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An existence theorem for the commutative neutrix product of distributions

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Abstract. In this paper we prove that the commutative neutrix product of the distributions x_+^{-r} and x_+^{-s} exists for $r, s = 1, 2, \dots$.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The distribution x_+^{-r} is defined by the equation

$$x_+^{-r} = \frac{(-1)^{r-1}(\ln x_+)^{(r)}}{(r-1)!}$$

for $r = 1, 2, \dots$ and not as in Gel'fand and Shilov [6]. If we denote Gel'fand and Shilov's definition of x_+^{-r} by $F(x_+, -r)$, it was proved in [4] that

$$x_+^{-r} = F(x_+, -r) + \frac{(-1)^r \phi(r-1)}{(r-1)!}\delta^{(r-1)}(x)$$

for $r = 1, 2, \dots$, where

$$\phi(r) = \begin{cases} \sum_{i=1}^{r} 1/i, & r \geq 1, \\ 0, & r = 0. \end{cases}$$

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Our definition of x_+^{-r} is more convenient to use because it satisfies the equation

$$(x_+^{-r})' = -rx_+^{-r-1}$$

for $r = 1, 2, \ldots$.

Further, the distribution $x_+^{-1} \ln x_+$ is defined by

$$x_+^{-1} \ln x_+ = \frac{1}{2}(\ln^2 x_+)'$$

and in general, the distribution $x_+^{-r} \ln x_+$ is defined inductively by the equation

$$x_+^{-r} \ln x_+ = \frac{x_+^{-r} - (x_+^{-r+1} \ln x_+)'}{r-1}$$

for $r = 2, 3, \ldots$.

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition for the commutative neutrix product of two distributions was given in [3].

Definition 1. Let f and g be distributions in $\mathcal{D}'$ and let $f_n(x) = (f * \delta_n)(x)$, $g_n(x) = (g * \delta_n)(x)$. We say that the neutrix product $f \,\Box\, g$ of f and g exists and is equal to the distribution h on the interval (a, b) if

$$\operatorname*{N-lim}_{n \to \infty} \langle f_n(x) g_n(x), \varphi(x) \rangle = \langle h(x), \varphi(x) \rangle$$

for all functions φ in $\mathcal{D}$ with support contained in the interval (a, b), where N is the neutrix, see van der CORPUT [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \qquad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity. Further, if

$$\lim_{n\to\infty} \langle f_n(x) g_n(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

we simply say that the product $f.g$ exists and equals h, see [2].

Before proving our main result, we note the following lemmas which are easily proved by induction.

Lemma 1. *If φ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$, then*

$$(1) \qquad \langle x_+^{-r}, \varphi(x)\rangle = \int_0^1 x^{-r}\Big[\varphi(x) - \sum_{i=0}^{r-1} \frac{x^i}{i!}\varphi^{(i)}(0)\Big]dx - \sum_{i=0}^{r-2} \frac{\varphi^{(i)}(0)}{i!(r-i-1)} - \frac{\phi(r-1)}{(r-1)!}\varphi^{(r-1)}(0),$$

for $r = 1, 2, \dots$.

Lemma 2.

$$(2) \qquad \int_{-1}^1 v^i \rho^{(r)}(v)\, dv = \begin{cases} 0, & 0 \le i < r, \\ (-1)^r r!, & i = r \end{cases}$$

for $r = 0, 1, 2, \dots$.

The following theorem was proved in [5].

Theorem 1. *The neutrix product $x^{-r} \,\Box\, x^{-s}$ exists and*

$$x^{-r} \,\Box\, x^{-s} = x^{-r-s}$$

for $r, s = 1, 2, \dots$.

The limits involved in the proof of Theorem 1 were easily evaluated. However, in the following, we are going to consider the neutrix product

$x_+^{-r} \,\Box\, x_+^{-s}$. For this neutrix product, the limits are more complicated and so we only prove the existence of the limits and thus the existence of the neutrix product $x_+^{-r} \,\Box\, x_+^{-s}$.

We now prove the following theorem.

Theorem 2. *The neutrix product* $x_+^{-r} \,\Box\, x_+^{-s}$ *exists for* $r, s = 1, 2, \ldots$.

