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A note on equivalence of means

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Abstract. Given a real interval I, a relation, denoted by '$\sim$', is defined on the set of means on $I \times I$ by setting $M \sim N$ when there exists a surjective continuous function f solving the functional equation

$$f(M(x,y)) = N(f(x), f(y)), \quad x, y \in I.$$

A surjective and continuous solution to this equation turns out to be injective and so, '$\sim$' is an equivalence. This fact seems to be not properly noticed in the literature on means.

1. Introduction

Let I be a real interval. A function $M : I^2 \to I$ is said to be *internal on* I ([1]) when

$$x < M(x,y) < y, \quad x, y \in I,\ x < y.$$

Throughout this note, a continuous function M which is internal on I is called a *continuous mean on* I. Given two functions M and N defined on the square $I^2 \subseteq \mathbb{R}^2$, the functional equation

$$f\left(M(x,y)\right) = N\left(f(x), f(y)\right), \quad x, y \in I, \tag{1}$$

has been studied by a huge number of authors from the first decades of the present century. For a related bibliography we refer to the classical book [1], ps. 62, 79, 145. When M and N are continuous means on I,

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equation (1) is implicit in the paper [6] (cited in [5], p. 353) by G. Pietra, where M and N are called *corresponding means* ("medie corrispondenti") provided that a (*continuous* and) *strictly monotone* solution f to equation (1) exists on I. Indeed, equation (1) is not formulated by Pietra, who is mainly interested in particular correspondences: the arithmetic mean corresponds to the geometric mean through $f(x) = \exp x$, $x > 0$; to the harmonic mean through $f(x) = 1/x$, $x > 0$, and so on.When f is bijective, equation (1) can be rewritten in the form

$$M = f^{-1} \circ N \circ (f \times f), \tag{2}$$

where $f \times f$ denotes the Cartesian product of f by itself. In [2], two means M and N satisfying (2) for a bijective f have been named *conjugated means* and invariance under conjugacy of important classes of means is studied. On the other hand, the simultaneous existence of continuous solutions to equation (1) and to the following one

$$g\left(N(x,y)\right) = M\left(g(x), g(y)\right), \quad x, y \in I, \tag{3}$$

(without any monotonicity requirement on f or g) is taken by J. M. Borwein and P. B. Borwein as a starting point for defining an equivalence relation on the set of means on $\mathbb{R}^+$. As a matter of fact, in p. 239 of [3], a mean N on $\mathbb{R}^+$ is said to *dominate* another mean M (written $N \succ M$) when there exists a continuous solution f to (1) with $I = \mathbb{R}^+$. Since domination relation is transitive, an equivalence "$\sim$" is then introduced on the set of means by defining $N \sim M$ if and only if $N \succ M$ and $M \succ N$ but, with the purpose of proving interesting results, a more restrictive notion of equivalence is proposed soon after by requiring the function f to be one to one; i.e., injective. The same authors then consider ([3], p. 241) the notion of *strong equivalence* of means, so calling two means M and N on $\mathbb{R}^+$ when there exists a *bijective* and continuous solution to equation (1) (with $I = \mathbb{R}^+$).

In this regard, first we observe that, without injectivity, the definition of domination relation is meaningless because constants are trivial continuous solutions to (1) whatever be the means M and N. However, this inconvenience can be quickly remedied by excluding the constant solutions to (1). In fact, we will show in this note that if constant solutions are put aside, a continuous solution to equation (1) must be injective, so that only surjectivity needs to be required in order to obtain an equivalence. In other words, domination by continuous and onto functions is a concept that does not really differ from the notion of conjugacy of means as considered in [2].

In finishing this introduction, we would like to stress the interesting problem of finding conditions on the means M and N in order that equation (1) admits a continuous solution. As far as we know, this problem have been not solved up to date in its full generality. However, some particular conditions for existence of continuous solutions are known. For instance, it follows from Theorem 4, p. 79, of [1], that if any mean in equation (1), say M, is a quasiarithmetic mean, then a continuous (and strictly monotone) solution f exists if and only if N is also a quasiarithmetic mean. Other results on existence of solutions to equation (1) can be found in [4]. It should be added that the discussion of the next section does not depend on any result of existence of solutions for equation (1).

