Publ. Math. Debrecen
58 / 1-2 (2001), 1–7

On proximal properties of proper symmetrizations of relators

By JÓZSEF MALA (Budapest)

Abstract. Relators (i.e. arbitrary sets of relations) are investigated from a proximal point of view. Various types of symmetries and symmetrizations are defined for a relator and their connections to other proximal properties such as proximal filteredness and proximal finiteness are studied. We show that if a relator is proximally equivalent to its proper symmetrization, then it is proximally filtered and proximally symmetric but the converse is not true in general, unless we restrict ourselves to proximally finite relators.

0. Introduction

A nonvoid family $\mathcal{R}$ of binary relations on a set X is a relator on X and the ordered pair $(X, \mathcal{R})$ is a relator space. If the set X is not important to mention we simply say that $\mathcal{R}$ is a relator.

Definition 0.1. Let $\mathcal{R}$ be a relator on X. The relators

$$\mathcal{R}^{\star} = \{S \subset X^2 \mid \exists R \in \mathcal{R} : R \subset S\},$$

$$\mathcal{R}^{\#} = \{S \subset X^2 \mid \forall A \subset X : \exists R \in \mathcal{R} : R(A) \subset S(A)\},$$

and

$$\mathcal{R}^{\wedge} = \{S \subset X^2 \mid \forall x \in X : \exists R \in \mathcal{R} : R(x) \subset S(x)\}$$

are called the uniform, proximal and topological refinements of $\mathcal{R}$, respectively.

Mathematics Subject Classification: 00A05, 54E05, 54E15.
Key words and phrases: proximity, relator.
This work was supported by Magyar Ösztöndíj Bizottság.

The following chain of inclusions is obviously true for each relator $\mathcal{R}$:

$$\mathcal{R} \subset \mathcal{R}^{\star} \subset \mathcal{R}^{\#} \subset \mathcal{R}^{\wedge}.$$

Definition 0.2. Let $\mathcal{R}$ be a relator on X. Then the relator $\mathcal{R}$ is called

i) uniformly filtered if for each R, S in $\mathcal{R}$ there exists a T in $\mathcal{R}$ such that $T \subset R \cap S$;

ii) proximally filtered if for each A, a subset of X, and for each R, S in $\mathcal{R}$ there exists a T in $\mathcal{R}$ such that $T(A) \subset R(A) \cap S(A)$;

iii) topologically filtered if for each x in X and for each R, S in $\mathcal{R}$ there exists a T in $\mathcal{R}$ such that $T(x) \subset R(x) \cap S(x)$.

It is obvious that uniform filteredness implies proximal filteredness and proximal filteredness implies topological filteredness.

Definition 0.3. Let $\mathcal{R}$ be a relator on X. Then the relator $\mathcal{R}$ is called

i) strongly symmetric if $R \in \mathcal{R}$ implies $R = R^{-1}$;

ii) properly symmetric if $R \in \mathcal{R}$ implies $R^{-1} \in \mathcal{R}$;

iii) uniformly symmetric if for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S \subset R^{-1}$;

iv) proximally symmetric if for each $A \subset X$ and for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(A) \subset R^{-1}(A)$;

v) topologically symmetric if for each $x \in X$ and for each $R \in \mathcal{R}$ there exists $S \in \mathcal{R}$ such that $S(x) \subset R^{-1}(x)$;

vi) weakly symmetric if $\bigcap \mathcal{R}$, that is, the intersection of all the members of $\mathcal{R}$, is a symmetric relation.

It is easy to see that each statement in Definition 0.3 from i) to v) implies its successor.

Definition 0.4. Let $\mathcal{R}$ and $\mathcal{S}$ be relators. Then we say that $\mathcal{R}$ is uniformly (resp. proximally, topologically) finer than $\mathcal{S}$ if $\mathcal{S} \subset \mathcal{R}^{\star}$ (resp. $\mathcal{S} \subset \mathcal{R}^{\#}$, $\mathcal{S} \subset \mathcal{R}^{\wedge}$) holds. We also say then that $\mathcal{S}$ is uniformly (resp. proximally, topologically) coarser than $\mathcal{R}$.