Proof. We first of all consider the case $s = 1$ and put

$$(x_+^{-r})_n = x_+^{-r} * \delta_n(x) = \frac{(-1)^{r-1}}{(r-1)!} \int_{-1/n}^{1/n} \ln(x-t)_+ \delta_n^{(r)}(t)\, dt,$$

for $r = 1, 2, \ldots$. Then

$$(3) \qquad (-1)^{r-1}(r-1)! \int_{-1}^{1} (x_+^{-r})_n (x_+^{-1})_n x^k\, dx$$

$$= \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_t^{1/n} \delta_n'(s) \int_s^{1/n} x^k \ln(x-t)\ln(x-s)\, dx\, ds\, dt$$

$$+ \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_{-1/n}^{t} \delta_n'(s) \int_t^{1/n} x^k \ln(x-t)\ln(x-s)\, dx\, ds\, dt$$

$$+ \int_{-1/n}^{1/n} \delta_n^{(r)}(t) \int_{-1/n}^{1/n} \delta_n'(s) \int_{1/n}^{1} x^k \ln(x-t)\ln(x-s)\, dx\, ds\, dt$$

$$= n^{r-k} \int_{-1}^{1} \rho^{(r)}(v) \int_v^1 \rho'(u)$$

$$\times \int_u^1 w^k \ln[(w-v)/n]\ln[(w-u)/n]\, dw\, du\, dv$$

$$+ n^{r-k} \int_{-1}^{1} \rho^{(r)}(v) \int_{-1}^{v} \rho'(u)$$

$$\times \int_v^1 w^k \ln[(w-v)/n]\ln[(w-u)/n]\, dw\, du\, dv$$

$$+ n^{r-k} \int_{-1}^{1} \rho^{(r)}(v) \int_{-1}^{1} \rho'(u)$$

$$\times \int_1^n w^k \ln[(w-v)/n]\ln[(w-u)/n]\, dw\, du\, dv = I_1 + I_2 + I_3,$$

where the substitutions $ns = u$, $nt = v$ and $nx = w$ have been made.

It follows immediately that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = \operatorname*{N-lim}_{n\to\infty} I_2 = 0, \tag{4}$$

for $k = 0, 1, 2, \ldots, r-1$.

Now

$$\begin{aligned}
&\int_1^n w^k \ln[(w-v)/n] \ln[(w-u)/n]\, dw \\
&\quad = \int_1^n w^k [\ln(w-v) - \ln n][\ln(w-u) - \ln n]\, dw \\
&\quad = \ln^2 n \int_1^n w^k\, dw - 2\ln n \int_1^n w^k \ln(w-v)\, dw \\
&\qquad + \int_1^n w^k \ln(w-v)\ln(w-u)\, dw
\end{aligned} \tag{5}$$

and it follows immediately that

$$\operatorname*{N-lim}_{n\to\infty} n^{r-k} \ln^2 n \int_1^n w^k\, dw = 0 \tag{6}$$

for $k = 0, 1, 2, \ldots$.

Further, by expanding $\ln(w-v)$ in powers of v/w, it also follows that

$$\operatorname*{N-lim}_{n\to\infty} n^{r-k} \ln n \int_1^n w^k \ln(w-v)\, dw = 0 \tag{7}$$

for $k = 0, 1, 2, \ldots$.

Finally, we have

$$\begin{aligned}
&\int_1^n w^k \ln(w-v)\ln(w-u)\, dw \\
&\quad = \int_1^n w^k \Big[\ln w - \sum_{i=1}^{\infty} \frac{v^i}{iw^i}\Big]\Big[\ln w - \sum_{j=1}^{\infty} \frac{u^j}{jw^j}\Big] dw \\
&\quad = \int_1^n w^k \ln^2 w\, dw - 2\sum_{i=1}^{\infty} \frac{v^i}{i} \int_1^n w^{k-i} \ln w\, dw \\
&\qquad + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{v^i u^j}{ij} \int_1^n w^{k-i-j}\, dw
\end{aligned} \tag{8}$$

and it follows that

$$\operatorname*{N-lim}_{n\to\infty} n^{r-k}\int_1^n w^k\ln(w-v)\ln(w-u)\,dw = -\sum_{j=1}^{r}\frac{v^{r-j+1}u^j}{j(r-k)(r-j+1)}.$$