2. Results

Let F be a function defined on $I \times I$ such that $F(I \times I) \subseteq I$. We inductively define a family $\{F^d : d \in [0,1],\ d \text{ dyadic}\}$ of "dyadic iterates" of F as follows. Firstly, we set

$$F^0(x,y) \equiv x, \quad F^1(x,y) \equiv y. \tag{4}$$

Now, assume that $F^{\frac{j}{2^n}}$ is known for $n \geq 0$ and for every $0 \leq j \leq 2^n$; if $k = 2h$ with $0 \leq h \leq 2^n$, then we set

$$F^{\frac{k}{2^{n+1}}}(x,y) = F^{\frac{h}{2^n}}(x,y), \tag{5}$$

while if $k = 2h+1$ with $0 \leq h \leq 2^n - 1$,

$$F^{\frac{k}{2^{n+1}}}(x,y) = F\left(F^{\frac{h}{2^n}}(x,y), F^{\frac{h+1}{2^n}}(x,y)\right). \tag{6}$$

When F is an internal function, the family $\{F^d\}$ is monotone in the sense specified by the following lemma.

Lemma 1. *Let F be an internal function on I and d_1, $d_2 \in [0,1]$ be dyadic numbers with $d_1 < d_2$. Then*

$$F^{d_1}(x,y) < F^{d_2}(x,y) \tag{7}$$

for every $x, y \in I$, $x < y$.

Proof. Without lost of generality, we inductively prove that inequality (7) holds for $d_1 = \frac{p}{2^n}$ and $d_2 = \frac{q}{2^n}$, with $p < q$. For $n = 0$ the statement is trivial from (4). Suppose the statement is true for a non-negative integer n. Then, for $k = 2h$ we would have

$$\begin{aligned} F^{\frac{k+1}{2^{n+1}}}(x,y) &= F\left(F^{\frac{h}{2^n}}(x,y), F^{\frac{h+1}{2^n}}(x,y)\right) \\ &> F^{\frac{h}{2^n}}(x,y) = F^{\frac{k}{2^{n+1}}}(x,y) \end{aligned} \tag{8}$$

where we have applied the inductive hypothesis and the internality of F. In a similar way, if $k = 2h+1$, we obtain

$$\begin{aligned} F^{\frac{k+1}{2^{n+1}}}(x,y) &= F^{\frac{2h+2}{2^{n+1}}}(x,y) = F^{\frac{h+1}{2^n}}(x,y) \\ &> F\left(F^{\frac{h}{2^n}}(x,y), F^{\frac{h+1}{2^n}}(x,y)\right) = F^{\frac{k}{2^{n+1}}}(x,y). \end{aligned} \tag{9}$$

Now, if $k < l$, from (8) and (9) we deduce

$$F^{\frac{k}{2^{n+1}}}(x,y) < F^{\frac{k+1}{2^{n+1}}}(x,y) < \cdots < F^{\frac{l-1}{2^{n+1}}} < F^{\frac{l}{2^{n+1}}}(x,y),$$

which completes the induction. $\square$

Our next result links dyadic iterations with the functional equation (1).

Lemma 2. *Let M and N be internal functions on I and suppose that there exists a function f such that*

$$f\left(M(x,y)\right) = N\left(f(x), f(y)\right).$$

Then

$$f\left(M^d(x,y)\right) = N^d\left(f(x), f(y)\right) \tag{10}$$

for every dyadic number $d \in [0,1]$.

Proof. The proof follows from a simple inductive argument whose details are omitted. $\square$

The essential property of the set of dyadic iterates $\{M^d(x,y)\}$ is established in the following:

Proposition 3. *Let M be a continuous mean on I and $x, y \in I$ be real numbers such that $x < y$. Then, the set*

$$A = \left\{M^d(x,y) : 0 \leq d \leq 1 \text{ is a dyadic number}\right\}$$

is dense in the interval $[x,y]$.

