = \lambda_{j'} \text{ if } \ell_j = \ell_{j'},\ j, j' \in \mathcal{J} \text{ and } \sigma(\lambda_j) = \lambda_{\sigma(j)} \\ \text{for all } j \in \mathcal{J} \text{ and } \sigma \in \mathrm{Gal}(G/K), \end{cases}$$

where $\{\sigma(1), \ldots, \sigma(f)\}$ is the same permutation of $\{1, \ldots, f\}$ as in (2.4) and (2.5). Defining the product of $\boldsymbol{\lambda} \in \mathbf{M}$ and $\boldsymbol{\mu} \in \mathbf{M}$ coordinatewise, $\mathbf{M}$ becomes a K-subalgebra of G^f. It is easy to see that $\dim_K \mathbf{M} = r$, where r denotes the maximal number of pairwise linearly independent linear functions in $\{\ell_j\}_{j \in \mathcal{J}}$. The element $\mathbf{1} = (1, \ldots, 1)$ of $\mathbf{M}$ is the unit element of $\mathbf{M}$. We denote by $\mathbf{M}^*$ the multiplicative group of invertible elements of $\mathbf{M}$. For $i = 1, \ldots, k$, we define $N_i(\boldsymbol{\lambda})$ as the product of all coordinates λ_j of $\boldsymbol{\lambda} \in \mathbf{M}$ with $j \in \mathcal{J}_i$. It is clear that $N_i(\ )$ is multiplicative and that $N_i(\boldsymbol{\lambda}) \in K$ for all $\boldsymbol{\lambda} \in \mathbf{M}$, $i = 1, \ldots, k$. The linear mapping

$$\text{(2.7)} \qquad \Psi : K\mathfrak{M} \to G^f \quad : \quad \mathbf{x} \mapsto (\ell_1(\mathbf{x}), \ldots, \ell_f(\mathbf{x}))$$

is injective because $F_1 \cdots F_k$ is of maximal rank. In view of (2.4) and (2.6), $\Psi(K\mathfrak{M})$ is contained in $\mathbf{M}$. Further, $\mathcal{M} := \Psi(\mathfrak{M})$ is an R-lattice in $\mathbf{M}$, and Ψ induces an isomorphism between $\mathfrak{M}$ and $\mathcal{M}$ (as well as between $K\mathfrak{M}$ and $K\mathcal{M}$). We say that $\mathcal{M}$ is *full* in $\mathbf{M}$ if $\mathcal{K}\mathcal{M} = \mathbf{M}$.

It will be more convenient to consider the system of equations (2.2) in the form

$$\text{(2.2')} \qquad \alpha_i N_i(\boldsymbol{\mu}) \in \beta_i R^* \quad \text{in } \boldsymbol{\mu} \in \mathcal{M},\ i = 1, \ldots, k.$$

(2.7) establishes a one-to-one correspondence between the solutions $\mathbf{x}$ of (2.2) and the solutions $\boldsymbol{\mu}$ of (2.2'). If in particular $k = 1$, $v = 1$ and $e_1 = 1$ then (omitting everywhere the subscripts), (2.2') reduces to the norm form equation

$$\text{(2.2'')} \qquad \alpha N_{M/K}(\mu) \in \beta R^* \quad \text{in } \mu \in \mathcal{M}$$

where $\mathcal{M}$ denotes now the R-module $\{\ell(\mathbf{x}) : \mathbf{x} \in \mathfrak{M}\}$ and M is a suitable subfield of G containing K and $K\mathcal{M}$.

A partition $I = \{A_1, \ldots, A_h\}$ of $\mathcal{J}$ (into non-empty subsets $A_1, \ldots, A_h$) will be called *symmetric* with respect to $\mathrm{Gal}(G/K)$ or simply sysmmetric if

$$\text{(2.8)} \qquad \begin{cases} j, j' \in \mathcal{J} \text{ belong to the same subset if } \ell_j = \ell_{j'} \text{ in (2.3)} \\ \text{and } \sigma(A_1), \ldots, \sigma(A_h) \text{ is a permutation of } A_1, \ldots, A_h \\ \text{for each } \sigma \in \mathrm{Gal}(G/K). \end{cases}$$

For a symmetric partition $I = \{A_1, \dots, A_h\}$ of $\mathcal{J}$, we denote by $\mathbf{L} = \mathbf{L}(I)$ the subset of $\mathbf{M}$ consisting of *all* elements $\boldsymbol{\lambda}$ of $\mathbf{M}$ for which $\lambda_j = \lambda_{j'}$ whenever j and j' belong to the same subset in the partition I of $\mathcal{J}$. Partition now $\mathcal{J}$ into $\mathrm{Gal}(G/K)$-orbits such that the elements of the subsets A_ℓ and A_m of the partition I belong to the same orbit if and only if $\sigma(A_\ell) = A_m$ for some $\sigma \in \mathrm{Gal}(G/K)$. Pick a full set of representatives, say $\{j_1, \dots, j_b\}$, of these orbits. Then it is easy to see that for given a with $1 \le a \le b$, the coordinates λ_{j_a} in $\boldsymbol{\lambda} \in \mathbf{L}$ form a subfield L_{j_a} of G containing K, that $\mathbf{L}$ is a K-subalgebra of $\mathbf{M}$ with unit element $\mathbf{1}$ and that

(2.9) $\mathbf{L} = \mathbf{L}(I)$ is isomorphic to the K-algebra $L_{j_1} \oplus \cdots \oplus L_{j_b}$,

(where the fields $L_{j_1}, \dots, L_{j_b}$ are not necessarily distinct). We note that for $k = 1$ and $v = 1$ we have $b = 1$ in (2.9). If in particular $I = \{\mathcal{J}\}$, then $b = 1$, $L_{j_1} = K$ and the corresponding subalgebra $\mathbf{L}(\mathcal{J})$ is denoted by $\mathbf{K}$. Sometimes we shall identify $\mathbf{K}$ by K. Further, if I_0 denotes that partition of $\mathcal{J}$ for which $j, j' \in \mathcal{J}$ belong to the same subset if and only if $\ell_j = \ell_{j'}$ in (2.3), then $\mathbf{M} = \mathbf{L}(I_0)$. Finally, it is not difficult to show that for any K-subalgebra $\mathbf{L}$ of $\mathbf{M}$ with $\mathbf{1}$ there is a symmetric partition I of $\mathcal{J}$ such that $\mathbf{L} = \mathbf{L}(I)$. Consequently, the K-subalgebras $\mathbf{L} = \mathbf{L}(I)$ of $\mathbf{M}$ under consideration are precisely the subalgebras $\mathbf{L}$ of $\mathbf{M}$ containing $\mathbf{K}$.

Let $\mathbf{L}$ be a subalgebra of $\mathbf{M}$ containing $\mathbf{1}$; then $\mathbf{L} = \mathbf{L}(I)$ with a symmetric partition I of $\mathcal{J}$. Let $R_{\mathbf{L}}$ denote the set of those elements $\boldsymbol{\lambda} = (\lambda_j)$ of $\mathbf{L}$ for which all coordinates λ_j are integral over R. It is easy to see that $R_{\mathbf{L}}$ is a subring of $\mathbf{L}$ with unit element $\mathbf{1}$. Further, for the unit group $R^*_{\mathbf{L}}$ of $R_{\mathbf{L}}$, we get by (2.9) that

(2.10) $R^*_{\mathbf{L}}$ is isomorphic to $R^*_{L_{J_1}} \times \cdots \times R^*_{L_{j_b}}$,

where $R^*_{L_{j_a}}$ denotes the unit group of the subring of L_{j_a} consisting of all integral elements of L_{j_a} over R. The group $R^*_{L_{j_a}}$ is finitely generated for each a (cf. [19], [17]) hence, by (2.10), $R^*_{\mathbf{L}}$ is also finitely generated.

For every solution $\boldsymbol{\mu}'$ of (2.2') and every subalgebra $\mathbf{L}$ of $\mathbf{M}$ containing $\mathbf{1}$ for which $\boldsymbol{\mu}'\mathbf{L} \subseteq K\mathcal{M}$, all elements of $(\boldsymbol{\mu}' R^*_{\mathbf{L}}) \cap \mathcal{M}$ are solutions of (2.2'). Such a subset of solutions $(\boldsymbol{\mu}' R^*_{\mathbf{L}}) \cap \mathcal{M}$ of (2.2') is called an $(\mathcal{M}, \mathbf{L})$-*family of solutions in a wider sense* or simply a *wide* $(\mathcal{M}, \mathbf{L})$-*family of solutions*[2)].

To state our results in a quantitative form, we need some further notation and assumptions. Let m be the minimal cardinality of the sets of generators of $\mathfrak{M}$. A set of generators of minimal cardinality is called *minimal.* We assume that the decomposable forms $F_1, \dots, F_k$ are *integral* on $\mathfrak{M}$, i.e. that for some set of generators $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ of $\mathfrak{M}$, the polynomial $F_i(\sum_{j=1}^m X_j \mathbf{a}_j)$ has its coefficients in R for $i = 1, \dots, k$. This notion of

[2)] Later, we shall also define families of solutions in a restricted sense.

integrality is independent of the choice of $\mathbf{a}_1, \dots, \mathbf{a}_m$. In what follows, we fix a minimal set of generators $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ of $\mathfrak{M}$. We denote by m_i the number of those variables in the polynomial $F_i(\sum_{j=1}^m X_j \mathbf{a}_j)$ which have non-zero coefficients, $i = 1, \dots, k$. It is clear that $m_i \le m$ for $i = 1, \dots, k$.

By the assumption made on R, there exists a divisor theory for R. For a non-zero element $\alpha \in R$ and for a positive integer q, we denote by $\tau_q(\alpha)$ the number of factorizations of the principal divisor (α) into q integral divisors in R, and by $\omega(\alpha)$ the number of distinct prime divisors of (α). Denote by R_G the integral closure of R in G. Then both R_G and R_G^* are finitely generated (cf. [17], [19]). For all $\mathbf{x} \in \mathfrak{M}$, there is at least one tuple $(x_1, \dots, x_m) \in R^m$ for which

$$\mathbf{x} = x_1 \mathbf{a}_1 + \cdots + x_m \mathbf{a}_m \tag{2.11}$$

holds. A solution $\mathbf{x} \in \mathfrak{M}$ of (2.2) is called *primitive* (with respect to $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$) if $\mathbf{x}$ has a representation of the form (2.11) such that the divisors $(x_1), \dots, (x_m)$ are relatively prime in R.

For $i = 1, \dots, k$, t_i (resp. r_i) denotes the maximal number of (pairwise) linearly independent linear functions in $\{\ell_j\}_{j \in \mathcal{J}_i}$, $q_i := \min\{m_i - 1, t_i\}$ and u_i is the maximum of the degrees of the irreducible factors of F_i over K. We note that $r \le r_1 + \cdots + r_k$. Throughout this paper, C_k and C_k^* will denote the following expressions:

$$C_k = \prod_{i=1}^{k} \binom{r_i}{q_i}^{\omega(\beta_i)} \tau_{q_i+1}(\beta_i^{u_i}) \tag{2.12}$$

and

$$C_k^* = \prod_{i=1}^{k} \binom{r_i}{q_i}^{\omega(\beta_i)} \tau_{q_i}(\beta_i^{u_i}). \tag{2.13}$$

For any finitely generated multiplicative subgroup Γ of G^* and for any integer $p \ge 2$, there is a number $B(p, \Gamma)$ with the following property: if $\alpha_1, \dots, \alpha_i \in G^*$ with $2 \le i \le p$, then the solutions of the equation

$$\alpha_1 x_1 + \cdots + \alpha_i x_i = 1 \text{ in } x_1, \dots, x_i \in \Gamma \tag{2.14}$$

are contained in the union of at most $B(p, \Gamma)$ $(i-1)$-dimensional linear subspaces of K^i. The existence of such a bound $B(p, \Gamma)$ was proved by Evertse and Győry [8]. Equations of the form (2.14) are called *unit equations.*

§3. Results for systems of decomposable form equations in the general case

In this section, we keep the notation and assumptions made in § 2.

Theorem 1. *The set of solutions of* (2.2') *is the union of at most*

$$C_k \prod_{i=2}^{n} B(i, R_G^*) \tag{3.1}$$

wide families of solutions. Further, if $\mathcal{M}$ is full in **M** *then the set of solutions of* (2,2') *is contained in the union of at most C_k wide* $(M, \mathbf{M})$-*families of solutions.*

Let U be an arbitrary subgroup of R^*, and consieder the generalization

$$\alpha_i N_i(\boldsymbol{\mu}) \in \beta_i U \text{ in } \boldsymbol{\mu} \in \mathcal{M},\ i = 1, \dots, k \tag{3.2}$$

of (2.2'). For any subalgebra $\mathbf{L}$ of $\mathbf{M}$ containing $\mathbf{1}$, denote by $U_{\mathbf{L}}$ the subgroup of $R_{\mathbf{L}}^*$ consisting of all elements $\boldsymbol{\lambda}$ with $N_i(\boldsymbol{\lambda}) \in U$ for $i = 1, \dots, k$. Then $U_{\mathbf{L}}$ is also finitely generated. Further, for every solution $\boldsymbol{\mu}$ of (3.2) with $\boldsymbol{\mu}\mathbf{L} \subseteq K\mathcal{M}$, all elements of $(\boldsymbol{\mu}U_{\mathbf{L}}) \cap \mathcal{M}$ are also solutions of (3.2). Such a set of solutions is called a *wide* $(\mathcal{M}, \mathbf{L})$-*family of solutions* of (3.2).

From Theorem 1, it is easy to deduce a similar result for (3.2). Indeed, by Theorem 1, all solutions of (3.2) are contained in the union of finitely many sets of the form $\boldsymbol{\mu}R_{\mathbf{L}}^* \cap \mathcal{M}$ where $\boldsymbol{\mu}$ is a solution of (2.2') and $\mathbf{L}$ is a subalgebra of $\mathbf{M}$ with $\mathbf{1}$ and with $\boldsymbol{\mu}\mathbf{L} \subseteq K\mathcal{M}$. But if $\boldsymbol{\mu}R_{\mathbf{L}}^*$ has an element which satisfies (3.2) then we may assume that $\boldsymbol{\mu}$ itself is such an element. Hence all elements of $\boldsymbol{\mu}R_{\mathbf{L}}^*$ satifsfying (3.2) belong to $\boldsymbol{\mu}U_{\mathbf{L}}$. This implies the following.

Theorem 2. *The set of solutions of* (3.2) *is the union of finitely many wide families of solutions of* (3.2). *Further, the number of these families of solutions is bounded above by* (3.1) *in general, and by C_k if $\mathcal{M}$ is full in* **M**.

For $U = R^*$, Theorem 2 gives Theorem 1 and hence Theorems 1 and 2 are equivalent. Another interesting special case is when $U = \{1\}$. Denote by $E_{\mathbf{L}}$ the subgroup of $R_{\mathbf{L}}^*$ consisting of elements $\boldsymbol{\lambda}$ with $N_i(\boldsymbol{\lambda}) = 1$ for $i = 1, \dots, k$. Then, by Theorem 2, the set of solutions of the system of equations

$$\alpha_i N_i(\boldsymbol{\mu}) = \beta_i \text{ in } \boldsymbol{\mu} \in \mathcal{M}, \quad i = 1, \dots, k, \tag{3.3}$$

is the union of finitely many sets of the form $(\boldsymbol{\mu}E_{\mathbf{L}}) \cap \mathcal{M}$ where $\boldsymbol{\mu}$ is a solution of (3.3) with $\boldsymbol{\mu}\mathbf{L} \subseteq K\mathcal{M}$. Further, Theorem 2 furnishes an upper bound for the number of these sets.