Hence

$$\operatorname*{N-lim}_{n\to\infty} n^{r-k}\int_{-1}^1 \rho^{(r)}(v)\int_{-1}^1 \rho'(u)\int_1^n w^k\ln(w-v)\ln(w-u)\,dw\,du\,dv = \frac{(-1)^r(r-1)!}{r-k}, \tag{9}$$

for $k = 0,1,2,\dots,r-1$ on using equation (2). It follows from equations (5), (6), (7) and (9) that

$$\operatorname*{N-lim}_{n\to\infty} I_3 = \frac{(-1)^r(r-1)!}{r-k}. \tag{10}$$

It now follows from equations (3), (4) and (10) that

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^1 (x_+^{-r})_n(x_+^{-1})_n x^k\,dx = -\frac{1}{r-k}, \tag{11}$$

for $k = 0,1,2,\dots,r-1$.

We now deal with the case $k = r$. Equation (3) still holds but this time it follows that

$$\operatorname*{N-lim}_{n\to\infty} I_1 = \int_{-1}^1 \rho^{(r)}(v)\int_v^1 \rho'(u)\int_u^1 w^r\ln|(w-v)| \times\ln|(w-u)|\,dw\,du\,dv, \tag{12}$$

$$\operatorname*{N-lim}_{n\to\infty} I_2 = \int_{-1}^1 \rho^{(r)}(v)\int_{-1}^v \rho'(u)\int_v^1 w^r\ln|(w-v)| \times\ln|(w-u)|\,dw\,du\,dv. \tag{13}$$

Further, equation (8) is replaced by the equation

$$\int_1^n w^r \ln(w-v)\ln(w-u)\,dw = \int_1^n w^r \ln^2 w\,dw$$
$$-2\sum_{i=1}^{\infty}\frac{v^i}{i}\int_1^n w^{r-i}\ln w\,dw + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{v^iu^j}{ij}\int_1^n w^{r-i-j}\,dw.$$

It follows that

$$\operatorname*{N-lim}_{n\to\infty}\int_1^n w^r \ln(w-v)\ln(w-u)\,dw = g_r(u,v),$$

say and so

$$\operatorname*{N-lim}_{n\to\infty} I_3 = \int_{-1}^1 \rho^{(r)}(v)\int_{-1}^1 \rho'(u) g_r(u,v)\,du\,dv. \tag{14}$$

We therefore see from equations (3), (12), (13) and (14) that

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^1 (x_+^{-r})_n(x_+^{-1})_n x^r\,dx$$

exists and we put

$$\operatorname*{N-lim}_{n\to\infty}\int_{-1}^1 (x_+^{-r})_n(x_+^{-1})_n x^r\,dx = L_{r,1}. \tag{15}$$

When $k = r+1$, it follows as for equation (3) that for any continuous function ψ

$$\int_{-1/n}^{1/n} (x_+^{-r})_n(x_+^{-1})_n x^{r+1}\psi(x)\,dx$$
$$= n^{-1}\int_{-1}^1 \rho^{(r)}(v)\int_v^1 \rho'(u)\int_u^1 w^{r+1}\psi(w/n)\ln|(w-v)/n|$$
$$\times \ln|(w-u)/n|\,dw\,du\,dv$$
$$+\, n^{-1}\int_{-1}^1 \rho^{(r)}(v)\int_{-1}^v \rho'(u)\int_u^1 w^{r+1}\psi(w/n)\ln|(w-v)/n|$$
$$\times \ln|(w-u)/n|\,dw\,du\,dv$$

and it follows that

$$\lim_{n\to\infty}\int_{-1/n}^{1/n}(x_+^{-r})_n(x_+^{-1})_n x^{r+1}\psi(x)\,dx = 0. \tag{16}$$

Next, when $x \geq 1/n$, we have

$$\begin{aligned}(-1)^{r-1}(r-1)!(x_+^{-r})_n &= \int_{-1/n}^{1/n}\ln|x-t|\delta_n^{(r)}(t)\,dt\\ &= n^r\int_{-1}^{1}\ln|x-v/n|\rho^{(r)}(v)\,dv\\ &= n^r\int_{-1}^{1}\Big[\ln x-\sum_{i=1}^{\infty}\frac{v^i}{in^ix^i}\Big]\rho^{(r)}(v)\,dv\\ &= -\sum_{i=r}^{\infty}\int_{-1}^{1}\frac{v^i}{in^{i-r}x^i}\,\rho^{(r)}(v)\,dv.\end{aligned}$$