If $\mathcal{R}$ is uniformly (resp. proximally, topologically) finer than $\mathcal{S}$ and vice versa, then we say that $\mathcal{R}$ and $\mathcal{S}$ are uniformly (resp. proximally, topologically) equivalent.

It is easy to see that two relators are uniformly (resp. proximally, topologically) equivalent if and only if their uniform (resp. proximal, topological) refinements are identical.

Definition 0.5. The relator $\mathcal{R}^{-1} = \{R^{-1} : R \in \mathcal{R}\}$ is called the inverse of $\mathcal{R}$.

If a property P holds for $\mathcal{R}^{-1}$, then we say that $\mathcal{R}$ is inversely P.

From proximal viewpoint, the following theorem (due to Árpád Száz) is basic in the systhematic study of relators (see [3]):

Theorem 0.6. *For each relator $\mathcal{R}$ it is true that $(\mathcal{R}^{\#})^{-1} = (\mathcal{R}^{-1})^{\#}$.*

It is interesting to note that the corresponding statement for the uniform refinement is almost obvious, while the proof of Theorem 0.6 is far from trivial. As for the topological refinement, it is not true (see [1]).

As an easy consequence of Theorem 0.6 we can state the still basic

Corollary 0.7. *A relator is proximally symmetric if and only if it is inversely proximally symmetric.*

Proof. An easy consequence of the definition is that a relator $\mathcal{R}$ is proximally symmetric if and only if $\mathcal{R}^{-1} \subset \mathcal{R}^{\#}$. Now suppose $\mathcal{R}$ is proximally symmetric. Then obviously $(\mathcal{R}^{-1})^{-1} \subset (\mathcal{R}^{\#})^{-1}$ holds and applying Száz's theorem we have $(\mathcal{R}^{-1})^{-1} \subset (\mathcal{R}^{-1})^{\#}$, whence $\mathcal{R}$ is proximally symmetric.

Applying the same argument to $\mathcal{R}^{-1}$ we obtain the "if part" of the theorem. □

Corollary 0.7 is basic in proving:

Theorem 0.8. *A proximally symmetric relator is proximally filtered if and only if it is inversely proximally filtered.*

Proof. Suppose that the proximally symmetric relator $\mathcal{R}$ on X is proximally filtered. Let $R, S \in \mathcal{R}$ and $A \subset X$. Then by the proximal symmetry of $\mathcal{R}$ we can find U, V in $\mathcal{R}$ such that $U(A) \subset R^{-1}(A)$ and $V(A) \subset S^{-1}(A)$. The proximal filteredness of $\mathcal{R}$ implies the existence of a W in $\mathcal{R}$ such that $W(A) \subset U(A) \cap V(A)$. Now using Corollary 0.7 we obtain that a T can be found in $\mathcal{R}$ with $T^{-1}(A) \subset W(A)$.

Summing up, we have $T^{-1}(A) \subset R^{-1}(A) \cap S^{-1}(A)$, hence $\mathcal{R}^{-1}$ is proximally filtered.

Applying the same argument to $\mathcal{R}^{-1}$ we obtain the "if part" of the theorem. □

Definition 0.9. If a relator is proximally equivalent to a finite (resp. singleton) relator then we say that it is proximally finite (resp. simple).

It is clear that $\mathcal{R}$ is uniformly (resp. proximally, topologically) simple if and only if $\mathcal{R}$ is uniformly (resp. proximally, topologically) equivalent to $\{\bigcap \mathcal{R}\}$.

Definition 0.10. The relator $\mathcal{R} \wedge \mathcal{R}^{-1} = \{R \cap S^{-1} \mid R, S \in \mathcal{R}\}$ (resp. $\mathcal{R} \triangle \mathcal{R}^{-1} = \{R \cap R^{-1} \mid R \in \mathcal{R}\}$) is called the proper (resp. strong) symmetrization of the relator $\mathcal{R}$.

It is easy to see that $\mathcal{R} \wedge \mathcal{R}^{-1}$ (resp. $\mathcal{R} \triangle \mathcal{R}^{-1}$) is the uniformly coarsest properly (resp. strongly) symmetric relator among the properly (resp. strongly) symmetric relators that are uniformly finer than $\mathcal{R}$. Note also that $\mathcal{R} \triangle \mathcal{R}^{-1} \subset \mathcal{R} \wedge \mathcal{R}^{-1}$.