We shall say that $\mathcal{M}$ is *degenerate* (resp. *non-degenerate*) if there exists (no) a subalgebra $\mathbf{L}$ of $\mathbf{M}$ with $\mathbf{1}$, different from $\mathbf{K} = \mathbf{K}(\mathcal{J})$, such that $\boldsymbol{\nu}\mathbf{L} \subset K\mathcal{M}$ for some $\boldsymbol{\nu} \in \mathbf{M}^*$. If $\mathcal{M}$ is non-degenerate then, by Theorem 1, all solutions of (2.2') are contained in the union of at most $C_k \Pi_{i=2}^n B(i, R_G^*)$ sets of the form $\boldsymbol{\mu}R_{\mathbf{K}}^*$. Since, by assumption, R is integrally closed, we may take $\boldsymbol{\mu}R^*$ instead of $\boldsymbol{\mu}R_{\mathbf{K}}^*$. A set of solutions of the form $\boldsymbol{\mu}R^*$ is called an

R^*-coset of solutions of (2.2'). Hence Theorem 1 implies the implication $(i) \Longrightarrow (ii)$ in Corollary 1 below. The implication $(ii) \Longrightarrow (i)$ follows from Theorem 4 of [6] and Lemma 8 of the present paper.

Corollary 1. *The following two statements are equivalent.*

(i) *$\mathcal{M}$ is non-degenerate;*

(ii) *For every $(\beta_1, \dots, \beta_k) \in (K^*)^k$ and every integrally closed subring R of K with 1 which has K as its quotient field and which is finitely generated over $\mathbb{Z}$, the system of equations (2.2') has only finitely many R^*-cosets of solutions.*

Moreover, if (i) *holds then under the notation and assumptions made in § 2 on $\mathcal{M}$ and $\beta_1, \dots, \beta_k$, the number of R^*-cosets of solutions is bounded above by the number occurring in (3.1) of Theorem 1.*

In order to compare Corollary 1 with earlier finiteness criteria concerning decomposable form equations, we now reformulate the finiteness condition (i) for the system of equations (2.2). If $\mathbf{x}$ is a solution of (2.2) then $\mathbf{x}R^*$ is called an R^*-coset of solutions of (2.2). If in particular $\mathbf{x}$ is a primitive solution then every element of $\mathbf{x}R^*$ is also a primitive solution. We recall that $\ell_1, \dots, \ell_f$ are the linear functions in the factorization (2.3) of $F_1 \dots F_k$ over G. Denote by $\mathcal{L}$ the collection of linear functions $\{\ell_1, \dots, \ell_f\}$. A non-zero subspace H of $K\mathfrak{M}$ is called *$\mathcal{L}$-admissible* (or simply admissible) if no linear function in $\mathcal{L}$ vanishes identically on H. Further, an $\mathcal{L}$-admissible subspace H of $K\mathfrak{M}$ is called *$\mathcal{L}$-non-degenerate* or *$\mathcal{L}$-degenerate* (or simply non-degenerate or degenerate) according as $\mathcal{L}$ does or does not contain a subset of at least three linear functions which are linearly dependent on H, but pairwise linearly independent on H. In particular, H is degenerate if H has dimension 1. We shall show in §7 (cf. Lemma 8) that $\mathcal{M}$ is non-degenerate if and only if every admissible subspace of $K\mathfrak{M}$ of dimension ≥ 2 is non-degenerate. Hence, for $k = 1$ and $K\mathfrak{M} = K^n$, our Corollary 1 implies Theorem 1 of [6] and (the case $\mathcal{L} = \emptyset$ of) Corollary 1 of [10]. Further, under the above assumptions concerning $F_1, \dots, F_k$ and $\beta_1, \dots, \beta_k$, we have

Corollary 1'. *Suppose that every admissible subspace of $K\mathfrak{M}$ of dimension ≥ 2 is non-degenerate. Then the number of R^*-cosets of solutions of (2.2) is bounded above by the number occurring in* (3.1) *of Theorem 1. Further, the number of R^*-cosets of primitive solutions is at most*

$$C_k^* \prod_{i=2}^{n} B(i, R_G^*).$$

In certain important special cases, Theorems 1 and 2 can be made more precise. Let $\mathbf{L}$ be a subalgebra of $\mathbf{M}$ with $\mathbf{1}$ and denote by $\mathcal{M}^{\mathbf{L}}$ the set of elements $\boldsymbol{\mu} \in \mathcal{M}$ such that $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} \in K\mathcal{M}$ for every $\boldsymbol{\lambda} \in \mathbf{L}$. It is easy to see that in this case even $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} \in K\mathcal{M}^{\mathbf{L}}$ holds. Further, it is not difficult to show that $\mathcal{M}^{\mathbf{L}}$ is an R-sublattice of $\mathcal{M}$ and $\mathcal{M}^{\mathbf{K}} = \mathcal{M}$. If $\mathcal{M}$ is full in $\mathbf{M}$

then $\mathcal{M}^{\mathbf{M}} = \mathcal{M}$. Suppose now that $\boldsymbol{\mu} \cdot \mathbf{L} \subseteq K\mathcal{M}$ for some subalgebra $\mathbf{L}$ of $\mathbf{M}$ with $\mathbf{1}$ and some solution $\boldsymbol{\mu}$ of (3.2). Then $\boldsymbol{\mu} \in \mathcal{M}^{\mathbf{L}}$ and hence we have $\mathcal{M}^{\mathbf{L}} \cap \mathbf{M}^* \neq \mathbf{0}$. We shall say that a subalgebra $\mathbf{L}$ of $\mathbf{M}$ with $\mathbf{1}$ is *admissible* with respect to $\mathcal{M}$ or simply admissible if $\mathcal{M}^{\mathbf{L}} \cap \mathbf{M}^* \neq \mathbf{0}$ and if there is no subalgebra $\mathbf{L}'$ with $\mathbf{1}$ in $\mathbf{M}$ such that $\mathbf{L} \subsetneqq \mathbf{L}'$ and $K\mathcal{M}^{\mathbf{L}'} = K\mathcal{M}^{\mathbf{L}}$. If $\mathcal{M}$ is full then $\mathbf{M}$ is admissible. Denote by $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}}$ the set of those $\boldsymbol{\lambda} \in \mathbf{L}$ for which $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} \in \mathcal{M}^{\mathbf{L}}$ for all $\boldsymbol{\mu} \in \mathcal{M}^{\mathbf{L}}$. Then $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}}$ is a subring of $R_{\mathbf{L}}$ with $\mathbf{1}$ and $K\mathcal{D}_{\mathcal{M}}^{\mathbf{L}} = \mathbf{L}$. Further, $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}}$ contains R as a subring (identifying the elements a of R with $(a, \ldots, a)$). $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}}$ is called the *ring of coefficients* of $\mathcal{M}^{\mathbf{L}}$. Denote by $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$ the unit group of $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}}$, and put

$$U'_{\mathbf{L}} = U_{\mathbf{L}} \cap \mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}.$$

Consider a full set of representatives $(\boldsymbol{\varepsilon}_j)_{j \in \mathcal{I}}$ of $U_{\mathbf{L}}$ modulo $U'_{\mathbf{L}}$. It is easy to see that the wide family of solutions $(\boldsymbol{\mu} U_{\mathbf{L}}) \cap \mathcal{M}$ of (3.2) splits into disjoint subsets of the form $(\boldsymbol{\mu}\boldsymbol{\varepsilon}_j)U'_{\mathbf{L}}$ where j runs over the set of indices sucht that $\boldsymbol{\mu} \cdot \boldsymbol{\varepsilon}_j \in \mathcal{M}^{\mathbf{L}}$. If $\boldsymbol{\mu}' \in \mathcal{M}^{\mathbf{L}}$ is a solution of (3.2) then so are all elements of $\boldsymbol{\mu}' \cdot U'_{\mathbf{L}}$. Such a set is called a *family of solutions* of (3.2), or more precisely an $(\mathcal{M}, \mathbf{L})$-family of solutions. If in particular $U = R$, then $U'_{\mathbf{L}} = \mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$. Hence, if $\boldsymbol{\mu} \in \mathcal{M}^{\mathbf{L}}$ is a solution of (2.2') then $\boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$ is called a family of solutions of (2.2').

From Theorem 2 it follows the following

Theorem 3. *Suppose that*

(FN) *for all admissible subalgebras* $\mathbf{L}$ *of* $\mathbf{M}$, $U'_{\mathbf{L}}$ *is of finite index in* $U_{\mathbf{L}}$.

Then the set of solutions of (3.2) *is the union of at most* $\sum_{\mathbf{L}} [U_{\mathbf{L}} : U'_{\mathbf{L}}]$ *families of solutions, where the sum is taken over at most* $C_k \Pi_{i=2}^{n} B(i, R_G^*)$ *admissible subalgebras* $\mathbf{L}$ *of* $\mathbf{M}$ *for which* (3.2) *has an* $(\mathcal{M}, \mathbf{L})$*-family of solutions (and among which there can be identical subalgebras* $\mathbf{L}$*). Further, if* $\mathcal{M}$ *is full in* $\mathbf{M}$ *then the set of solutions of (3.2) is the union of at most* $[U_{\mathbf{M}} : U'_{\mathbf{M}}] \cdot C_k$ $(\mathcal{M}, \mathbf{M})$*-families of solutions.*

Theorem 3 is a quantitative generalization of Theorem 9 of Laurent [18] concerning norm form equations. As was pointed out by Laurent [18] in the case of norm form equations, the finiteness condition (FN) does not hold in general, for the case of arbitrary finitely generated field K and arbitrary finitely generated and integrally closed subring R.

Theorem 3 can be applied in an obvious way to the solutions of (2.2') and of (2.2) as well, provided that the finiteness condition corresponding to (FN) is satisfied.

In the following section, we shall show that in the number field case and for an important class of rings R, the finiteness condition (FN) always holds. Further, we shall make more precise and explicit the results of § 3.

§4. Results for systems of decomposable form equations in the number field case

In this section, we specialize our results to the algebraic number field case. We keep the notation of § 2. Throughout this section, let in particular K be an algebraic number field, and $R = O_S$, the ring of S-integers in K, where S is a finite subset of the set of places $M(K)$ of K with cardinality s which contains the set of infinite places $M_\infty(K)$ of K. We recall that $O_S = \{\alpha \in K : |\alpha|_v = 1 \text{ for all } v \in M(K) \setminus S\}$. In this case, the system of equations (2.2') takes the form

$$\alpha_i N_i(\boldsymbol{\mu}) \in \beta_i O_S^* \quad \text{in } \boldsymbol{\mu} \in \mathcal{M},\ i = 1, \ldots, k. \tag{4.1}$$

Denote by d the degree of K over $\mathbb{Q}$, by D the degree of the normal closure of G over $\mathbb{Q}$, and by $\mathfrak{I}_{\mathbf{L}}$ the index $[R_{\mathbf{L}}^* : \mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}]$ where $\mathbf{L}$ is an admissible subalgebra of $\mathbf{M}$ with $\mathbf{1}$. We shall show (cf. Lemma 9) that in this special situation the finiteness condition (FN) of Theorem 3 holds. Further, in this case one can derive an explicit upper bound for $\Pi_{i=2}^n B(i, R_G^*)$ (cf. Lemma 10) by using a recent explicit estimate of Schlickewei [26]. Thus, using Theorem 3 we get the following.

Theorem 4. *Under the above notation and assumptions, the set of solutions of* (4.1) *is the union of at most* $\sum_{\mathbf{L}} \mathfrak{I}_{\mathbf{L}}$ *families of solutions, where the sum is taken over at most*

$$C_k \exp\{2^{37nD} \cdot s^6 \cdot \log(4sD)\} \tag{4.2}$$

admissible subalgebras $\mathbf{L}$ *of* $\mathbf{M}$ *for which* (4.1) *has an* $(\mathcal{M}, \mathbf{L})$*-family of solutions (and among which there can be identical subalgebras* $\mathbf{L}$*). Further, if* $\mathcal{M}$ *is full in* $\mathbf{M}$ *then the set of solutions of* (4.1) *is the union of at most* $\mathfrak{I}_{\mathbf{M}} \cdot C_k$ $(\mathcal{M}, \mathbf{M})$*-families of solutions.*

From Theorem 3 one can deduce a similar result for the solutions of the system of equations corresponding to (3.2).

A family of solutions of (4.1) is called *maximal* if it is not properly contained in another family of solutions. If $\boldsymbol{\mu}_1 \mathcal{D}_{\mathcal{M}}^{\mathbf{L}_1*}$ and $\boldsymbol{\mu}_2 \mathcal{D}_{\mathcal{M}}^{\mathbf{L}_2*}$ are families of solutions of (4.1) such that $\boldsymbol{\mu}_1 \mathcal{D}_{\mathcal{M}}^{\mathbf{L}_1*} \subseteq \boldsymbol{\mu}_2 \mathcal{D}_{\mathcal{M}}^{\mathbf{L}_2*}$ then $\boldsymbol{\mu}_1 \mathcal{D}_{\mathcal{M}}^{\mathbf{L}_2*} = \boldsymbol{\mu}_2 \mathcal{D}_{\mathcal{M}}^{\mathbf{L}_2*}$ and so $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}_1*} \subseteq \mathcal{D}_{\mathcal{M}}^{\mathbf{L}_2*}$. Further, if $\boldsymbol{\mu}_1 \mathcal{D}_{\mathcal{M}}^{\mathbf{L}_1*}$ is properly contained in $\boldsymbol{\mu}_2 \mathcal{D}_{\mathcal{M}}^{\mathbf{L}_2*}$ then $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}_1*}$ is properly contained in $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}_2*}$. $\mathbf{M}$ has only finitely many admissible subalgebras $\mathbf{L}$, hence there are only finitely many groups $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$. This implies that every family of solutions is contained in a maximal family of solutions. Using a generalization of some arguments of Schmidt [29] concerning norm form equations, we shall deduce from Theorem 4 the following Theorem 4'. Thus Theorem 4 and 4' are in fact equivalent.

Theorem 4'. *The system of equations* (4.1) *has at most* $\sum_{\mathbf{L}} \mathfrak{I}_{\mathbf{L}}$ *maximal families of solutions, where the sum is taken over at most*

$$C_k n^r \exp\{2^{37nD} \cdot s^6 \cdot \log(4sD)\} \tag{4.3}$$

admissible subalgebras $\mathbf{L}$ *of* $\mathbf{M}$ *for which* (4.1) *has a maximal* $(\mathcal{M}, \mathbf{L})$*-family of solutions (and among which there can be identical subalgebras* $\mathbf{L}$*). Further, if* $\mathcal{M}$ *is full in* $\mathbf{M}$ *then* (4.1) *has at most* $\mathfrak{I}_{\mathbf{M}} \cdot C_k$ *maximal families of solutions and all these are* $(\mathcal{M}, \mathbf{M})$*-families of solutions.*

It is easy to show that in our bounds, the factors $\mathfrak{I}_{\mathbf{L}}$, $\mathfrak{I}_{\mathbf{M}}$ cannot be omitted. Further, $D \leq (dr)!$.

For $k = 1$, some qualitative versions of Theorems 4 and 4' have been established independently by Evertse (private communication). He uses different terminology which is however equivalent to ours.

From Theorem 4 we shall deduce the following.

Corollary 2. *Suppose that the system of equations* (4.1) *has only finitely many* O_S^**-cosets of solutions. Then the number of its* O_S^**-cosets of solutions is at most*

$$C_k \exp\{2^{37nD} \cdot s^6 \cdot \log(5sD)\}. \tag{4.4}$$

We say that a solution $\boldsymbol{\mu}$ of (4.1) is *degenerate* if it is contained in an $(\mathcal{M}, \mathbf{L})$-family of solutions of (4.1) with some admissible subalgebra $\mathbf{L}$ of $\mathbf{M}$ different from $\mathbf{K}$, and *non-degenerate* otherwise. If $\boldsymbol{\mu}$ is a degenerate (resp. non-degenerate) solution of (4.1) then so is every element of the O_S^*-coset $\boldsymbol{\mu} O_S^*$. Since the O_S^*-cosets of solutions are precisely the $(\mathcal{M}, \mathbf{K})$-families of solutions, we get from Theorem 4 as an immediate consequence that the number of O_S^*-cosets of non-degenerate solutions of (4.1) is bounded above by the number occurring in (4.2).