It follows that

$$|(r-1)!(x_+^{-r})_n| \leq \sum_{i=r}^{\infty}\int_{-1}^{1}\frac{|v|^i}{in^{i-r}x^i}|\rho^{(r)}(v)|dv \leq \sum_{i=r}^{\infty}\frac{K_r}{in^{i-r}x^i},$$

where

$$K_r = \int_{-1}^{1}|\rho^{(r)}(v)|\,dv$$

for $r = 1, 2\ldots.$

If now $n^{-1} < \eta < 1$, then

$$\begin{aligned}&(r-1)!\int_{1/n}^{\eta}|(x_+^{-r})_n(x_+^{-1})_n x^{r+1}|\,dx\\ &\qquad\leq K_1K_r\sum_{i=r}^{\infty}\sum_{j=1}^{\infty}\int_{1/n}^{\eta}\frac{n^{r-i-j+1}x^{r-i-j-1}}{ij}\,dx\\ &\qquad= \frac{K_1K_r}{n}\sum_{i=r}^{\infty}\sum_{j=1}^{\infty}\int_{1}^{n\eta}\frac{w^{r-i-j+1}}{ij}\,dw\end{aligned}$$

$$= \frac{K_1 K_r}{n}\left[\frac{\ln w}{r+1} + \frac{\ln w}{2r} + \sum_{i=r}^{\infty} \sum_{\substack{j=1 \\ i+j\neq r+2}}^{\infty} \frac{w^{r-i-j+2}}{ij(r-i-j+2)}\right]_1^{n\eta}$$

and it follows that

$$\lim_{n\to\infty} \int_{1/n}^{\eta} |(x_+^{-r})_n (x_+^{-1})_n x^{m+r}|\, dx \le \frac{K_1 K_r \eta}{r!}$$

for $r = 1, 2, \ldots$.

Thus, if ψ is a continuous function

$$\lim_{n\to\infty} \left| \int_{1/n}^{\eta} (x_+^{-r})_n (x_+^{-1})_n x^{r+1} \psi(x)\, dx \right| = O(\eta) \tag{17}$$

for $r = 1, 2, \ldots$.

Now let φ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$. By Taylor's Theorem we have

$$\varphi(x) = \sum_{k=0}^{r} \frac{x^k \varphi^{(k)}(0)}{k!} + \frac{x^{r+1}\varphi^{(r+1)}(\xi x)}{(r+1)!},$$

where $0 < \xi < 1$. Thus

$$\begin{aligned}
\langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x)\rangle &= \int_{-1}^{1} (x_+^{-r})_n (x_+^{-1})_n \varphi(x)\, dx \\
&= \sum_{k=0}^{r} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} (x_+^{-r})_n (x_+^{-1})_n x^k\, dx \\
&\quad + \int_{-1/n}^{1/n} \frac{(x_+^{-r})_n (x_+^{-1})_n x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!}\, dx \\
&\quad + \int_{1/n}^{\eta} \frac{(x_+^{-r})_n (x_+^{-1})_n x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!}\, dx \\
&\quad + \int_{\eta}^{1} \frac{(x_+^{-r})_n (x_+^{-1})_n x^{r+1} \varphi^{(r+1)}(\xi x)}{(r+1)!}\, dx.
\end{aligned}$$

On using the equations (11), (15), (16) and (17) and noting that the sequence $\{(x_+^{-r})_n (x_+^{-1})_n\}$ converges uniformly to x^{-r-1} on the interval