Remark 0.11. If $\mathcal{R}$ is a weakly symmetric relator then $\{\bigcap \mathcal{R}\}$ is uniformly, proximally and topologically finer than $\mathcal{R} \wedge \mathcal{R}^{-1}$. Indeed, in this case $\bigcap \mathcal{R} \subset R \cap S^{-1}$ is true for all $R, S \in \mathcal{R}$.

In the sequel we will use the following theorem (see [2]):

Theorem 0.12. *A proximally finite, proximally filtered and inversely topologically filtered relator is proximally simple.*

1. The proper symmetrization of a relator

In the following we investigate the proximal equivalence between a relator and its proper symmetrization. We begin with the following

Theorem 1.1. *If $\mathcal{R} \wedge \mathcal{R}^{-1}$ and $\mathcal{R}$ are proximally equivalent, then $\mathcal{R}$ is proximally filtered and proximally symmetric.*

Proof. Assume now that the condition of the theorem holds for the relator $\mathcal{R}$ on X. If $R \in \mathcal{R}$ and $A \subset X$, since $R \cap R^{-1} \in \mathcal{R} \wedge \mathcal{R}^{-1}$, there exists $S \in \mathcal{R}$ such that $S(A) \subset (R \cap R^{-1})(A) \subset R^{-1}(A)$, therefore $\mathcal{R}$ is proximally symmetric.

If $R, S \in \mathcal{R}$ and $A \subset X$, then using Corollary 0.7 we obtain that there exists $T \in \mathcal{R}$ such that $T^{-1}(A) \subset S(A)$. Now, by the hyphotesis of the theorem, there exists $U \in \mathcal{R}$ such that $U(A) \subset (R \cap T^{-1})(A)$. Since $(R \cap T^{-1})(A) \subset R(A) \cap T^{-1}(A)$ is always true, using the chain of obtained inclusions we have $U(A) \subset R(A) \cap S(A)$, hence $\mathcal{R}$ is proximally filtered. □

The following example shows that in Theorem 1.1 the converse implication is not true in general. However, it is interesting to point out that if we restrict ourselves to proximally finite relators, then the converse implication becomes true as we will see later.

Example 1.2. Let $\mathcal{R} = \{R \subset \mathbb{N}^2 \mid \mathbb{N} \setminus R(A)$ is finite for each infinite subset A of $\mathbb{N}\}$. Then

i) $\mathcal{R}$ is properly symmetric;

ii) $\mathcal{R}$ is proximally filtered;

iii) $\mathcal{R} \wedge \mathcal{R}^{-1}$ and $\mathcal{R}$ are not proximally equivalent.

Proof. To prove i), let $R \in \mathcal{R}$, $A \subset \mathbb{N}$ and let $B = \mathbb{N} \setminus R^{-1}(A)$. Then we have $R(B) \cap A = \emptyset$ and if A is infinite then B must be finite by the definition of $\mathcal{R}$. This shows that $R^{-1} \in \mathcal{R}$, therefore i) holds.

To prove ii), let $R, S \in \mathcal{R}$ and $A \subset X$. If $T = A \times (R(A) \cap S(A)) \cup (\mathbb{N} \setminus A) \times \mathbb{N}$, then $T(A) = R(A) \cap S(A)$, so it is enough to show that $T \in \mathcal{R}$. Let $B \subset X$. Then $T(B) = R(A) \cap S(A)$ if $\emptyset \neq B \subset A$ and $T(B) = \mathbb{N}$ if $B \not\subset A$. Assume now that B is infinite. If $B \subset A$, then A is infinite and hence $\mathbb{N} \setminus T(B) = (\mathbb{N} \setminus R(A)) \cup (\mathbb{N} \setminus S(A))$ is finite. Otherwise $\mathbb{N} \setminus T(B) = \emptyset$. It follows that $T \in \mathcal{R}$.

Finally prove iii). First we state that $\mathcal{R}^{\#} \subset \mathcal{R}$. To show this, let $S \in \mathcal{R}^{\#}$. Then for each $A \subset \mathbb{N}$ there exists $R \in \mathcal{R}$ such that $R(A) \subset S(A)$. Using the definition of $\mathcal{R}$ we obtain that $\mathbb{N} \setminus S(A)$ is finite for each infinite subset A of $\mathbb{N}$, that is, $S \in \mathcal{R}$, hence $\mathcal{R}^{\#} \subset \mathcal{R}$, as was to be shown.