It is easy to see that $\mathcal{M}$ is degenerate if and only if there is a subalgebra $\mathbf{L}$ of $\mathbf{M}$ with $\mathbf{1}$ which is different from $\mathbf{K}$ and is admissible with respect to $\mathcal{M}$, and non-degenerate otherwise. If $\mathcal{M}$ is non-degenerate then every solution of (4.1) is non-degenerate. Hence Corollary 3 below immediately follows from the above consequence of Theorem 4.

Corollary 3. *If* $\mathcal{M}$ *is non-degenerate, then the number of* O_S^**-cosets of solutions of* (4.1) *is bounded above by the number occurring in* (4.2).

Remark 1. If we consider the system of equations (4.1) in the equivalent form (2.2), then, by Corollary 1', we get

$$C_k^* \exp\{2^{37nD} \cdot s^6 \cdot \log(4sD)\} \tag{4.5}$$

as an upper bound for the number of O_S^*-cosets of primitive solutions of this system of equations.

Remark 2. For a consequence of Corollary 3 for norm form equations, see Corollary 4 in § 5. Qualitative versions of Corollary 3 were earlier established by SCHMIDT [28] (in case $K = \mathbb{Q}$, $O_S = \mathbb{Z}$), SCHLICKEWEI [25] (in case $K = \mathbb{Q}$) and LAURENT [18] (in the general case) for norm form equations, and by EVERTSE and GYŐRY [6] for decomposable form equations. In the case when $K = \mathbb{Q}$, $O_S = \mathbb{Z}$, $k = 1$ and (4.1) is a norm form equation, SCHMIDT [32] has recently derived in the non-degenerate case the bound

$$(4.6) \qquad \binom{r}{m-1}^{\omega(\beta)} \tau_m(\beta^r) r^{c_1} \quad \text{with } c_1 = \min(2^{29n} \cdot r^2, (2n)^{n \cdot 2^{n+4}})$$

for the number of primitive solutions of (4.1). In terms of r, this bound is better than (4.2) and (4.5) in the special case under consideration.

Remark 3. Very recently EVERTSE (private communication) has obtained another version of Corollary 3 for $k = 1$ with a bound which is better than (4.2) and (4.5) in terms of D but is in general weaker in terms of β. On combining our method of proof with that of Evertse, both Evertse's bound and our bound can be improved. Namely, the factor $\exp\{2^{37nD} \cdot s^6 \cdot \log(4sD)\}$ in our bounds (4.2), (4.5), (4.3) and (4.4) can be replaced by

$$(4.7) \qquad \exp\{2^{38nd} \cdot s^6 \cdot \log(8sD)\}.$$

Corollary 1 in § 3 provides a finiteness criterion for the number of O_S^*-cosets of solutions of (4.1) in the case when S and $\beta_1, \dots, \beta_k$ in (4.1) vary. We shall now deduce from Theorem 4 a finiteness criterion in the case when S and hence O_S is fixed, only $\beta_1, \dots, \beta_k$ vary. In order to formulate our criterion, we have to introduce some further concepts and notation. Let $\mathbf{L} = \mathbf{L}(I)$ be an admissible subalgebra of $\mathbf{M}$ where $I = \{A_1, \dots, A_h\}$ is a symmetric partition of $\mathcal{J}$. Consider again the partition of $\mathcal{J}$ into $\mathrm{Gal}(G/K)$-orbits such that the elements of A_ℓ and A_m belong to the same orbit if and only if $\sigma(A_\ell) = A_m$ for some $\sigma \in \mathrm{Gal}(G/K)$. If $\{j_1, \dots, j_b\}$ is a full set of representatives of these orbits then we have (2.9) where L_{j_a} is the subfield of G consisting of the coordinates λ_{j_a} in $\boldsymbol{\lambda} \in \mathbf{L}$, $a = 1, \dots, b$. The following cases will play an important role in our criterion:

$$(4.8) \quad \begin{cases} (4.8\text{a}) & b = 1,\ L_{j_1} = K; \\ (4.8\text{b}) & b = 1,\ K \text{ is totally real, } L_{j_1} \text{ is a totally imaginary} \\ & \text{quadratic extension of } K, \text{ and each place in} \\ & S \text{ has a unique extension to } L_{j_1}; \\ (4.8\text{c}) & b \geq 2,\ K = \mathbb{Q},\ S = M_\infty(\mathbb{Q}) \text{ and } L_{j_a} \text{ is either } \mathbb{Q} \\ & \text{or an imaginary quadratic field for } a = 1, \dots, b; \\ (4.8\text{d}) & b \geq 2,\ K \text{ is an imaginary quadratic field,} \\ & S = M_\infty(K) \text{ and } L_{j_a} = K \text{ for } a = 1, \dots, b. \end{cases}$$

One can easily show that each of the cases (4.8a) to (4.8d) can appear.

We shall say that $\mathcal{M}$ is *degenerate* (resp. *non-degenerate*) *with respect to* O_S if there is (resp. there is no) a subalgebra $\mathbf{L}$ of $\mathbf{M}$ with $\mathbf{1}$ which is admissible with respect to $\mathcal{M}$ and is *not* of the type (4.8a), (4.8b), (4.8c) or (4.8d). It is easy to see that $\mathcal{M}$ is non-degenerate if and only if it is non-degenerate with respect to O_S for all finite sets of places S on K containing $M_\infty(K)$. By means of Theorem 4 and Corollary 2 we shall prove the following criterion.

Theorem 5. *The following two statements are equivalent:*

(i) *$\mathcal{M}$ is non-degenerate with respect to O_S;*

(ii) *For all $\beta_1, \dots, \beta_k$ in $O_S \setminus \{0\}$, the system of equations* (4.1) *has only finitely many O_S^*-cosets of solutions.*

Moreover, if (i) *holds then, under the notation and assumptions concerning $F_1, \dots, F_k$, $\beta_1, \dots, \beta_k$ and S, the number of O_S^*-cosets of solutions of* (4.1) *is bounded above by the number occurring in* (4.4).

We remark that the last assertion of Theorem 5 follows immediately from the equivalence of (i) and (ii) and from Corollary 2 above.

§5. Applications to norm form equations

In this section, we apply the results of §§ 3 and 4 to norm form equations. We shall keep the notation of §§ 2, 3 and 4. First consider the general case. Let K and R be as in § 2, i.e. let K be a finitely generated extension field of $\mathbb{Q}$ and R a finitely generated and integrally closed integral domain in K with quotient field K. Let M be a finite extension of K of degree r, G the normal closure of M over K, R_G the integral closure of R in G, $\mathcal{M}$ an R-lattice in the integral closure of R in M with $\dim_K K\mathcal{M} = n \geq 2$, m the minimal number of generators of $\mathcal{M}$, $q = \min\{m-1, n\}$, $\beta \in R \setminus \{0\}$,

$$C = \binom{r}{q}^{\omega(\beta)} \tau_{q+1}(\beta^r)$$

and

$$C^* = \binom{r}{q}^{\omega(\beta)} \tau_q(\beta^r).$$

Consider the norm form equations

$$N_{M/K}(\mu) \in \beta R^* \text{ in } \mu \in \mathcal{M} \tag{5.1}$$

and more generally

$$N_{M/K}(\mu) \in \beta U \text{ in } \mu \in \mathcal{M}, \tag{5.2}$$

where U is a subgroup of R^*. We may assume without loss of generality that there are no $\mu \in M^*$ and a proper subfield M' of M with $\mu\mathcal{M} \subseteq M'$. We say that $\mathcal{M}$ is *full* in M if $K\mathcal{M} = M$.

For an intermediate field L with $K \subseteq L \subseteq M$, denote by R_L the integral closure of R in L, and by U_L the subgroup of R_L^* consisting of all elements λ with $N_{M/K}(\lambda) \in U$. For a solution μ of (5.2) with $\mu L \subseteq K\mathcal{M}$, the set $(\mu U_L) \cap \mathcal{M}$ is called a *wide* $(\mathcal{M}, L)$*-family of solutions* of (5.2). For (5.1), wide families of solutions can be defined in a similar way. From Theorem 2, we get the following

Theorem 6. *The set of solutions of* (5.2) *is the union of at most* $C\Pi_{i=2}^n B(i, R_G^*)$ *wide families of solutions. Further, if* $\mathcal{M}$ *is full then this bound can be replaced by* C.

A similar statement follows from Theorem 1 for equation (5.1). Theorem 6 implies Theorem 8 of Laurent [18] (see also Theorems 6 and 6' of [6]).

Next consider equations (5.1) and (5.2) in the special case when K is an algebraic number field of degree d and $R = O_S$, where S is a finite subset of $M(K)$ with cardinality s which contains $M_\infty(K)$. Then (5.1) takes e.g. the form

$$N_{M/K}(\mu) \in \beta O_S^* \text{ in } \mu \in \mathcal{M}. \tag{5.3}$$

In this case, a more precise result follows from Theorem 4'. Then $m = n$ or $m = n + 1$ according as $\mathcal{M}$ is free or not (see e.g. [16], [20]). For every intermediate field L with $K \subseteq L \subseteq M$, let $\mathcal{M}^L = \{\mu \in \mathcal{M} : \mu L \subseteq K\mathcal{M}\}$. Then $\mathcal{M}^L$ is an O_S-sublattice of $\mathcal{M}$. We say that L is *admissible* if $\mathcal{M}^L \neq \{0\}$ and if there is no subfield L' in M with $L \subsetneqq L'$ and $K\mathcal{M}^{L'} = K\mathcal{M}^L$. In what follows, we assume that L is admissible. Then $\mathcal{D}_{\mathcal{M}}^L := \{\lambda \in L : \lambda\mu \in \mathcal{M}^L \text{for all } \mu \in \mathcal{M}\}$ is a subring of $O_{S,L}$, the integral closure of O_S in L. $\mathcal{D}_{\mathcal{M}}^L$ is called the *ring of coefficients* of $\mathcal{M}^L$. If $\mu \in \mathcal{M}^L$ is a solution of (5.3) then so is every element of $\mu\mathcal{D}_{\mathcal{M}}^{L*}$ which is called an $(\mathcal{M}, L)$*-family of solutions.* $\mu\mathcal{D}_{\mathcal{M}}^{L*}$ is called *maximal* if it is not properly contained in another family of solutions. Put $\mathfrak{I}_L := [O_{S,L}^* : \mathcal{D}_{\mathcal{M}}^{L*}]$. It follows from Lemma 9 that $\mathfrak{I}_L$ is finite. Denote by D the degree of the normal closure of G over $\mathbb{Q}$. We get from Theorem 4' the following

Theorem 7. *Equation* (5.3) *has at most* $\sum_{\mathbf{L}} \mathfrak{I}_{\mathbf{L}}$ *maximal families of solutions, where the sum is taken over at most*

$$n^r \binom{r}{q}^{\omega(\beta)} \tau_{q+1}(\beta^r) \exp\{2^{37nD} \cdot s^6 \cdot \log(4sD)\}$$

admissible subfields L *of* M *for which* (5.3) *has a maximal family of solutions (and among which there can be identical subfields* L*). Further, if* $\mathcal{M}$

is full then (5.3) *has at most*

$$\binom{r}{q}^{\omega(\beta)} \tau_{q+1}(\beta^r)\mathfrak{I}_M$$

maximal families of solutions, and all these are $(\mathcal{M}, M)$*-families of solutions.*

For $K = \mathbb{Q}$, this gives a quantitative version of well-known finiteness results of Schmidt [29] (case $O_S = \mathbb{Z}$) and Schlickewei [25] on norm form equations.

We deduce now Theorem 6 from Theorem 2, and Theorem 7 from Theorem 4. In the general case, denote by $\lambda \to \lambda^{(i)}$, $i = 1, \dots, r$, the K-isomorphisms of M in G. For every $\sigma \in \mathrm{Gal}(G/K)$ there is a permutation of $\{1, \dots, r\}$, denoted also by σ, sucht that $\sigma(\lambda^{(i)}) = \lambda^{(\sigma(i))}$ for all $\lambda \in M$ and for $i = 1, \dots, r$. Denote by $\mathbf{M}$ the set of tuples $\{\boldsymbol{\lambda} \in G^r : \sigma(\lambda_i) = \lambda_{\sigma(i)}$ for $i = 1, \dots, r$, $\sigma \in \mathrm{Gal}(G/K)\}$. Define the K-algebra isomorphism $\Psi : M \to \mathbf{M}$ by $\Psi(\lambda) = \boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(r)})$. By (2.9), this is indeed an isomorphism. Further, by (2.9) again, Ψ establishes a bijective mapping from the fields L with $K \subseteq L \subseteq M$ to the subalgebras $\mathbf{L}(I)$ of $\mathbf{M}$ such that $\Psi(L) = \mathbf{L}(I)$ where I is a symmetric partition of $\{1, \dots, r\}$. Let $\Psi(\mathcal{M}) = \mathcal{M}_1$. It is easy to see that L is admissible if and only if $\mathbf{L}(I)$ is admissible. Further, in the number field case, $\Psi(\mathcal{M}^L) = \mathcal{M}_1^{\mathbf{L}}$, $\Psi(\mathcal{D}_{\mathcal{M}}^L) = \mathcal{D}_{\mathcal{M}_1}^{\mathbf{L}}$, $\Psi(\mathcal{D}_{\mathcal{M}_1}^{L*}) = \mathcal{D}_{\mathcal{M}_1}^{\mathbf{L}*}$ if L is admissible. From these it follows immediately that Ψ maps the wide families of solutions of (5.2) onto the wide families of solutions of the equation $N(\boldsymbol{\mu}) \in \beta U$ in $\boldsymbol{\mu} \in \mathcal{M}_1$ and Theorem 6 follows from Theorem 2. Further, in the number field case (when $R = O_S$), Ψ maps the maximal families of solutions of (5.3) onto the maximal families of solutions of the equation $N(\boldsymbol{\mu}) \in \beta O_S^*$ in $\boldsymbol{\mu} \in \mathcal{M}_1$ and Theorem 7 follows from Theorem 4'.

Consider again the number field case (when K is a number field and $R = O_S$). From Theorem 5, one can deduce in a similar way the following criterion. If μ is a solution of (5.3), then μO_S^* is called an *O_S^*-coset of solution* of (5.3). We say that $\mathcal{M}$ is *degenerate with respect to* O_S if M contains an admissible subfield $L \supsetneqq K$ which is *not* of the following type:

$$(5.4)\quad \begin{cases} K \text{ is totally real, } L \text{ is a totally imaginary quadratic extension} \\ \text{of } K \text{ and each place in } S \text{ has a unique extension to } L. \end{cases}$$

Otherwise $\mathcal{M}$ is called *non-degenerate with respect to* O_S.

Together with (2.9), Theorem 5 implies the next theorem.

Theorem 8. *The following two statements are equivalent:*

(i) *$\mathcal{M}$ is non-degenerate with respect to O_S;*

(ii) *For every $\beta \in O_S\setminus\{0\}$, equation (5.3) has only finitely many O_S^*-cosets of solutions.*

Moreover, if (i) *holds, then the number of O_S^*-cosets of solutions of* (5.3) *it at most*

$$\binom{r}{q}^{\omega(\beta)} \tau_{q+1}(\beta^r)\exp\{2^{37nD}\cdot s^6\cdot\log(5sD)\}. \tag{5.5}$$

Apart from the form of the bound, this can be considered as a generalization of results of Schmidt [28], [32] obtained in the case $K=\mathbb{Q}$, $O_S=\mathbb{Z}$.