$[\eta, 1]$, it follows that

$$\underset{n\to\infty}{\text{N–lim}}\, \langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x)\rangle$$
$$= -\sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1}\frac{\varphi^{(r)}(0)}{r!} + \int_\eta^1 \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!}\,dx + O(\eta),$$

but since η can be made arbitrarily small, it follows that

$$\begin{aligned}
&\underset{n\to\infty}{\text{N–lim}}\, \langle (x_+^{-r})_n (x_+^{-1})_n, \varphi(x)\rangle \\
&\quad = -\sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1}\frac{\varphi^{(r)}(0)}{r!} + \int_0^1 \frac{\varphi^{(r+1)}(\xi x)}{(r+1)!}\,dx \\
&\quad = \int_0^1 x^{-r-1}\Big[\varphi(x) - \sum_{k=0}^{r} \frac{x^k\varphi^{(k)}(0)}{k!}\Big]dx - \sum_{k=0}^{r-1} \frac{\varphi^{(k)}(0)}{(r-k)k!} + L_{r,1}\frac{\varphi^{(r)}(0)}{r!} \\
&\quad = \langle x_+^{-r-1}, \varphi(x)\rangle + \frac{(-1)^r}{r!}\,[L_{r,1} + \phi(r)]\,\langle \delta^{(r)}(x), \varphi(x)\rangle
\end{aligned}$$

on using equation (1).

The neutrix product $x_+^{-r} \,\Box\, x_+^{-1}$ therefore exists and

$$x_+^{-r} \,\Box\, x_+^{-1} = x_+^{-r-1} + \frac{(-1)^r}{r!}[L_{r,1} + \phi(r)]\delta^{(r)}(x)$$

on the interval $[-1, 1]$. However, the product $x_+^{-r}.x_+^{-1}$ obviously exists on any interval not containing the origin, and so the neutrix product $x_+^{-r} \,\Box\, x_+^{-1}$ exists on the real line for $r = 1, 2, \ldots$.

Suppose now that $x_+^{-r} \,\Box\, x_+^{-s}$ exists and is of the form

$$x_+^{-r} \,\Box\, x_+^{-s} = x_+^{-r-s} + a_{r,s}\delta^{(r+s-1)}(x)$$

for $r = 1, 2, \ldots$ and for some positive integer s. Then the derivative of $x_+^{-r} \,\Box\, x_+^{-s}$ exists, and

$$\begin{aligned}
(x_+^{-r} \,\Box\, x_+^{-s})' &= -(r+s)x_+^{-r-s-1} + a_{r,s}\delta^{(r+s)}(x) \\
&= -s x_+^{-r} \,\Box\, x_+^{-s-1} - r x_+^{-r-1} \,\Box\, x_+^{-s} \\
&= -s x_+^{-r} \,\Box\, x_+^{-s-1} - r x_+^{-r-s-1} - r a_{r+1,s}\delta^{(r+s)}(x).
\end{aligned}$$

The product $x_+^{-r} \,\Box\, x_+^{-s-1}$ therefore exists and

$$x_+^{-r} \,\Box\, x_+^{-s-1} = x_+^{-r-s-1} - \frac{r a_{r+1,s} + a_{r,s}}{s} \delta^{(r+s)}(x)$$
$$= x_+^{-r-s-1} + a_{r,s+1} \delta^{(r+s)}(x).$$

It follows by induction that the product $x_+^{-r} \,\Box\, x_+^{-s}$ exists for $r, s = 1, 2, \ldots$.

Defining the distribution x_-^{-r} by

$$x_-^{-r} = (-x)_+^{-r}$$

for $r = 1, 2, \ldots$, we have □

Corollary 2.1. *The neutrix product $x_-^{-r} \,\Box\, x_-^{-s}$ exists for $r, s = 1, 2, \ldots$.*

Proof. With the above notation we have

$$x_+^{-r} \,\Box\, x_+^{-s} = x_+^{-r-s} + a_{r,s} \delta^{(r+s-1)}(x). \tag{18}$$

Replacing x in this equation by $-x$, we get

$$x_-^{-r} \,\Box\, x_-^{-s} = x_-^{-r-s} - (-1)^{r+s} a_{r,s} \delta^{(r+s-1)}(x), \tag{19}$$

proving the existence of neutrix product $x_-^{-r} \,\Box\, x_-^{-s}$. □

Corollary 2.2.

$$x_+^{-r} \,\Box\, x_+^{-s} + (-1)^{r+s} x_-^{-r} \,\Box\, x_-^{-s} = x^{-r-s} \tag{20}$$

for $r, s = 1, 2, \ldots$.

