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Infinite dimensional singular discrete control systems and feedback law

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Abstract. The aim of this paper is to find a feedback law which steers the initial state to the origin in uniform finite time for the infinite dimensional singular discrete control systems where the state and control spaces are Hilbert spaces.

1. Introduction

We begin with two infinite dimensional Hilbert spaces H and U as the state space and the control space, respectively. Let Y be a closed subspace of H, then H can be written in the decomposition form:

$$H = Y \oplus Y^{\perp}, \tag{1}$$

where $Y^{\perp}$ is the orthogonal complement of Y. Let $\mathcal{L}(H)$ be the set of all bounded linear operators from H into H and $\mathcal{L}(U,H)$ the set of all bounded linear operators from U into H. Given $A \in \mathcal{L}(H)$, A can be represented in the matrix operator form:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{2}$$

where $A_{11} : Y \to Y$, $A_{12} : Y^{\perp} \to Y$, $A_{21} : Y \to Y^{\perp}$, $A_{22} : Y^{\perp} \to Y^{\perp}$, see [6].

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Consider the infinite dimensional autonomous singular discrete control systems.

$$Ex_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, 2, \dots, \tag{3}$$

where $x_k \in H$, $u_k \in U$, the operators $E, A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(U, H)$. The operator E is assumed to be singular and given in the form

$$E = \begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix}, \tag{4}$$

where $E_{11} : Y \to Y$ and it is invertible, furthermore it is assumed that:

$$\begin{pmatrix} E_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}^{-1} \tag{5}$$

exists if and only if A_{22}^{-1} exists.

Consider the feedback law

$$u_k = Mx_k, \tag{6}$$

where $M \in \mathcal{L}(H, U)$. Our aim in this work is to find a feedback law that drives the initial state x_0 to the origin in finite time.

This work has been undertaken in order to establish a link between operator theory and control theory and it can be considered as a generalization of the work in [7] to the infinite dimensional case. Also, the system (3) is very important for many fields such as economics and robotics. Furthermore the concepts of reachability, controllability and observability have been studied extensively for the system (3) in the two cases of finite and infinite dimensional spaces, for example see [1–5] and [8].

2. The main work

If we write the vector x_k as

$$x_k = \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix}, \tag{7}$$

then the system (3) can be written in the form

$$\begin{pmatrix} E_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi_{k+1} \\ \eta_{k+1} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u_k, \tag{8}$$

where $B_1 : U \to Y$, $B_2 : U \to Y^\perp$. This gives

$$\begin{cases} E_{11}\xi_{k+1} = A_{11}\xi_k + A_{12}\eta_k + B_1 u_k, \\ 0 = A_{21}\xi_k + A_{22}\eta_k + B_2 u_k. \end{cases} \tag{9}$$

Multiplying the first equation in (9) on the left by E_{11}^{-1}, yields

$$\begin{cases} \xi_{k+1} = E_{11}^{-1}A_{11}\xi_k + E_{11}^{-1}A_{12}\eta_k + E_{11}^{-1}B_1 u_k, \\ 0 = A_{21}\xi_k + A_{22}\eta_k + B_2 u_k. \end{cases} \tag{10}$$

The second equation in (10) gives

$$\eta_k = -A_{22}^{-1}A_{21}\xi_k - A_{22}^{-1}B_2 u_k. \tag{11}$$

Using the equation (11) in the first equation of (10), we get

$$\begin{aligned} \xi_{k+1} &= E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21})\xi_k \\ &\quad + E_{11}^{-1}(B_1 - A_{12}A_{22}^{-1}B_2)u_k = C\xi_k + Du_k, \end{aligned} \tag{12}$$

where

$$\begin{cases} C = E_{11}^{-1}(A_{11} - A_{12}A_{22}^{-1}A_{21}), \\ D = E_{11}^{-1}(B_1 - A_{12}A_{22}^{-1}B_2). \end{cases} \tag{13}$$

Using (11), the vector x_k given in (7) can be written as follows

$$\begin{aligned} x_k &= \begin{pmatrix} \xi_k \\ \eta_k \end{pmatrix} = \begin{pmatrix} \xi_k \\ -A_{22}^{-1}A_{21}\xi_k - A_{22}^{-1}B_2 u_k \end{pmatrix} \\ &= \begin{pmatrix} E_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}E_{11}^{-1} & A_{22}^{-1} \end{pmatrix} \begin{pmatrix} E_{11}\xi_k \\ -B_2 u_k \end{pmatrix} \\ &= \begin{pmatrix} E_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} E_{11}\xi_k \\ -B_2 u_k \end{pmatrix} = FE_{11}\xi_k - GB_2 u_k, \end{aligned} \tag{14}$$

where

$$(F\ G) = \begin{pmatrix} E_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} {E_{11}}^{-1} & 0 \\ -A_{22}^{-1}A_{21}E_{11} & A_{22}^{-1} \end{pmatrix}. \tag{15}$$

Now we try to find the feedback law

$$u_k = L\xi_k \tag{16}$$

which steers the initial state ξ_k to the origin in uniform time, where $L : Y \to U$.

A necessary and sufficient condition for the existence of such a feedback law is that the system (12) is exactly reachable, i.e. there exists an integer n such that:

$$\text{range}[C, CD, \dots, C^{n-1}D] = Y, \tag{17}$$

see [5]. We assume that (12) is exactly reachable, i.e. there exists a feedback operator $L : Y \to U$.

If L is known, then (12) and (13) become

$$\xi_{k+1} = C\xi_k + Du_k = (C + DL)\xi_k = S\xi_k, \tag{18}$$

and

$$x_k = FE_{11}\xi_k - GB_2u_k = (FE_{11} - GB_2L)\xi_k = T\xi_k, \tag{19}$$

where

$$\begin{cases} S = C + DL, \\ T = FE_{11} - GB_2L. \end{cases} \tag{20}$$

It should be noted that for some choice of the feedback operator L, the operator S is nilpotent, i.e. there exists an integer ν such that $S^\nu = 0$, $S^{\nu-1} \neq 0$.

The solution of (18) is

$$\xi_k = S^k\xi_0 \tag{21}$$

where ξ_0 is any initial vector, and if the feedback operator L has been chosen such that the operator S is nilpotent, then it follows that

$$\xi_\nu = 0. \tag{22}$$

Returning to the original feedback law

$$u_k = Mx_k, \tag{23}$$

we get that M takes the form

$$M = [L\ 0]. \tag{24}$$

Thus we get the feedback law $u_k = Mx_k$ which steers the vector x_k from an initial vector x_0 to the origin in uniform time steps.

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