In the number field case, $\mathcal{M}$ is called *degenerate* if M contains an admissible subfield L with $L \supsetneqq K$ and *non-degenerate* otherwise. Suppose that $\mathcal{M}$ is contained in the ring of integers of M. It is easy to show that $\mathcal{M}$ is non-degenerate if and only if it is non-degenerate with respect to O_S for all finite subsets S of $M(K)$ containing $M_\infty(K)$. Hence our Theorem 8 implies that, in the number field case, equation (5.3) has finitely many O_S^*-cosets of solutions for all finite subsets S of $M(K)$ containing $M_\infty(K)$ and for all $\beta\in O_S\setminus\{0\}$ if and only if $\mathcal{M}$ is non-degenerate. Further, as a consequence of Corollary 3 we get

Corollary 4. *If $\mathcal{M}$ is non-degenerate, then the number of O_S^*-cosets of solutions of* (5.3) *is bounded above by the number occurring in* (5.5).

Theorem 4 and Corollary 2 in § 4 have similar consequences for norm form equations.

As was mentioned after the enunciation of Corollary 3, qualitative versions of Corollary 4 were earlier established by Schmidt [28] (in case $K=\mathbb{Q}$, $O_S=\mathbb{Z}$), Schlickewei [25] (in case $K=\mathbb{Q}$) and Laurent [18] (in the general case). Further, in the case $K=\mathbb{Q}$, $O_S=\mathbb{Z}$, Schmidt [32] has recently derived the bound (4.6) for the number of solutions of (5.3) in the non-degenerate case. Using (4.7) from § 4, the bound (5.5) in Corollary 4 can be replaced by

$$\binom{r}{q}^{\omega(\beta)} \tau_{q+1}(\beta^r)\exp\{2^{38nd}\cdot s^6\cdot\log(8sD)\}.$$

Further, considering (5.3) in the form (2.2) and taking into consideration only the O_S^*-cosets of primitive solutions, $\tau_{q+1}(\beta^r)$ can be replaced by $\tau_q(\beta^r)$ in view of Corollary 1'. Finally, we note that in the case $n=2$, better bounds have been derived by Evertse [5], Bombieri and Schmidt [2] and Bombieri [1] for the number of O_S^*-cosets of solutions of (5.3).

§6. Applications to generalized systems of unit equations

In this section, we restrict ourselves to the number field case, that is K denotes an algebraic number field and O_S its ring of S-integers. Let

$m \geq 2$ be an integer. For any m-tuple $\lambda = (\lambda_1, \dots, \lambda_m)$ of non-negative integers, put $x^\lambda = x_1^{\lambda_1} \dots x_m^{\lambda_m}$. Consider the following generalization of S-unit equations

$$P_i(x) = \sum_{\lambda \in \mathcal{L}_i} p_i(\lambda) x^\lambda = 0 \quad \text{in } x \in (O_S^*)^m \text{ for } i \in I \tag{6.1}$$

where I is a finite index set, $p_i(\lambda) \in K$, and $\mathcal{L}_i$ is the support of P_i (i.e. the set of exponents λ for which the coefficient $p_i(\lambda)$ of x^λ in P_i is non-zero). Denote by $\mathcal{L}$ the disjoint union $\mathcal{L} = \coprod_{i \in I} \mathcal{L}_i$ of the sets $\mathcal{L}_i$. Consider a partition $\mathcal{P}$ of $\mathcal{L}$; it induces partitions $\mathcal{L}_i = \coprod_{j \in I_i} \mathcal{L}_{i,j}$ on each $\mathcal{L}_i$. The partition $\mathcal{P}$ is said to be *compatible* with a $\gamma \in O_S^m$ if

$$\sum_{\lambda \in \mathcal{L}_{i,j}} p_i(\lambda) \gamma^\lambda = 0 \quad \text{for each } i \in I \text{ and } j \in I_i.$$

Further, $\mathcal{P}$ is said to be *maximal compatible* with γ if $\mathcal{P}'$ is not compatible with γ for any refinement $\mathcal{P}'$ of $\mathcal{P}$. Denote by $H_\mathcal{P}$ that subgroup of $(O_S^*)^m$ whose elements γ have the property that for each $\mathcal{L}_{i,j}$, $\gamma^\lambda = \gamma^{\lambda'}$ if λ, $\lambda' \in \mathcal{L}_{i,j}$. If γ is a solution of (6.1) then so are all elements of $\gamma H_\mathcal{P}$, where $\mathcal{P}$ is a partition of $\mathcal{L}$, compatible with γ. Laurent [18] showed that the set of solutions of (6.1) is the union of finitely many sets of the form $\gamma H_\mathcal{P}$ where γ is a solution and $\mathcal{P}$ is a partition of $\mathcal{L}$ which is maximal compatible with γ. In fact Laurent proved this result in a more general case, for subgroups of finite rank of $\mathbb{C}^*$ instead of O_S^*.

As a consequence of Theorem 4', the following quantitative version can be proven. As in § 4, denote by d the degree of K and by s the cardinality of S. Further, let r denote the cardinality of $\mathcal{L}$, and k the rank of the matrix $(p_i(\lambda))_{i \in I, \lambda \in \mathcal{L}}$ formed from the coefficients $p_i(\lambda)$ in (6.1). Put $n = r - k$.

Theorem 9. *The set of solutions of* (6.1) *is the union of at most*

$$n^r \exp\{2^{37nd!} \cdot s^6 \cdot \log(4sd!)\} \tag{6.2}$$

sets of the form $\gamma H_\mathcal{P}$ where γ is a solution of (6.1) *and $\mathcal{P}$ is a partition of $\mathcal{L}$ which is maximal compatible with γ. Further, if $K/\mathbb{Q}$ is normal then $d!$ can be replaced by d.*

For a deduction of Theorem 9 from Theorem 4', see our recent paper [14].

Of particular interest is the special case when (6.1) consists of a single linear equation, i.e. when it is an S-unit equation

$$\alpha_1 x_1 + \dots + \alpha_n x_n = 1 \quad \text{in } x = (x_1, \dots, x_n) \in (O_S^*)^n \tag{6.3}$$

where $\alpha_1, \dots, \alpha_n \in K^*$. Then Theorem 9 describes, in a quantitative form, the structure of the set of solutions of (6.3). Further, it gives an explicit upper bound for the number of non-degenerate solutions. A solution x of (6.3) is called *non-degenerate* if there is no vanishing subsum in $\alpha_1 x_1 + \cdots + \alpha_n x_n$ and *degenerate* otherwise. Theorem 9 provides the bound (6.2) with $r = n+1$ for the number of the sets of solutions of the form $\gamma H_{\mathcal{P}_0}$ of (6.3) where $\mathcal{P}_0 = \{\mathcal{L}\}$, and $\mathcal{P}_0$ is maximal compatible with γ. But these solutions γ are just the non-degenerate solutions and $H_{\mathcal{P}_0} = (1, \ldots, 1)$. Hence Theorem 9 implies the following recent theorem of Schlickewei [26] which has been established with a slightly better bound.

Corollary 5. *Equation* (6.3) *has at most*

$$\exp\{2^{38nd!} \cdot s^6 \cdot \log(4sd!)\}$$

non-degenerate solutions. Further, if $K/\mathbb{Q}$ is normal then $d!$ can be replaced by d.

We should, however, remark that the proof of Theorem 9 depends on (another version of) this result of Schlickewei (cf. Lemma 10 of the present paper). We note that Theorem 9 can also be proven (with a slightly different bound) by combining the proof of Laurent with Corollary 5. Finally, we mention that using Theorem 4' with the improved bound (4.7), the second factor in (6.2) can be replaced by (4.7) with the choice $D = d!$.

A less explicit version of Theorem 9 can also be deduced from Theorem 3 in the general case when K is an arbitrary finitely generated field over $\mathbb{Q}$ and O_S^* is replaced by an arbitrary finitely generated subgroup of K^*.

§7. Proofs of Theorem 1 and Corollary 1'

To prove Theorem 1, we need several lemmas. In Lemmas 1 to 7 it will be more convenient to consider the system of equations (2.2') in the form (2.2).

We keep the notation of §§ 2 and 3. Since F_i maps $K\mathfrak{M}$ to K for all i with $1 \le i \le k$, $F_i(\mathbf{x})$ can be factorized as

$$F_i(\mathbf{x}) = \alpha_i \prod_{j=1}^{r_i} \ell_{i,j}(\mathbf{x})^{k_{i,j}} \text{ for all } \mathbf{x} \in K\mathfrak{M}, \tag{7.1}$$

where $\alpha_i \in K \setminus \{0\}$, $\ell_{i,1}, \ldots, \ell_{i,r_i} : K\mathfrak{M} \to G$ are pairwise linearly independent linear functions and $k_{i,j}$ are positive integers such that

$$\sigma(\ell_{i,j}) = \ell_{i,\sigma(j)}, \ k_{i,j} = k_{i,\sigma(j)} \text{ for } j = 1, \ldots, r_i, \ \sigma \in \mathrm{Gal}(G/K), \tag{7.2}$$

where $(\sigma(1), \ldots, \sigma(r_i))$ is a permutation of $(1, \ldots, r_i)$ for each $\sigma \in \mathrm{Gal}(G/K)$.

We recall that R_G denotes the integral closure of R in G. By a theorem of Nagata [19], R_G is finitely generated. Further, R and R_G are integrally closed in K and G, respectively. Hence both in R and in R_G there exists a divisor theory. Then for every prime divisor $\mathfrak{p}$ of R there is an additive valuation $v = v_{\mathfrak{p}}$ on K such that, for every $\alpha \in K^*$, $v(\alpha)$ denotes the exponent of $\mathfrak{p}$ in the prime divisor decomposition of the principal divisor (α). A similar assertion holds for the prime divisors $\mathfrak{P}$ of R_G and the corresponding additive valuations $w_{\mathfrak{P}}$ on G. Denote by $\mathbb{M}_K$ and $\mathbb{M}_G$ the sets of additive valuations with value group $\mathbb{Z}$, which correspond to the divisor theory of R and of R_G, respectively. Then we have

$$R = \{\alpha \in K : v(\alpha) \geq 0 \text{ for all } v \in \mathbb{M}_K\}$$

and

$$R_G = \{\alpha \in G : w(\alpha) \geq 0 \text{ for all } w \in \mathbb{M}_G\}.$$

Further, every $w \in \mathbb{M}_G$ is an extension of some $v \in \mathbb{M}_K$.

For every $\mathbf{y} = (y_1, \dots, y_m) \in G^m$ and every $w \in \mathbb{M}_G$, we put

$$w(\mathbf{y}) = \min_{1 \leq j \leq m} w(y_j).$$

Further, for every polynomial $P \in G[X_1, \dots, X_m]$ and every $w \in \mathbb{M}_G$, we denote by $w(P)$ the minimum of the w-values of the coefficients of P. Let $\{\mathbf{a}_1, \dots, \mathbf{a}_m\}$ denote the minimal set of generators of $\mathfrak{M}$ which has been fixed in § 2, and for $\mathbf{x} \in V = K\mathfrak{M}$, put $\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m$ where $x_1, \dots, x_m \in K$. Here $x_1, \dots, x_m$ are not in general uniquely determined. For each i with $1 \leq i \leq k$, consider

$$\ell^*_{i,j}(\mathbf{X}) = X_1\ell_{i,j}(\mathbf{a}_1) + \cdots + X_m\ell_{i,j}(\mathbf{a}_m) \text{ for } j = 1, \dots, r_i \tag{7.3}$$

and

$$F^*_i(\mathbf{X}) = \alpha_i \prod_{j=1}^{r_i} \ell^*_{i,j}(\mathbf{X})^{k_{i,j}} \tag{7.4}$$

in $G[X_1, \dots, X_m]$ and $K[X_1, \dots, X_m]$, respectively. By the assumption made on F_i, F^*_i has m_i variables with non-zero coefficients and has its coefficients in R. We identify $\mathbf{x} \in V$ with the set of tuples $(x_1, \dots, x_m) \in K^m$ with $\mathbf{x} = x_1\mathbf{a}_1 + \cdots + x_m\mathbf{a}_m$. Then

$$\ell^*_{i,j}(\mathbf{x}) = \ell_{i,j}(\mathbf{x}) \text{ and } F^*_i(\mathbf{x}) = F_i(\mathbf{x}) \text{ for all } i, j \text{ and } \mathbf{x} \in V. \tag{7.5}$$

By a generalization of Gauss' lemma (cf. [17]) it follows from (7.4) that

$$w(\alpha_i) + \sum_{j=1}^{r_i} k_{i,j} w(\ell^*_{i,j}) = w(F^*_i) \text{ for all } w \in \mathbb{M}_G. \tag{7.6}$$

Further, for each solution $\mathbf{x} \in \mathfrak{M}$ of (2.2), we get from (2.2), (7.1) and (7.5) that

$$w(\alpha_i) + \sum_{j=1}^{r_i} k_{i,j} w(\ell^*_{i,j}(\mathbf{x})) = w(F^*(\mathbf{x})) = w(\beta_i) \tag{7.7}$$

for all $w \in \mathbb{M}_G$. Together with the fact that F_i^* has its coefficients from R, (7.6) and (7.7) imply that

$$\sum_{j=1}^{r_i} k_{i,j}\{w(\ell_{i,j}^*(\mathbf{x})) - w(\ell_{i,j}^*)\} = w(\beta_i) - w(F_i^*) \leq w(\beta_i). \tag{7.8}$$

On the other hand, it follows from $(x_1, \dots, x_m) \in R^m$ that

$$0 \leq w(\ell_{i,j}^*(\mathbf{x})) - w(\ell_{i,j}^*). \tag{7.9}$$

We partition the set of solutions $\mathbf{x} \in \mathfrak{M}$ of (2.2) into *classes* such that two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ of (2.2) belong to the same class if and only if

$$w(\ell_{i,j}(\mathbf{x}_1)) = w(\ell_{i,j}(\mathbf{x}_2)) \text{ for all } i, j \text{ and } w \in \mathbb{M}_G. \tag{7.10}$$

This is equivalent to the fact that

$$\ell_{i,j}(\mathbf{x}_2)/\ell_{i,j}(\mathbf{x}_1) \in R_G^* \text{ for all } i \text{ and } j. \tag{7.11}$$

For a solution $\mathbf{x}$ of (2.2), we put

$$w_{i,j} := w(\ell_{i,j}(\mathbf{x})) \text{ for all } i \text{ and } j. \tag{7.12}$$

For $w(\beta_i) = 0$, (7.5), (7.8) and (7.9) imply that $w_{i,j} = w(\ell_{i,j}^*)$ for all j, while if $w(\beta_i) > 0$, then $w_{i,1}, \dots, w_{i,r_i}$ can assume only finitely many values. Hence there are only finitely many classes of solutions.

We derive now an upper bound for the number of classes of solutions of (2.2). Given an arbitrary but fixed $w \in \mathbb{M}_G$, we derive first an upper bound for the number of tuples $\mathbf{w}_i = (w_{i,1}, \dots, w_{i,r_i}) \in \mathbb{Z}^{r_i}$ for which there is a solution $\mathbf{x}$ of (2.2) with (7.12). In the proof we shall need the following three lemmas which are appropriate modifications of Lemmas 11 to 13 of Schmidt [32]. In Lemmas 1 to 3 below, we fix an element w of $\mathbb{M}_G$.

Lemma 1. *Let $\ell_1(\mathbf{x}), \dots, \ell_h(\mathbf{x})$ be linearly dependent linear forms in m variables with coefficients in G. Given integers $w_1, \dots, w_h$, let $\mathfrak{X}$ be the set of $\mathbf{x} \in G^m$ for which*

$$w(\ell_i(\mathbf{x})) \geq w_i \text{ for } i = 1, \dots, h.$$

Then $\mathfrak{X}$ may already be defined by $h-1$ of these inequalities.