Proof. Equation (20) follows immediately from equations (18) and (19). □

Theorem 3. *The neutrix product $x_+^{-r} \,\Box\, \ln x_+$ exists for $r = 1, 2, \ldots$. In particular, the product $x_+^{-1} . \ln x_+$ exists and*

$$x_+^{-1} . \ln x_+ = x_+^{-1} \ln x_+. \tag{21}$$

Proof. We put

$$(\ln x_+)_n = \ln x_+ * \delta_n(x) = \int_{-1/n}^{1/n} \ln(x-t)_+ \delta_n(t)\, dt$$

and

$$(x_+^{-1})_n = x_+^{-1} * \delta_n(x) = \int_{-1/n}^{1/n} \ln(x-t)_+ \delta_n'(t)\, dt.$$

Since $\ln x_+$ and $\ln^2 x_+$ are locally summable functions, it follows that

$$\lim_{n\to\infty} (\ln x_+)_n^2 = \ln^2 x_+.$$

Thus, for arbitrary φ in $\mathcal{D}$, we have

$$\begin{aligned}\lim_{n\to\infty} \langle [(\ln x_+)_n^2]', \varphi(x)\rangle &= 2 \lim_{n\to\infty} \langle (\ln x_+)_n (x_+^{-1})_n, \varphi(x)\rangle \\ &= \langle (\ln x_+^2)', \varphi(x)\rangle = 2\langle x_+^{-1} \ln x_+, \varphi(x)\rangle\end{aligned}$$

and equation (21) follows.

Now suppose that the neutrix product $x_+^{-r} \Box \ln x_+$ exists and is of the form

$$x_+^{-r} \Box \ln x_+ = x_+^{-r} \ln x_+ + a_{r,0}\, \delta^{(r-1)}(x)$$

for some positive integer r. Then the derivative of $x_+^{-r} \Box \ln x_+$ exists and

$$\begin{aligned}(x_+^{-r} \Box \ln x_+)' &= -r x_+^{-r-1} \ln x_+ + x_+^{-r-1} + a_{r,0}\, \delta^{(r)}(x) \\ &= -r x_+^{-r-1} \Box \ln x_+ + x_+^{-r} \Box x_+^{-1} \\ &= -r x_+^{-r-1} \Box \ln x_+ + x_+^{-r-1} + a_{r,1} \delta^{(r)}(x).\end{aligned}$$

The product $x_+^{-r-1} \Box \ln x_+$ therefore exists and

$$\begin{aligned}x_+^{-r-1} \Box \ln x_+ &= x_+^{-r-1} \ln x_+ + \frac{a_{r,1} - a_{r,0}}{r} \delta^{(r)}(x) \\ &= x_+^{-r-1} \ln x_+ + a_{r+1,1} \delta^{(r)}(x).\end{aligned}$$

It follows by induction that the product $x_+^{-r} \Box \ln x_+$ exists for $r = 1, 2, \ldots$. $\Box$

Corollary 3.1. *The neutrix product* $x_-^{-r} \,\Box\, \ln x_-$ *exists for* $r = 1, 2, \ldots$. *In particular, the product* $x_-^{-1}.\ln x_-$ *exists and*

$$x_-^{-1}.\ln x_- = x_-^{-1} \ln x_-. \tag{22}$$

Proof. With the above notation we have

$$x_+^{-r} \,\Box\, \ln x_+ = x_+^{-r} \ln x_+ + a_{r,0}\,\delta^{(r-1)}(x). \tag{23}$$

Replacing x in this equation by $-x$, we get

$$x_-^{-r} \,\Box\, \ln x_- = x_-^{-r} \ln x_- - (-1)^r a_{r,0}\,\delta^{(r-1)}(x), \tag{24}$$

proving the existence of neutrix product $x_-^{-r} \,\Box\, \ln x_-$. The particular case $r = 1$ of course reduces to the product

$$x_-^{-1}.\ln x_- = x_-^{-1} \ln x_-.$$ □

Corollary 3.2.

$$x_+^{-r} \,\Box\, \ln x_+ + (-1)^r x_-^{-r} \,\Box\, \ln x_- = x^{-r} \ln |x| \tag{25}$$

for $r = 1, 2, \ldots$.

Proof. Equation (25) follows immediately from equations (23) and (24). □

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