Now it is obvious that $\emptyset \notin \mathcal{R}$, therefore by the previous argument $\emptyset \notin \mathcal{R}^{\#}$ is also true, so to prove iii) it is enough to show that $\emptyset \in (\mathcal{R} \wedge \mathcal{R}^{-1})^{\#}$.

If $<$ is the usual strict ordering on $\mathbb{N}$, then it is a member of $\mathcal{R}$, hence the intersection of it and its inverse, that is, the empty set is in $\mathcal{R} \wedge \mathcal{R}^{-1}$. Therefore $\emptyset \in (\mathcal{R} \wedge \mathcal{R}^{-1})^{\#}$. □

The proof of Example 1.2 also shows that the converse implication in Theorem 1.1 is not true even if we require the much stronger condition that $\mathcal{R}$ be properly symmetric and the weaker corollary that $\mathcal{R}\triangle\mathcal{R}^{-1}$ be proximally equivalent to $\mathcal{R}$.

At the end of the paper we will show that among the proximally finite relators the converse implication in Theorem 1.1 becomes true. In proving it we will need

Proposition 1.3. *Let $\mathcal{R}$ be a proximally simple and weakly symmetric relator. Then $\mathcal{R} \wedge \mathcal{R}^{-1}$ and $\mathcal{R}$ are proximally equivalent.*

Proof. Let $\mathcal{R}$ satisfy the conditions of the proposition. Then by Remark 0.11 we have $\mathcal{R} \wedge \mathcal{R}^{-1} \subset \{\bigcap \mathcal{R}\}^{\#}$. By the proximal simpleness of $\mathcal{R}$ we have $\{\bigcap \mathcal{R}\}^{\#} \subset \mathcal{R}^{\#}$ and combining the obtained inclusions we have that $\mathcal{R}$ is proximally finer than $\mathcal{R} \wedge \mathcal{R}^{-1}$.

On the other hand, it is easy to see that $\mathcal{R} \wedge \mathcal{R}^{-1}$ is always proximally finer than $\mathcal{R}$. □

Theorem 1.4. *If a proximally finite relator $\mathcal{R}$ is proximally symmetric and proximally filtered, then $\mathcal{R} \wedge \mathcal{R}^{-1}$ and $\mathcal{R}$ are proximally equivalent.*

Proof. Let the relator $\mathcal{R}$ satisfy the conditions of the theorem. Then by Theorem 0.8 $\mathcal{R}$ is inversely proximally filtered, hence it is inversely topologically filtered. Applying Theorem 0.12 we obtain that $\mathcal{R}$ is proximally simple, and since a proximal symmetric relator is weakly symmetric, we can use Proposition 1.1, whence the theorem holds. □

Example 1.2 naturally gives raise to the following (at the present unsolved)

Question 1.5. Does there exist a relator $\mathcal{R}$ which is strongly symmetric, proximally filtered, whereas $\mathcal{R} \wedge \mathcal{R}^{-1}$ and $\mathcal{R}$ are not proximally equivalent?

Acknowledgements. The author is deeply indebted to Árpád Száz for his encouragement and continuous help during the work on this topic. The author also wish to thank a referee for the careful reading of the manuscript and for the many suggestions.

References

[1] J. Mala and Á. Száz, Equations for families of relations can also be solved, *C. R. Math. Rep. Acad. Sci. Canada* **XII**, No. 4, August (1990), 109–112.

[2] J. Mala, Finitely generated quasi-proximities, *Period. Math. Hung.* **35**, No. 3 (1997), 193–197.

[3] Á. Száz, Inverse and symmetric relators, *Acta Math. Hung.* **60** (1–2) (1992), 157–176.

JÓZSEF MALA
BUDAPESTI KÖZGAZDASÁGTUDOMÁNYI ÉS ÁLLAMIGAZGATÁSI EGYETEM
MATEMATIKA TANSZÉK
H–1093 BUDAPEST, FŐVÁM TÉR 8.

E-mail: jozsef.mala@math.bke.hu

(Received August 3, 1996; revised March 20, 2000 and July 24, 2000)