Proof. This is in fact Lemma 11 of Schmidt [32]; we use here additive valuations instead of multiplicative ones. □

Lemma 2. *Let $\ell_1(\mathbf{x}), \dots, \ell_m(\mathbf{x})$ be linear forms in m variables and with coefficients in G. Suppose that there exists an $\mathbf{x}' \in R^m$ with $\mathbf{x}' \neq \mathbf{0}$ and*

$$w(\mathbf{x}') = w_0 \geq 0 \text{ and } w(\ell_i(\mathbf{x}')) \geq w_i,\ i = 1, \dots, m, \tag{7.13}$$

where $w_1, \dots, w_m$ are given integers. Then there is an i_0 with $1 \le i_0 \le m$ such that the solutions of

$$w(\mathbf{x}) \ge w_0 \text{ and } w(\ell_i(\mathbf{x})) \ge w_i \text{ in } \mathbf{x} \in R^m, \quad i = 1, \dots, i_0 - 1, i_0 + 1, \dots, m \tag{7.14}$$

imply that

$$w(\ell_i(\mathbf{x})) \ge w_i \text{ for } i = 1, \dots, m. \tag{7.15}$$

Proof. We follow the argument of the proof of Lemma 12 of [32]. By Lemma 1, it suffices to deal with the case when $\ell_1, \dots, \ell_m$ are linearly independent. Then there are $\gamma_{i,j} \in G$, $i \le i, j \le m$, such that

$$X_j = \gamma_{j1}\ell_1(\mathbf{x}) + \cdots + \gamma_{jm}\ell_m(\mathbf{x}), \; j = 1, \dots, m. \tag{7.16}$$

One may suppose without loss of generality that

$$w(\gamma_{11}) + w_1 = \min_{1 \le i,j \le m} \{w(\gamma_{ji}) + w_i\}. \tag{7.17}$$

For the given $\mathbf{x}' = (x_1', \dots, x_m')$, we have by (7.16), (7.13) and (7.17)

$$w(x_j') \ge \min_i \{w(\gamma_{ji}) + w_i\} \ge w(\gamma_{11}) + w_1 \text{ for } j = 1, \dots, m.$$

Hence, by (7.13), we have

$$w_0 \ge w(\gamma_{11}) + w_1. \tag{7.18}$$

This implies that $\gamma_{11} \ne 0$. Let $\mathbf{x} \in R^m$ satisfy (7.14). Then for $i_0 = 1$, (7.16), (7.14), (7.17) and (7.18) give

$$w(\gamma_{11}) + w(\ell_1(\mathbf{x})) \ge \min\{w(\gamma_{12}) + w_2, \dots, w(\gamma_{1m}) + w_m, w(x_1)\} \ge w(\gamma_{11}) + w_1$$

whence $w(\ell_1(\mathbf{x})) \ge w_1$ and (7.15) follows. □

Lemma 3. *Let $\ell_1, \dots, \ell_r$ be linear forms in m variables and with coefficients in G, and put $q = \min\{m-1, t\}$, where t is the maximal number of linearly independent forms among $\ell_1, \dots \ell_r$. Let $w_1, \dots, w_r$ be integers, and suppose that there is an $\mathbf{x}' \in R^m$ with $\mathbf{x}' \ne \mathbf{0}$ and*

$$w(\mathbf{x}') = w_0 \ge 0 \text{ and } w(\ell_i(\mathbf{x}')) \ge w_i, \; i = 1, \dots, r.$$

Then there are q among these forms, say $\ell_{i_1}, \dots, \ell_{i_q}$, such that every $\mathbf{x} \in R^m$ with

$$w(\mathbf{x}) \ge w_0 \text{ and } w(\ell_{i_j}(\mathbf{x})) \ge w_{i_j}, \; j = 1, \dots, q$$

satisfies

$$w(\ell_i(\mathbf{x})) \ge w_i \text{ for } i = 1, \dots, r.$$

Proof. The assertion follows by $r-t$ applications of Lemma 1 and, if $t=m$, a further application of Lemma 2. $\square$

In the next lemma, let again α_i, $\ell_{i,1},\dots,\ell_{i,r_i},k_{i,1},\dots,k_{i,r_i}$ be as in (7.1) with property (7.2). For $i=1,\dots,k$, define the fields $M_{i,j}$ ($j=1,\dots,r_i$) by

$$\mathrm{Gal}(G/M_{i,j})=\{\sigma\in\mathrm{Gal}(G/K):\sigma(j)=j\}. \tag{7.19}$$

For a non-zero divisor $\mathfrak{a}$ of R, we define $\tau_{q+1}(\mathfrak{a})$ and $\tau_q(\mathfrak{a})$ in a similar way as for principal divisors. Let $\mathfrak{p}$ be a prime divisor of R, $v=v_{\mathfrak{p}}\in\mathbb{M}_K$ the associated additive valuation and $w\in\mathbb{M}_G$ an extension of v.

Lemma 4. *For $i=1,\dots,k$, the number of tuples $\mathbf{w}_i=(w_{i,1},\dots,w_{i,r_i})\in\mathbb{Z}^{r_i}$ for which there exists a solution $\mathbf{x}$ of (2.2) with*

$$w_{i,j}=w(\ell_{i,j}(\mathbf{x}))\ \textit{for}\ j=1,\dots,r_i \tag{7.20}$$

is at most

$$\binom{r_i}{q_i}\tau_{q_i+1}(\mathfrak{p}^{u_iv(\beta_i)}). \tag{7.21}$$

Further, if we restrict ourselves to primitive solutions, this bound can be replaced by

$$\binom{r_i}{q_i}\tau_{q_i}(\mathfrak{p}^{u_iv(\beta_i)}). \tag{7.22}$$

Proof. Fix a subscript i with $1\le i\le k$. For simplicity, we omit everywhere i. For $v(\beta)=0$, it follows from (7.8), (7.9) and (7.5) that $\mathbf{w}=(w(\ell_1^*),\dots,w(\ell_r^*))\in\mathbb{Z}^r$ is the only tuple for which (2.2) can have a solution $\mathbf{x}$ with (7.20). Hence we assume that $v(\beta)>0$ and that (2.2) is solvable. First we deal with the general (not necessarily primitive) solutions. Let $\mathbf{w}=(w_1,\dots,w_r)\in\mathbb{Z}^r$ be a tuple for which there is a solution $\mathbf{x}$ of (2.2) with the property (7.20). Then, by (7.5) and (7.7), we have

$$w(\alpha)+\sum_{j=1}^{r}k_jw_j=w(\beta). \tag{7.23}$$

It suffices to give an upper bound for the number of tuples $\mathbf{w}=(w_1,\dots,w_r)\in\mathbb{Z}^r$ for which (7.23) holds and for which there is an $\mathbf{x}\in\mathfrak{M}$ with $\mathbf{x}\neq\mathbf{0}$ and with the property (7.20).

Let $\mathbf{w}=(w_1,\dots,w_r)\in\mathbb{Z}^r$ be such a tuple, and denote by w_0 the minimum of values $w(\mathbf{x})$ when $\mathbf{x}$ runs over all tuples $\mathbf{0}\neq\mathbf{x}=(x_1,\dots,x_m)\in R^m$ for which the corresponding $\mathbf{x}\in\mathfrak{M}$ satisfy (7.20), i.e., by (7.5),

$$w_j=w_j(\ell_j(\mathbf{x}))=w(\ell_j^*(\mathbf{x}))\ \text{for}\ j=1,\dots,r. \tag{7.24}$$

By Lemma 3, we get linear forms $\ell^*_{j_1},\dots,\ell^*_{j_q}$ among the forms $\ell^*_1,\dots,\ell^*_r$ with the property specified in Lemma 3. Denote by $w_{j_1},\dots,w_{j_q}$ the corresponding components in the tuple $\mathbf{w}$. Following SCHMIDT [32], we call $(\ell^*_{j_1},\dots,\ell^*_{j_q},w_{j_1},\dots,w_{j_q})$ an anchor of the solutions under consideration which comes from the tuple $\mathbf{w}=(w_1,\dots,w_r)$. We show now that different tuples yield different anchors. Assume, to the contrary, that $\mathbf{w}'=(w'_1,\dots,w'_r)\in\mathbb{Z}^r$ is another tuple for which there is an $\mathbf{x}\in\mathfrak{M}$ with

$$w(\ell_j(\mathbf{x}))=w(\ell^*_j(\mathbf{x}))=w'_j \text{ for } j=1,\dots,r \tag{7.25}$$

and

$$w(\alpha)+\sum_{j=1}^{r}k_jw'_j=w(\beta). \tag{7.26}$$

Further, assume that the anchor coming from $\mathbf{w}'$ coincides with the above-considered anchor coming from $\mathbf{w}$. Then $w'_{j_i}=w_{j_i}$ for $i=1,\dots,q$. Let $\min\limits_{\mathbf{x}\in R^m} w(\mathbf{x})=w'_0$, where the minimum is taken over all tuples $\mathbf{0}\neq\mathbf{x}=(x_1,\dots,x_m)\in R^m$ for which the corresponding $\mathbf{x}\in\mathfrak{M}$ satisfy (7.25). Suppose that $w_0\geq w'_0$. Then for every $\mathbf{0}\neq\mathbf{x}\in R^m$ which satisfies (7.24) we have

$$w(\mathbf{x})\geq w'_0 \text{ and } w(\ell^*_{j_i}(\mathbf{x}))\geq w'_{j_i} \text{ for } j=1,\dots,q.$$

Hence, it follows from Lemma 3 that $w_j\geq w'_j$ for $j=1,\dots,r$. Finally, together with (7.23) and (7.26) this implies that $w'_j=w_j$ for $j=1,\dots,r$ which proves our claim.

It remained to estimate the number of possible anchors. The number of q-tuples $(j_1,\dots,j_q)$ with $1\leq j_1<\dots<j_q\leq r$ is $\binom{r}{q}$. Hence it will be enough to prove that for given $j_1,\dots,j_q$, the number of possibilities for $w_{j_1},\dots,w_{j_q}$ is at most $\tau_{q+1}(\mathfrak{p}^{uv(\beta)})$. For convenience, we estimate the number of anchors $(\ell^*_1,\dots,\ell^*_q,w_1,\dots,w_q)$, i.e. the number of possibilities for $w_1,\dots,w_q$. For this purpose it suffices to estimate the number of possibilities for $v_j:=w_j-w(\ell^*_j)$, $j=1,\dots,q$.

For the tuples $(v_1,\dots,v_q)$ under consideration, we have by (7.8) and (7.9)

$$\sum_{j=1}^{q}v_j\leq w(\beta)=e\cdot v(\beta) \text{ and } v_j\geq 0 \text{ for } j=1,\dots,q, \tag{7.27}$$

where e denotes the ramification index of w with respect to v. Denote by $w^{(j)}$ the extension of v to the extension field M_j of K defined by (7.19), and by $e^{(j)}_1$, $e^{(j)}_2$ the ramification index of $w^{(j)}$ with respect to v, and the ramification index of w with respect to $w^{(j)}$. Then $e_1^{(j)} e_2^{(j)} = e$. Further, in view of $v_j = w(\ell_j(\mathbf{x})) - w(\ell_j^*)$, there is a non-negative integer $v_j^{(j)}$ such that $v_j = e_2^{(j)} v_j^{(j)}$. Hence, by (7.27),

$$\sum_{j=1}^{q} \frac{v_j^{(j)}}{e_1^{(j)}} \leq v(\beta) \text{ and } v_j^{(j)} \geq 0 \text{ for } j = 1, \ldots, q.$$

But $e_1^{(j)} \leq [M_j : K]$ and $[M_j : K] \leq u$ (see e.g. [3]), where u is the maximum of the degrees of the irreducible factors of F over K. Thus it suffices to estimate from above the number of $(v_1^{(1)}, \ldots, v_q^{(q)}) \in \mathbb{Z}^q$ for which

$$\sum_{j=1}^{q} v_j^{(j)} \leq u \cdot v(\beta) \text{ and } v_j^{(j)} \geq 0 \text{ for } j = 1, \ldots, q. \tag{7.28}$$

However, the number of these tuples is equal to $\tau_{q+1}(\mathfrak{p}^{u \cdot v(\beta)})$.

Next we consider only the primitive solutions. Then following the above arguments, it is easy to see that if in (7.28) $v_1^{(1)}, \ldots, v_q^{(q)}$ and $v_1'^{(1)}, \ldots, v_q'^{(q)}$ come from anchors and $v_j'^{(j)} \leq v_j^{(j)}$ for $j = 1, \ldots, q$ then $v_j'^{(j)} = v_j^{(j)}$ for $j = 1, \ldots, q$. This implies that for given $v_1^{(1)} \ldots, v_{q-1}^{(q-1)}$, there is at most one possibility for $v_q^{(q)}$ in (7.28). Thus, in this case the number of tuples satisfying (7.28) is equal to $\tau_q(\mathfrak{p}^{uv(\beta)})$. $\square$

We are now in a position to give upper bound for the number of classes of solutions of (2.2).

Lemma 5. *The number of classes of solutions of* (2.2) *is at most*

$$C_k = \prod_{i=1}^{k} \binom{r_i}{q_i}^{w(\beta_i)} \tau_{q_{i+1}}(\beta_i^{u_i}). \tag{7.29}$$

Further, if we restrict ourselves to primitive solutions, then this bound can be repleced by

$$C_k^* = \prod_{i=1}^{k} \binom{r_i}{q_i}^{w(\beta_i)} \tau_{q_i}(\beta_i^{u_i}). \tag{7.30}$$

Proof. Fix a subscript i with $1 \leq i \leq k$. For simplicity, we omit again i everywhere in the proof below.

By the definition of the classes of solutions, every class $\mathcal{C}$ of solutions $\mathbf{x}$ of (2.2) is uniquely determined by the tuples $(w_{\mathfrak{P}}(\ell_j(\mathbf{x}))_{1 \leq j \leq r}$ where

$\mathbf{x} \in \mathfrak{M}$ with $\mathbf{x} \in \mathcal{C}$ and where $\mathfrak{P}$ runs over all prime divisors of R_G. But, for a prime divisor $\mathfrak{P}$ for which $\mathfrak{P} \nmid (\beta)$ in R_G, the corresponding tuple $(w_{\mathfrak{P}}(\ell_j(\mathbf{x}))_{1 \le j \le r}$ coincides with the tuple $(w_{\mathfrak{P}}(\ell_j^*))_{1 \le j \le r}$ by (7.5), (7.8) and (7.9). Further, G/K being a normal extension, the prime divisors of (β) in R_G are conjugate to each others (see e.g. [3], Ch. III). This implies that for each $\sigma \in \mathrm{Gal}(G/K)$, $(w_{\sigma(\mathfrak{P})}(\ell_j(\mathbf{x}))_{1 \le j \le r}$ is uniquely determined by $(w_{\mathfrak{P}}(\ell_j(\mathbf{x})))_{1 \le j \le r}$. Thus, the number of classes of solutions of (2.2) is equal to the number of tuples of vectors

$$(w_{\mathfrak{P}_1}(\ell_j(\mathbf{x})))_{1 \le j \le r}, \dots, (w_{\mathfrak{P}_s}(\ell_j(\mathbf{x})))_{1 \le j \le r},$$

where $\mathbf{x} \in \mathfrak{M}$ runs over all solutions $\mathbf{x}$ of (2.2) and $\{\mathfrak{P}_1, \dots, \mathfrak{P}_s\}$ is a maximal set of prime divisors of (β) in R_G which are not conjugate to each others. Here $s = w(\beta)$. By appling now Lemma 4, using the fact that $\tau_{q+1}(\ )$ (and similarly $\tau_q(\ )$) is multiplicative, and repeating the above argument for $i = 1, \dots, k$, we get (7.29). In case of primitive solutions, we get (7.30) in a similar way, using (7.22) instead of (7.21) in Lemma 4.
□

Next we consider the solutions of (2.2) contained in an $\mathcal{L}$-admissible, $\mathcal{L}$-non-degenerate subspace (cf. § 3) of $K\mathfrak{M}$. We recall that $B(p, R_G^*)$ was defined in 2.

Lemma 6. *Let $\mathcal{C}$ be a class of solutions of* (2.2), *and let H be an admissible, non-degenerate subspace of $K\mathfrak{M}$ with dimension $p \ge 2$. All solutions $\mathbf{x}$ of* (2.2) *in $\mathcal{C}$ with $\mathbf{x} \in H$ are contained in the union of at most $B(p, R_G^*)$ admissible, proper subspaces of H.*

Proof. We shall use some argument from the proof of Lemma 2 of [6]. Let $\mathbf{x}_0$ be a fixed solution of (2.2) from $\mathcal{C}$. Put

$$\ell_i'(\mathbf{x}) = \ell_i(\mathbf{x})/\ell_i(\mathbf{x}_0) \text{ for } \mathbf{x} \in K\mathfrak{M},\ i = 1, \dots, f.$$

Then

$$\ell_i'(\mathbf{x}) \in R_G^* \text{ for } i = 1, \dots, f, \tag{7.31}$$

and for all solutions $\mathbf{x} \in \mathcal{C}$ with $\mathbf{x} \in H$. Since by assumption H is non-degenerate, there exist linear functions $\ell_{i_1}', \dots, \ell_{i_h}'$ in $\{\ell_1', \dots, \ell_f'\}$ which are pairwise linearly independent on H such that

$$\sum_{j=1}^{h} c_j \ell_{i_j}'(\mathbf{x}) = 0 \text{ identically on } H \tag{7.32}$$

for some $c_1, \dots, c_h \in G^*$. Let h be the smallest integer with this property; then we have $3 \le h \le p+1$. From (7.32) it follows that

$$\sum_{j=1}^{h-1} (-c_j/c_h) \ell_{i_j}'(\mathbf{x})/\ell_{i_h}'(\mathbf{x}) = 1$$

for all solutions $\mathbf{x}$ with $\mathbf{x} \in \mathcal{C} \cap H$. Since R_G^* is finitely generated, it follows from Theorem 3 of [8] that the tuples $(\ell'_{i_1}(\mathbf{x})/\ell'_{i_h}(\mathbf{x}), \ldots, \ell'_{i_{h-1}}(\mathbf{x})/\ell'_{i_h}(\mathbf{x}))$ for which $\mathbf{x} \in \mathcal{C} \cap H$, are contained in at most $B(p, R_G^*)$ $(h-2)$-dimensional subspaces of G^{h-1}. For each of these subspaces of G^{h-1}, there are $\gamma_1, \ldots, \gamma_{h-1} \in G$, not all zero, such that

$$\sum_{j=1}^{h-1} \gamma_j \ell'_{i_j}(\mathbf{x})/\ell'_{i_h}(\mathbf{x}) = 0,$$

that is

$$\sum_{j=1}^{h-1} \gamma_j \ell'_{i_j}(\mathbf{x}) = 0 \tag{7.33}$$

for the corresponding solutions $\mathbf{x}$ with $\mathbf{x} \in \mathcal{C} \cap H$. By the minimality of h, (7.33) cannot hold for all $\mathbf{x} \in H$. Therefore, the $\mathbf{x} \in H$ satisfying (7.33) lie in a $(p-1)$-dimensional subspace H' of H which is admissible. □

Lemma 7. *Suppose that $K\mathfrak{M}$ is non-degenerate, and let $\mathcal{C}$ be a class of solutions of* (2.2)*. Then all solutions of* (2.2) *which belong to $\mathcal{C}$ are contained in the union of at most*

$$\prod_{i=2}^{n} B(i, R_G^*)$$

admissible, degenerate subspaces of $K\mathfrak{M}$.

Proof. Lemma 7 follows immediately by repeated applications of Lemma 6. □

Let H be a non-zero subspace of $K\mathfrak{M}$, and consider the linear mapping $\Psi : K\mathfrak{M} \to G^f$ introduced in (2.7). Then Ψ establishes an isomorphism between H and the subspace $\Psi(H)$ of $\mathbf{M}$.

Lemma 8. *Let H be a non-zero subspace of $K\mathfrak{M}$. Then the following two statements are equivalent:*
(i) *H is admissible and degenerate;*
(ii) *$\Psi(H) = \boldsymbol{\nu} \cdot \mathbf{L}$ for some $\boldsymbol{\nu} \in \Psi(H) \cap \mathbf{M}^*$ and some K-subalgebra $\mathbf{L} = \mathbf{L}(I)$ of $\mathbf{M}$ where I is a symmetric partition of $\mathcal{J}$.*
In particular, $K\mathfrak{M}$ is admissible and degenerate if and only if the R-lattice $\mathcal{M}$ is full in $\mathbf{M}$, and then $\Psi(K\mathfrak{M}) = \mathbf{M}$.

Remark 1. It follows from Lemma 8 that $K\mathfrak{M}$ has an admissible, degenerate subspace of dimension ≥ 2 if and only if $\mathcal{M}$ is degenerate (as defined in § 3).

Remark 2. We note that in (ii) the K-subalgebra $\mathbf{L}$ of $\mathbf{M}$ is uniquely determined by H.

Proof. We receall that $n\,(\geq 2)$ is the dimension of the K-vector space $K\mathfrak{M}$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis of $K\mathfrak{M}$, and put

$$\ell_j^*(\mathbf{X}) = \sum_{v=1}^{n} \ell_j(\mathbf{e}_v) X_v \text{ for } j \in \mathcal{J}. \tag{7.34}$$

We have $\ell_j^* = \ell_{j'}^*$ if $\ell_j = \ell_{j'}$, and, by (2.4),

$$\sigma(\ell_j^*) = \ell_{\sigma(j)}^* \quad \text{for all } j \quad \text{and } \sigma \in \mathrm{Gal}(G/K). \tag{7.35}$$

We indentify $\mathbf{x} \in K\mathfrak{M}$ with the tuple $(x_1, \dots, x_n) \in K^n$ for which $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$. Denote by $\mathcal{L}^*$ the collection of linear forms ℓ_j^* with $j \in \mathcal{J}$, and by H^* the set of tuples $(x_1, \dots, x_n) \in K^n$ for which $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n \in H$. Then H^* is a K-vector subspace of K^n having the same dimension, say p, as H. Further, H^* is $\mathcal{L}^*$-admissible if H so is, and H^* is $\mathcal{L}^*$-non-degenerate or $\mathcal{L}^*$-degenerate according as H is $\mathcal{L}$-non-degenerate or $\mathcal{L}$ degenerate.

Denote by r' the maximal number of pairwise linearly independent linear functions among the elements of $\mathcal{L}$ over H. Then we have $r' \geq p$.

Now we prove the implication (i) $\Rightarrow$ (ii). Suppose that H is $\mathcal{L}$-admissible and $\mathcal{L}$-degenerate. Let $j_1, \dots, j_v$ be a full set of representatives of the $\mathrm{Gal}(G/K)$-orbits $\mathcal{C}_1, \dots, \mathcal{C}_v$ of $\mathcal{J}$ defined in §2. It follows from the degeneracy of H that H is then also degenerate (and admissible) with respect to the collection of linear functions $\{\ell_{\sigma(j_w)}\}_{\sigma \in \mathrm{Gal}(G/K)}$ for $w = 1, \dots, v$. Consequently, H^* is admissible and degenerate with respect to the collection of linear forms $\{\ell_{\sigma(j_w)}^*\}_{\sigma \in \mathrm{Gal}(G/K)}$ for $w = 1, \dots, v$. By Lemma 3 of [6], we have

$$\begin{cases} \ell_{j_w}(H) = \ell_{j_w}^*(H^*) = \nu'_{j_w} L_{j_w} \text{ for some } \nu'_{j_w} \in \ell_{j_w}(H) \setminus \{0\} \\ \quad \text{and some field } L_{j_w} \text{ with } K \subseteq L_{j_w} \subseteq G \text{ for } w = 1, \cdots, v. \end{cases} \tag{7.36}$$

It follows from (7.36) that for each $j \in \mathcal{J}$,

$$\ell_j(H) = \ell_j^*(H^*) = \nu'_j L_j \text{ for some } \nu'_j \in \ell_j(H) \setminus \{0\} \tag{7.37}$$

where the L_j are subfields of G with $K \subseteq L_j$ and with the property

$$\sigma(L_j) = L_{\sigma(j)} \text{ for all } \sigma \in \mathrm{Gal}(G/K). \tag{7.38}$$

Further, using a well-known theorem (see e.g. [33], Ch. I, Theorem 14), ν'_j can be chosen in (7.37) so that $\boldsymbol{\nu}' = (\nu'_j)_{j \in \mathcal{J}} \in \Psi(H) \cap \mathbf{M}^*$.

If $r' \leq 2$ then, by the degeneracy of H, $r' = p$. Suppose now that $r' \geq 3$. Then $p \geq 2$. If $r' > p$ then there are $p + 1$ linear functions in

$\mathcal{L}$ which are linearly dependent but pairwise linearly independent on H. However, this contradicts the fact that H is degenerate, hence $r' = p$.

It follows from (7.37) that if ℓ_j and $\ell_{j'}$ are linearly dependent on H for some j and j' in $\mathcal{J}$ then $L_j = L_{j'}$. Consider now that partition I of $\mathcal{J}$ in which j and j' belong to the same subset if and only if ℓ_j and $\ell_{j'}$ are linearly dependent on H. By (7.36) and (7.38), I is symmetric. Further, it is easy to see that there is a $\boldsymbol{\nu} \in \Psi(H) \cap \mathbf{M}^*$ for which $\boldsymbol{\nu}^{-1}\Psi(H) \subseteq \mathbf{L}(I)$. On the other hand, it is not difficult to show that in view of $r' = p$, $\boldsymbol{\nu}^{-1}\Psi(H)$ contains a basis of $\mathbf{L} = \mathbf{L}(I)$ as K-vector space and hence $\Psi(H) = \boldsymbol{\nu} \cdot \mathbf{L}$. This proves (ii).

Next we prove the implication (ii) $\Rightarrow$ (i). Suppose that $\Psi(H) = \boldsymbol{\nu} \cdot \mathbf{L}$ for some $\boldsymbol{\nu} \in \Psi(H) \cap \mathbf{M}^*$ and some K-subalgebra $\mathbf{L} = \mathbf{L}(I)$ of $\mathbf{M}$ where I is a symmetric partition of $\mathcal{J}$. Then H is $\mathcal{L}$-admissible. We prove that H is $\mathcal{L}$-degenerate. It follows from $\Psi(H) = \boldsymbol{\nu} \cdot \mathbf{L}$ and (2.9) that $r' = p$. But the above arguments show that if we consider a maximal subset of $\mathcal{L}$ consisting of r' pairwise linearly independent linear functions over H then these are also linearly independent on H. Consequently, H is $\mathcal{L}$-degenerate indeed.

Finally, consider the case when $H = K\mathfrak{M}$. It follows from the above arguments that $K\mathfrak{M}$ is degenerate if and only if $n = \dim_K K\mathfrak{M}$ is equal to r, the maximal number of pairwise linearly independent linear functions in the factorization of $F_1 \dots F_k$ over G. This is, however, equivalent to the fact that $\dim_K K\mathcal{M} = \dim_K \mathbf{M}$, i.e. that $\mathcal{M}$ is full in $\mathbf{M}$. This implies that $\Psi(K\mathfrak{M}) = K\mathcal{M} = \mathbf{M}$. $\square$

Proof of Theorem 1. We recall that the linear mapping Ψ defined by (2.7) establishes a one-to-one correspondence between the solutions $\mathbf{x}$ of (2.2) and the solutions $\boldsymbol{\mu}$ of (2.2'). By Lemma 5, the solutions of (2.2) belong to at most C_k classes of solutions. Consider an arbitrary but fixed class of solutinons $\mathcal{C}$. If $K\mathfrak{M}$ is degenerate then all solutions of (2.2) in $\mathcal{C}$ are already contained in a single degenerate subspace H of $K\mathfrak{M}$, namely in $K\mathfrak{M}$ itself. Otherwise, if $K\mathfrak{M}$ is non-degenerate then, by Lemma 7, all solutions in $\mathcal{C}$ are contained in the union of at most

$$\prod_{i=2}^{n} B(i, R_G^*)$$

admissible, degenerate subspaces of $K\mathfrak{M}$. Let H be such a subspace. Then it follows from Lemma 8 that $\Psi(H) = \boldsymbol{\mu} \cdot \mathbf{L}$ for some K-subalgebra $\mathbf{L} = \mathbf{L}(I)$ of $\mathbf{M}$ where I is a symmetric partition of $\mathcal{J}$, and some $\boldsymbol{\mu} = (\mu_j) \in \Psi(H) \cap \mathbf{M}^*$ which can be chosen to be a solution of the system of equations (2.2'). Further, if $\mathcal{M}$ is full in $\mathbf{M}$ then $K\mathfrak{M}$ is degenerate, i.e. $H = K\mathfrak{M}$ is the only subspace under consideration. Consider now all those solutions $\boldsymbol{\mu}' = (\mu_j')$ of (2.2') which belong both to $\Psi(\mathcal{C})$ and to $\boldsymbol{\mu} \cdot \mathbf{L}$. Then $\mu_j'/\mu_j \in R_G^*$ for $j = 1, \dots, f$. On the other hand, we have $\boldsymbol{\mu}' \in \boldsymbol{\mu} \cdot \mathbf{L}$.

Hence $\boldsymbol{\mu}' \in \boldsymbol{\mu} \cdot R^*_{\mathbf{L}}$, i.e. all the $\boldsymbol{\mu}'$ considered above belong to the wide family of solutions $(\boldsymbol{\mu} \cdot R^*_{\mathbf{L}}) \cap \mathcal{M}$. This completes the proof of Theorem 1. □

Proof of Corollary 1'. In view of Lemma 8, the first assertion is an immediate consequence of Corollary 1. The second assertion (concerning primitive solutions) follows from Lemmas 5, 7 and 8, using the argument of the proof of Theorem 1. □

§8. Proofs of Theorems 4, 4', 5 and Corollary 2

Theorem 4 follows immediately from Theorem 3 and Lemmas 9 and 10 below. In what follows, we use the notation of §4 and recall that in this case R is O_S, the ring of S-integers of the number field K. In this special case, the rings R_G, $R_{\mathbf{L}}$ and R_{L_j}, defined in §2, will be denoted by $O_{S,G}$, $O_{S,\mathbf{L}}$ and O_{S,L_j}, respectively. Let $\mathbf{L}$ be an admissible subalgebra of $\mathbf{M}$ containing $\mathbf{1}$. Consider the unit group $\mathcal{D}^{\mathbf{L}*}_{\mathcal{M}}$ of the coefficient ring $\mathcal{D}^{\mathbf{L}}_{\mathcal{M}}$ of $\mathcal{M}^{\mathbf{L}}$.

Lemma 9. *$\mathcal{D}^{\mathbf{L}*}_{\mathcal{M}}$ is a subgroup of finite index in $O^*_{S,\mathbf{L}}$.*

Proof. $O_{S,\mathbf{L}}$ consists of those elements $\boldsymbol{\lambda} = (\lambda_j)$ of $\mathbf{L}$ for which $\lambda_j \in O_{S,\mathbf{L}_j}$ where $O_{S,\mathbf{L}_j}$ is the integral closure of O_S in L_j. The ring $O_{S,\mathbf{L}_j}$ is a finitely generated O_S-module for each $j \in \mathcal{J}$. Consequently, by (2.9), $O_{S,\mathbf{L}}$ is an O_S-lattice. Hence there exists an $\alpha \in O_S \setminus \{0\}$ such that

$$\alpha \mathcal{M}^{\mathbf{L}} O_{S,\mathbf{L}} \subseteq \mathcal{M}^{\mathbf{L}}. \tag{8.1}$$

The ring $O_S/\mathfrak{a}$ is finite for every non-zero integral O_S-ideal $\mathfrak{a}$ in K. Since O_S is a Dedekind ring, the same finiteness assertion is true for $O_{S,L_j}/\mathfrak{a}_{L_j}$ where $\mathfrak{a}_{L_j}$ is an arbitrary non-zero integral O_{S,L_j}-ideal in L_j for $j \in \mathcal{J}$ (cf. [20]). This implies that there exists a positive integer a such that for every $\varepsilon_j \in O^*_{S,L_j}$, $\varepsilon_j^a \equiv 1 \pmod{\alpha}$ in O_{S,L_j} for all $j \in \mathcal{J}$. Let now $\boldsymbol{\varepsilon}$ be an arbitrary element of $O^*_{S,\mathbf{L}}$. Then we obtain that $\boldsymbol{\varepsilon}^a - \mathbf{1} \in \alpha \cdot O_{S,\mathbf{L}}$. Together with (8.1) this implies that $(\boldsymbol{\varepsilon}^a - \mathbf{1})\mathcal{M}^{\mathbf{L}} \subseteq \mathcal{M}^{\mathbf{L}}$, whence $\boldsymbol{\varepsilon}^a \in \mathcal{D}^{\mathbf{L}}_{\mathcal{M}}$. Further, it follows in a similar way that $\boldsymbol{\varepsilon}^{-a} \in \mathcal{D}^{\mathbf{L}}_{\mathcal{M}}$. Hence $\boldsymbol{\varepsilon}^a \in \mathcal{D}^{\mathbf{L}*}_{\mathcal{M}}$ and so $\mathcal{D}^{\mathbf{L}*}_{\mathcal{M}}$ is indeed of finite index in $O^*_{S,\mathbf{L}}$. □

In the special situation considered above, an explicit upper bound can be derived for the number $B(p, O^*_{S,G})$ (introduced in §2) by means of a recent result of Schlickewei [26] on S-unit equations.

Lemma 10. *The number $B(i, O^*_{S,G})$ can be chosen so that*

$$\prod_{i=2}^{n} B(i, O^*_{S,G}) \leq \exp\left\{2^{37nD} \cdot s^6 \cdot \log(4sD)\right\}. \tag{8.2}$$

Proof. Consider equation (2.14) over K with $\Gamma = O^*_{S,G}$. Then the number of extensions to G of the places in S is at most $g \cdot s$, where $g = [H : K]$. Hence, by Lemma 6.1 of Schlickewei [26], $B(i, O^*_{S,G})$ can be chosen so that

$$B(i, O^*_{S,G}) \leq 2(i+1)^{sg}(4sgD)^{2^{35iD}\cdot 4s^6},$$

whence (8.2) follows. $\square$

To prove Theorem 4', we need two further lemmas. Let I_1 and I_2 be symmetric partitions of $\mathcal{J}$. It is easy to show that

$$\mathbf{L}_2(I_2) \subseteq \mathbf{L}_1(I_1) \Longleftrightarrow I_1 \text{ is a refinement of } I_2. \tag{8.3}$$

The following two lemmas are generalizations of Lemmas 2 and 3 of Schmidt [29].

Lemma 11. *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be admissible subalgebras of $\mathbf{M}$ with $\mathbf{L}_1 \subseteq \mathbf{L}_2$. Then $\mathcal{D}^{\mathbf{L}_1}_{\mathcal{M}} \subseteq \mathcal{D}^{\mathbf{L}_2}_{\mathcal{M}}$ and $\mathcal{D}^{\mathbf{L}_1 *}_{\mathcal{M}} \subseteq \mathcal{D}^{\mathbf{L}_2 *}_{\mathcal{M}}$.*

Proof. It follows from $\mathbf{L}_1 \subseteq \mathbf{L}_2$ that $\mathcal{M}^{\mathbf{L}_2} \subseteq \mathcal{M}^{\mathbf{L}_1}$. Hence, for arbitrary $\boldsymbol{\lambda} \in \mathcal{D}^{\mathbf{L}_1}_{\mathcal{M}}$ and $\boldsymbol{\mu} \in \mathcal{M}^{\mathbf{L}_2}$ we have $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} \in \mathcal{M}^{\mathbf{L}_1} \subseteq \mathcal{M}$. Further, $\boldsymbol{\lambda} \in \mathcal{D}^{\mathbf{L}_1}_{\mathcal{M}} \subseteq \mathbf{L}_1 \subseteq \mathbf{L}_2$ implies that $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} \in K\mathcal{M}^{\mathbf{L}_2}$, whence $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} \in K\mathcal{M}^{\mathbf{L}_2} \cap \mathcal{M} = \mathcal{M}^{\mathbf{L}_2}$. Hence $\boldsymbol{\lambda} \in \mathcal{D}^{\mathbf{L}_2}_{\mathcal{M}}$ and so $\mathcal{D}^{\mathbf{L}_1}_{\mathcal{M}} \subseteq \mathcal{D}^{\mathbf{L}_2}_{\mathcal{M}}$ which implies $\mathcal{D}^{\mathbf{L}_1 *}_{\mathcal{M}} \subseteq \mathcal{D}^{\mathbf{L}_2 *}_{\mathcal{M}}$. $\square$

Lemma 12. *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be subalgebras of $\mathbf{M}$ with $\mathbf{1}$ such that $\mathbf{L}_1$ is admissible with respect to $\mathcal{M}$ and $\mathbf{L}_2$ is admissible with respect to $\mathcal{M}^{\mathbf{L}_1}$. Then $\mathbf{L}_2 \supseteq \mathbf{L}_1$, $\mathbf{L}_2$ is admissible with respect to $\mathcal{M}$, and $K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2} = K\mathcal{M}^{\mathbf{L}_2}$ and $(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2} = \mathcal{M}^{\mathbf{L}_2}$.*

Proof. As was mentioned in § 2, there are symmetric partitions $I_1 = \{A_1, \dots, A_h\}$, $I_2 = \{B_1, \dots, B_\ell\}$ of $\mathcal{J}$ such that $\mathbf{L}_1 = \mathbf{L}_1(I_1)$ and $\mathbf{L}_2 = \mathbf{L}_2(I_2)$. Put $I = \{A_i \cap B_j : \text{for } i = 1, \dots, h,\ j = 1, \dots, \ell\}$ and denote by $\mathbf{L} = \mathbf{L}(I)$ the subalgebra associated with I. Then, by (8.3), $\mathbf{L}_1 \subseteq \mathbf{L}$ and $\mathbf{L}_2 \subseteq \mathbf{L}$. Further, for all $\boldsymbol{\lambda} \in \mathbf{L}_1$, $\boldsymbol{\mu} \in \mathbf{L}_2$, we have $\boldsymbol{\lambda} \cdot \boldsymbol{\mu} \in \mathbf{L}$. Denote by I' that symmetric partitions of $\mathcal{J}$ for which $\mathbf{L}' = \mathbf{L}'(I')$ is the subalgebra of $\mathbf{M}$ generated by the products $\boldsymbol{\lambda} \cdot \boldsymbol{\mu}$ with $\lambda \in \mathbf{L}_1$, $\boldsymbol{\mu} \in \mathbf{L}_2$. Then $\mathbf{L}' \subseteq \mathbf{L}$. Further, $\mathbf{L}'$ contains $\mathbf{L}_1$ and $\mathbf{L}_2$, hence I' is a refinement both of I_1 and of I_2. This implies that I' is a refinement of I, i.e. that $\mathbf{L} \subseteq \mathbf{L}'$, whence $\mathbf{L}' = \mathbf{L}$.

For $\boldsymbol{\lambda} \in \mathbf{L}_1$, $\boldsymbol{\mu} \in \mathbf{L}_2$, we have $(\boldsymbol{\lambda} \cdot \boldsymbol{\mu})(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2} = \boldsymbol{\lambda}(\boldsymbol{\mu}(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2}) \subseteq \boldsymbol{\lambda} K\mathcal{M}^{\mathbf{L}_1} \subseteq K\mathcal{M}$ which yields $\mathbf{L}(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2} \subseteq K\mathcal{M}$. This implies that $(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2} \subseteq \mathcal{M}^{\mathbf{L}}$ and so $K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2} \subseteq K\mathcal{M}^{\mathbf{L}}$. On the other hand, $\mathcal{M}^{\mathbf{L}} \cdot \mathbf{L} \subseteq K\mathcal{M}^{\mathbf{L}} \subseteq K\mathcal{M}^{\mathbf{L}_1}$, whence

$$K\mathcal{M}^{\mathbf{L}} \subseteq \{\boldsymbol{\mu} \in K\mathcal{M}^{\mathbf{L}_1} : \boldsymbol{\mu} \cdot \mathbf{L} \subseteq K\mathcal{M}^{\mathbf{L}_1}\} = K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}} \subseteq K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2}$$

and so

$$K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2} = K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}} = K\mathcal{M}^{\mathbf{L}}.$$

But , by assumption, $\mathbf{L}_2$ is admissible with respect to $\mathcal{M}^{\mathbf{L}_1}$, hence we get that $\mathbf{L} = \mathbf{L}_2$ and so $K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2} = K\mathcal{M}^{\mathbf{L}_2}$. Further, $\mathbf{L}_1 \subseteq \mathbf{L}_2$. If $\mathbf{L}_2'$ is a subalgebra of $\mathbf{M}$ with $\mathbf{L}_2' \supseteq \mathbf{L}_2$ and $K\mathcal{M}^{\mathbf{L}_2'} = K\mathcal{M}^{\mathbf{L}_2}$ then one can see as above that $K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2'} = K\mathcal{M}^{\mathbf{L}_2'}$ and so $K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2'} = K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2}$. It follows now that $\mathbf{L}_2' = \mathbf{L}_2$, i.e. that $\mathbf{L}_2$ is admissible with respect to $\mathcal{M}$. Finally,

$$(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2} = K(\mathcal{M}^{\mathbf{L}_1})^{\mathbf{L}_2} \cap \mathcal{M}^{\mathbf{L}_1} = K\mathcal{M}^{\mathbf{L}_2} \cap \mathcal{M} = \mathcal{M}^{\mathbf{L}_2}$$

and the proof of Lemma 12 is completed. □

Proof of Theorem 4'. The number of admissible subalgebras of $\mathbf{M}$ is at most n^r. Indeed, if $\mathbf{L} = \mathbf{L}(I)$ is an admissible subalgebra of $\mathbf{M}$ associated with a symmetric partition I of $\mathcal{J}$, then there is a $\boldsymbol{\nu} \in K\mathcal{M}^{\mathbf{L}} \cap \mathbf{M}^*$ for which $\boldsymbol{\nu} \cdot \mathbf{L} \subseteq K\mathcal{M}$. It follows from (2.9) that $\mathbf{L}$ has dimension $\rho \leq n$ over K, where ρ is the number of sets in I and $n = \dim_K K\mathcal{M}$. But it is easy to see that the number of partitions I of $\mathcal{J}$ with at most n subsets is at most n^r which proves our claim.

Let $\mathbf{L}$ be an admissible subalgebra of $\mathbf{M}$. We shall show that under the notation and assumptions of Theorem 4', (4.1) has only finitely many maximal $(\mathcal{M}, \mathbf{L})$-families of solutions. Further, we shall give an upper bound for the number of these families.

If $\mathbf{L} = \mathbf{K}$ (i.e. if I consists of one subset) then $\mathfrak{I}_{\mathbf{K}} = 1$ and Theorem 4 gives the bound

$$C_k \exp\left\{2^{37nD} \cdot s^6 \cdot \log(4sD)\right\} \tag{8.4}$$

for the number of maximal $(\mathcal{M}, \mathbf{K})$-families of solutions. Hence it suffices to deal with the case when $\mathbf{K} \subsetneqq \mathbf{L}$. All $(\mathcal{M}, \mathbf{L})$-families of solutions are contained in $\mathcal{M}^{\mathbf{L}}$. Thus we consider these solutions of (4.1) in $\mathcal{M}^{\mathbf{L}}$ instead of $\mathcal{M}$, i.e. we deal with the solutions of

$$\alpha_i N_i(\boldsymbol{\mu}) \in \beta_i O_S^* \text{ in } \boldsymbol{\mu} \in \mathcal{M}^{\mathbf{L}},\ i = 1, \ldots, k. \tag{8.5}$$

We have $2 \leq \dim_K \mathcal{M}^{\mathbf{L}} \leq n$. It follows from Theorem 4 that the set of solutions of (8.5) is the union of at most

$$\sum_{\mathbf{L}'} \left[O^*_{S,\mathbf{L}'} : \mathcal{D}^{\mathbf{L}'*}_{\mathcal{M}^{\mathbf{L}}} \right] \tag{8.6}$$

families of solutions $\boldsymbol{\mu}\mathcal{D}^{\mathbf{L}'*}_{\mathcal{M}^{\mathbf{L}}}$, where the sum is extended to at most

$$C_k \exp\{2^{37nD} \cdot s^6 \cdot \log(4sD)\}$$

subalgebras $\mathbf{L}'$ of $\mathbf{M}$ which are admissible with respect to $\mathcal{M}^{\mathbf{L}}$ and for which (8.5) has an $(\mathcal{M}^{\mathbf{L}}, \mathbf{L}')$-family of solutions (and among which there can be identical subalgebras $\mathbf{L}'$). Let

$$\boldsymbol{\mu}_1\mathcal{D}_{\mathcal{M}^{\mathbf{L}}}^{\mathbf{L}'_1*}, \dots, \boldsymbol{\mu}_t\mathcal{D}_{\mathcal{M}^{\mathbf{L}}}^{\mathbf{L}'_t*} \tag{8.7}$$

be these families of solutions.

The family of solutions $\boldsymbol{\mu}_i\mathcal{D}_{\mathcal{M}^{\mathbf{L}}}^{\mathbf{L}'_i*}$ is contained in $(\mathcal{M}^{\mathbf{L}})^{\mathbf{L}'_i}$ for $i = 1, \dots, t$. By Lemma 12, we get that $\mathbf{L} \subseteq \mathbf{L}'_i$, that $\mathbf{L}'_i$ is admissible with respect to $\mathcal{M}$ and that $(\mathcal{M}^{\mathbf{L}})^{\mathbf{L}'_i} = \mathcal{M}^{\mathbf{L}'_i}$. Hence $\boldsymbol{\mu}_i\mathcal{D}_{\mathcal{M}^{\mathbf{L}}}^{\mathbf{L}'_i*} = \boldsymbol{\mu}_i\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'_i*}$ for $i = 1, \dots, t$. This implies that the set of solutions of (8.5) is the union of the families of solutions

$$\boldsymbol{\mu}_i\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'_i*}, \ i = 1, \dots, t. \tag{8.8}$$

Further,

$$\left[O_{S,\mathbf{L}'_i}^* : \mathcal{D}_{\mathcal{M}^{\mathbf{L}}}^{\mathbf{L}'_i*}\right] = \left[O_{S,\mathbf{L}'_i}^* : \mathcal{D}_{\mathcal{M}}^{\mathbf{L}'_i*}\right] \text{ for } i = 1, \dots, t,$$

hence, in (8.6), $\left[O_{S,\mathbf{L}'}^* : \mathcal{D}_{\mathcal{M}^{\mathbf{L}}}^{\mathbf{L}'*}\right]$ can be replaced by $\mathfrak{I}_{\mathbf{L}'}$. Let now $\boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$ be a maximal $(\mathcal{M}, \mathbf{L})$-family of solutions of (4.1). Then $\boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$ is contained in the union of the families of solutions (8.8). We may assume without loss of generality that $\boldsymbol{\mu} \in \boldsymbol{\mu}_1\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'_1*}$, whence $\boldsymbol{\mu}_1\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'_1*} = \boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'_1*}$. Since $\mathbf{L} \subseteq \mathbf{L}'_1$, by Lemma 11 we have $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*} \subseteq \mathcal{D}_{\mathcal{M}}^{\mathbf{L}'_1*}$. Thus we have $\boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*} \subseteq \boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'_1*} = \boldsymbol{\mu}_1\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'_1*}$. But, by assumption, $\boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$ is a maximal family, hence we have $\boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*} = \boldsymbol{\mu}_1\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'_1*}$. Consequently, $\boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$ is one of the families of solutions listed in (8.8). This completes the proof of the first assertion of the theorem.

Now suppose that $\mathcal{M}$ is full in $\mathbf{M}$. Then, by Lemma 8, $K\mathfrak{M}$ is $\mathcal{L}$-degenerate and $\Psi(K\mathfrak{M}) = \mathbf{M}$. If (4.1) is solvable, it follows from Theorem 4 that the set of solutions is contained in at most $C_k\mathfrak{I}_{\mathbf{M}}$ $(\mathcal{M}, \mathbf{M})$-families of solutions, say $\boldsymbol{\mu}_1\mathcal{D}_{\mathcal{M}}^{\mathbf{M}*}, \dots, \boldsymbol{\mu}_t\mathcal{D}_{\mathcal{M}}^{\mathbf{M}*}$. Let now $\boldsymbol{\mu}'\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'*}$ be a maximal family of solutions of (4.1) where $\mathbf{L}'$ is an admissible subalgebra of $\mathbf{M}$. Then we may suppose that $\boldsymbol{\mu}' \in \boldsymbol{\mu}_1\mathcal{D}_{\mathcal{M}}^{\mathbf{M}*}$, whence $\boldsymbol{\mu}'\mathcal{D}_{\mathcal{M}}^{\mathbf{M}*} = \boldsymbol{\mu}_1\mathcal{D}_{\mathcal{M}}^{\mathbf{M}*}$. Further, by Lemma 11, $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'*} \subseteq \mathcal{D}_{\mathcal{M}}^{\mathbf{M}*}$. Consequently, $\boldsymbol{\mu}'\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'*} \subseteq \boldsymbol{\mu}'\mathcal{D}_{\mathcal{M}}^{\mathbf{M}*} = \boldsymbol{\mu}_1\mathcal{D}_{\mathcal{M}}^{\mathbf{M}*}$. Since $\boldsymbol{\mu}'\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'*}$ is maximal, it follows that $\boldsymbol{\mu}'\mathcal{D}_{\mathcal{M}}^{\mathbf{L}'*} = \boldsymbol{\mu}_1\mathcal{D}_{\mathcal{M}}^{\mathbf{M}*}$ which completes the proof of Theorem 4'. $\square$

The following two lemmas will be used in the proofs of Corollary 2 and Theorem 5.

Lemma 13. *Let* $\mathbf{L}$ *be an admissible subalgebra of* $\mathbf{M}$. *Then* $O_{S,\mathbf{L}}^*/O_S^*$ *is finite if and only if one of the cases* (4.8a) *to* (4.8d) *holds.*

Proof. It follows from (2.10) and a generalized version of Dirichlet's unit theorem (see e.g. [15], p. 549) that if one of the cases (4.8a) to (4.8d) holds then $O^*_{S,\mathbf{L}}/O^*_S$ is finite. Conversely, assume now that $O^*_{S,\mathbf{L}}/O^*_S$ is finite. By assumption, $\mathbf{L} = \mathbf{L}(I)$ with some symmetric partition $I = \{A_1, \dots, A_h\}$ of $\mathcal{J}$. Consider the $\mathrm{Gal}(G/K)$-orbits of $\mathcal{J}$ introduced before (4.8), and let again $\{j_1, \dots, j_b\}$ be a full set of representatives of these orbits. Then, by (2.10), $O^*_{S,L_{j_a}}/O^*_S$ must be finite for $a = 1, \dots, b$.

We distinguish two cases:

$\underline{b = 1}$. If $L_{j_1} = K$ then, in view of $O_{S,K} = O_S$, $O^*_{S,K}/O^*_S$ is finite and we get (4.8a). If L_{j_1} is a proper extension of K, then using Dirichlet's unit theorem one can easily show that the case (4.8b) holds.

$\underline{b \geq 2}$. First we prove that O^*_S is finite. Suppose, on the contrary, that O^*_S is infinite. Each orbit considered above consists of sets of I of the same cardinality. Denote by q_{j_a} the common cardinality of these sets in the orbit represented by j_a. There exist rational integers $p_{j_1}, \dots, p_{j_b}$, not all zero, such that $\sum_{a=1}^{b} p_{j_a} q_{j_a} [L_{j_a} : K] = 0$. We note that not all of the p_{j_a} can be equal. Choose an element η of O^*_S which is not a root of unity, and for $t \in \mathbb{Z}$ put $\boldsymbol{\varepsilon}_t = (\eta_j)_{j \in \mathcal{J}}$, where $\eta_j = \eta^{t p_{j_a}}$ if $j \in \mathcal{J}$ is contained in the orbit represented by j_a. It is easy to see that $\boldsymbol{\varepsilon}_t \in O^*_{S,\mathbf{L}}$ and, for distinct t_1, t_2, we have $\boldsymbol{\varepsilon}_{t_1}/\boldsymbol{\varepsilon}_{t_2} \notin O^*_S$. However, this would imply that $O^*_{S,\mathbf{L}}/O^*_S$ is infinite which is impossible. Hence O^*_S is finite indeed. Then, by Dirichlet's unit theorem, K is equal to $\mathbb{Q}$ or an imaginary quadratic number field and $S = M_\infty(K)$. Further, $O^*_{S,L_{j_a}}/O^*_S$ being finite, $O^*_{S,L_{j_a}}$ must also be finite for $a = 1, \dots, b$. But $O_{S,L_{j_a}}$ is the ring of S-integers of L_{j_a}, $a = 1, \dots, b$, hence Dirichlet's unit theorem implies that one of the cases (4.8c), (4.8d) holds. □

Lemma 14. *Let $\mathbf{L}$ be an admissible subalgebra of $\mathbf{M}$ which is of the type* (4.8). *Then we have*

$$[O^*_{S,\mathbf{L}} : O^*_S] \begin{cases} = 1 \text{ if } \mathbf{L} \text{ is of the type } (4.8a), \\ \leq 40d^2 \text{ if } \mathbf{L} \text{ is of the type } (4.8b), \\ \leq 6^n \text{ if } \mathbf{L} \text{ is of the type } (4.8c), \\ = 1 \text{ if } \mathbf{L} \text{ is of the type } (4.8b). \end{cases}$$

Proof. For brevity, we put $\mathcal{J}_\mathbf{L} = [O^*_{S,\mathbf{L}} : O^*_S]$. By assumption, $\mathbf{L}$ is of the type (4.8). Hence, by Lemma 13, $\mathcal{J}_\mathbf{L} < \infty$. We shall use repeatedly (2.10). If $\mathbf{L}$ is of the type (4.8a) or (4.8d) then we have obviously $\mathcal{J}_\mathbf{L} = 1$. Consider now the case when $\mathbf{L}$ is of the type (4.8b). Then it follows e.g. from Theorem 1 of [11] that $\mathcal{J}_\mathbf{L} \leq 2 \times$ the number of roots of unity of L_{j_1}. Denote by ρ the maximum of the numbers of roots of unity in totally imaginary quadratic extensions of K. Then $\varphi(\rho) \le 2d$, where $\varphi(\ )$ denotes Euler's function. But, by a theorem of ROSSER and SCHOENFELD [23]

$$\frac{\rho}{\varphi(\rho)} < e^{\lambda} \log\log\rho + \frac{2,6}{\log\log\rho},$$

where $\lambda = 0,577\ldots$ denotes Euler's constant. Hence

$$\rho \le 20d \log\log(3d), \tag{8.9}$$

whence

$$\mathcal{J}_{\mathbf{L}} \le 2\rho \le 40d^2.$$

Finally, consider the case when $\mathbf{L}$ is of the type (4.8c). Since there are at most 6 roots of unity in an imaginary quadratic number field, we have $\mathcal{J}_{\mathbf{L}} \le 6^b$. Further, it is easy to see that $b \le n$. Thus $\mathcal{J}_{\mathbf{L}} \le 6^n$, and this completes the proof of Lemma 11. $\square$

PROOF OF COROLLARY 2. By Theorem 4, the solutions of (4.1) belong to at most

$$\sum_{\mathbf{L}} \mathfrak{I}_{\mathbf{L}} \tag{8.10}$$

families of solutions, where the sum is taken over at most

$$C_k \exp\left\{2^{37nD} \cdot s^6 \cdot \log(4sD)\right\} \tag{8.11}$$

admissible subalgebras $\mathbf{L}$ of $\mathbf{M}$ for which (4.1) has an $(\mathcal{M}, \mathbf{L})$-family of solutions (and among which there can be identical subalgebras $\mathbf{L}$). Every $(\mathcal{M}, \mathbf{L})$-family of solutions $\boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$ splits into $\left[\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*} : O_S^*\right]$ O_S^*-cosets of solutions. But, by assumption, (4.1) has only finitely many O_S^*-cosets of solutions, hence $\left[\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*} : O_S^*\right]$ must be finite for all admissible subalgebras $\mathbf{L}$ involved in (8.10). Thus

$$\left[O_{S,\mathbf{L}}^* : O_S^*\right] = \mathfrak{I}_{\mathbf{L}} \cdot \left[\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*} : O_S^*\right]$$

is also finite and hence, by Lemma 13, each of the $\mathbf{L}$ involved is of the type (4.8). Further, in view of (8.10) the solutions of (4.1) belong to at most

$$\sum_{\mathbf{L}} \left[O_{S,\mathbf{L}}^* : O_S^*\right] \tag{8.12}$$

O_S^*-cosets of solutions, where the sum is taken over the same admissible subalgebras $\mathbf{L}$ as in (8.10). By Lemma 14, we have

$$\left[O_{S,\mathbf{L}}^* : O_S^*\right] \le \max\{40d^2, 6^n\}$$

for each $\mathbf{L}$ involved in (8.10) and (8.12). Together with (8.11) this implies the assertion of Corollary 2. $\square$

Proof of Theorem 5. It suffices to prove the equivalence of the statements (i) and (ii). Then the bound (4.4) in Theorem 5 follows immediately from Corollary 2.

By Theorem 4, the solutions of (4.1) are contained in the union of finitely many sets of the form $\boldsymbol{\mu}\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$, where $\mathbf{L}$ is an admissible subalgebra of $\mathbf{M}$ and $\boldsymbol{\mu} \in \mathcal{M}^{\mathbf{L}}$ is a solution of (4.1). This implies that (4.1) has finitely many O_S^*-cosets of solutions for all $\beta_1, \dots, \beta_k \in O_S \setminus \{0\}$ if and only if $\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}/O_S^*$ is finite for all admissible subalgebras $\mathbf{L}$ of $\mathbf{M}$. But, by Lemma 9, $O_{S,\mathbf{L}}^*/\mathcal{D}_{\mathcal{M}}^{\mathbf{L}*}$ is finite. Hence the statement (ii) in Theorem 5 is equivalent to the fact that $O_{S,\mathbf{L}}^*/O_S^*$ is finite for all admissible subalgebras $\mathbf{L}$ of $\mathbf{M}$. Finally, in view of Lemma 13 we get that the statements (i) and (ii) of Theorem 5 are indeed equivalent. $\square$

References

[1] E. Bombieri, On the Thue-Mahler equation, *Lect. Notes in Math.* **1290** (1987), 213–243.

[2] E. Bombieri and W. M. Schmidt, On Thue's equation, *Invent. Math.* **88** (1987), 69–81.

[3] Z. I. Borevich and I. R. Shafarevich, Number Theory, *Acad. Press*, *New York*, 1967.

[4] J. H. Evertse, On sums of S-units and linear recurrences, *Comp. Math.* **53** (1984), 225–244.

[5] J. H. Evertse, On equations in S-units and the Thue-Mahler equation, *Invent. Math.* **75** (1984), 561–584.

[6] J. H. Evertse and K. Győry, Finiteness criteria for decomposable form equations, *Acta Arith.* **50** (1988), 357–379.

[7] J. H. Evertse and K. Győry, Decomposable form equations, New Advances in Transcendence Theory (A. Baker ed.), *Cambridge Univ. Press*, 1988, pp. 175–202.

[8] J. H. Evertse and K. Győry, On the numbers of solutions of weighted unit equations, *Comp. Math.* **66** (1988), 329–354.

[9] J. H. Evertse and K. Győry, Effective finiteness theorems for decomposable forms of given discriminant, *Acta Arith.* **53** (1992), 233–277.

[10] J. H. Evertse, I. Gaál and K. Győry, On the numbers of solutions of decomposable polynomial equations, *Archiv der Math.* **52** (1989), 337–353.

[11] K. Győry, Sur une classe des corps de nombres algébriques et ses applications, *Publ. Math.Debrecen* **22** (1975), 151–175.

[12] K. Győry, Résultats effectifs sur la représentation des entiers par des formes décomposables, *Queen's Papers in Pure and Applied Math.,* No **56** (1980), *Kingston, Canada.*

[13] K. Győry, On norm form, discriminant form and index form equations, Coll. Math. Soc. J. Bolyai 34. Topics in Classical Number Theory, Budapest, *North-Holland*, 1984, pp. 617–676.

[14] K. Győry, Some recent applications of S-unit equations, (*to appear*).

[15] H. Hasse, Number Theory, *Springer Verlag*, 1980.

[16] I. Kaplansky, Modules over Dedekind rings and valuation rings, *Trans. Amer. Math. Soc.* **72** (1952), 327–340.

[17] S. Lang, Fundamentals of diophantine geometry, *Springer Verlag*, 1983.

[18] M. Laurent, Equations diophantiennes exponentielles, *Invent. Math* **78** (1984), 299–327.

[19] M. Nagata, A general theory of algebraic geometry over Dedekind domains I, *Amer. J. Math.* **78** (1956), 78–116.

[20] W. Narkiewicz, Elementary and Analytic Theory of Algebraic Numbers, *Springer Verlag* and *Polish Scientific Publishers,* 1990.

[21] A. J. van der Poorten and H. P. Schlickewei, The growth conditions for recurrence sequences, *Report 82.0041* (1982), *Macquarie University, N.S.W. Australia.*

[22] A. J. van der Poorten and H. P. Schlickewei, Additive relations in fields, *J. Austral. Math. Soc. (Series A)* **51** (1991), 154–170.

[23] J. B. Rosser and L. Schoenfeld, Approximate formulas for some functions of prime numbers, *Illinois J. Math.* **6** (1962), 64–94.

[24] H. P. Schlickewei, The $\mathfrak{p}$-adic Thue-Siegel-Roth-Schmidt theorem, *Archiv der Math.* **29** (1977), 267–270.

[25] H. P. Schlickewei, On norm form equations, *J. Number Theory* **9** (1977), 370–380.

[26] H. P. Schlickewei, S-unit equations over number fields, *Invent. Math.* **102** (1990), 95–107.

[27] H. P. Schlickewei, The quantitative Subspace Theorem for number fields, *Comp. Math.* **82** (1992), 245–273.

[28] W. M. Schmidt, Linearformen mit algebraischen Koeffizienten II, *Math. Ann.* **191** (1971), 1–20.

[29] W. M. Schmidt, Norm form equations, *Annals of Math.* **96** (1972), 526–551.

[30] W. M. Schmidt, Simultaneous approximation to algebraic numbers by elements of a number field, *Monatsh. Math.* **79** (1975), 55-66.

[31] W. M. Schmidt, The subspace theorem in diophantine approximations, *Comp. Math.* **69** (1989), 121–173.

[32] W. M. Schmidt, The number of solutions of norm form equations, *Trans. Amer. Math. Soc.* **317** (1990), 197–227.

[33] O. Zariski and P. Samuel, Commutative Algebra I, *van Nostrand Comp.*, 1958.

K. GYŐRY
MATHEMATICAL INSTITUTE
KOSSUTH LAJOS UNIVERSITY
H–4010 DEBRECEN, P.O. BOX 12
HUNGARY

(Received November 20, 1991)