Proof. If A were not dense in $[x,y]$, there would exist numbers a, b with $x \leq a < b \leq y$ such that

$$A \cap (a,b) = \emptyset, \tag{11}$$

being (a,b) maximal in the sense that, for every small enough ε,

$$A \cap (a-\varepsilon, b) \neq \emptyset \quad \text{and} \quad A \cap (a, b+\varepsilon) \neq \emptyset.$$

For $n \in \mathbb{N}$, define

$$k_n = \max\left\{k \in \mathbb{N} : M^d(x,y) \leq a,\ d = \frac{k}{2^n},\ 0 \leq k \leq 2^n\right\}$$

and put

$$d_n = \frac{k_n}{2^n},\ e_n = \frac{k_n+1}{2^n} \quad \text{and} \quad f_n = \frac{2k_n+1}{2^{n+1}}.$$

Sequences $\{d_n\}$ and $\{e_n\}$ turn out to be monotone ones: $\{d_n\}$ is increasing and $\{e_n\}$ is decreasing. From these facts and Lemma 1, it follows that sequences $\left\{M^{d_n}(x,y)\right\}$ and $\{M^{e_n}(x,y)\}$ are convergent; then, by the maximality of the interval (a,b), we have

$$\lim_{n\to\infty} M^{d_n}(x,y) = a \quad \text{and} \quad \lim_{n\to\infty} M^{e_n}(x,y) = b.$$

On the other hand, from definition of $M^d(x,y)$ for dyadic d, we have

$$M^{f_n}(x,y) = M\left(M^{d_n}(x,y), M^{e_n}(x,y)\right)$$

and, in view of the continuity of M,

$$\lim_{n\to\infty} M^{f_n}(x,y) = M(a,b).$$

Hence, using the internality of M, we obtain

$$a < \lim_{n\to\infty} M^{f_n}(x,y) < b,$$

so that there exists n_0 such that the inequalities

$$a < M^{f_n}(x,y) < b$$

hold for every $n \geq n_0$, which is in contradiction with (11). This proves that A is dense in $[x,y]$, as we claimed. $\square$

In the next result, density of dyadic iterates of an internal function is exploited in order to prove a monotonicity property of continuous solutions to equation (1).

Proposition 4. *Let M and N be two continuous means on $[x_0, y_0]$ and suppose that there exists a continuous function f such that*

$$f\left(M(x,y)\right) = N\left(f(x), f(y)\right), \quad x_0 \leq x < y \leq y_0. \tag{12}$$

If $f(x_0) < f(y_0)$ ($f(x_0) > f(y_0)$), then f is strictly increasing (decreasing) on $[x_0, y_0]$, while f reduces to a constant provided that the equality $f(x_0) = f(y_0)$ holds.

Proof. Assume that a continuous function f satisfies equation (12) and that $f(x_0) < f(y_0)$. By Lemma 2, equation (10) holds. Then, if $d_1 < d_2$ are dyadic numbers, from Lemma 1 we deduce

$$\begin{aligned} f\left(M^{d_1}(x_0,y_0)\right) &= N^{d_1}\left(f(x_0), f(y_0)\right) \\ &< N^{d_2}\left(f(x_0), f(y_0)\right) = f\left(M^{d_2}(x_0,y_0)\right); \end{aligned} \tag{13}$$

i.e., f is strictly increasing on the set

$$A = \left\{M^d(x_0,y_0) : 0 \leq d \leq 1 \text{ dyadic number}\right\}$$

which is dense in $[x_0, y_0]$ by Proposition 3. Therefore f is increasing on $[x_0, y_0]$. To see that f is strictly increasing, let us take $x, y \in [x_0, y_0]$, $x < y$, and two dyadic number d_1, d_2 such that $x < M^{d_1}(x_0,y_0) < M^{d_2}(x_0,y_0) < y$; then we have

$$f(x) \leq f(M^{d_1}(x_0,y_0)) < f(M^{d_2}(x_0,y_0)) \leq f(y).$$

In the case in which $f(x_0) > f(y_0)$, inequality (13) is reversed and f turns out to be decreasing. Finally, if $f(x_0) = f(y_0) = C$, then it follows from Lemma 2 that $f(x) = C$ for every $x \in A$ and therefore $f \equiv C$ on $[x_0, y_0]$. $\square$

A sort of converse of Proposition 4 is now established.

Proposition 5. *Let f be a strictly monotone function satisfying equation* (12) *for two continuous means M and N. Then f is continuous.*

Proof. Assume that f is strictly monotone. From Lemma 3, the set

$$\left\{N^d\left(f(x_0), f(y_0)\right) : d \text{ dyadic number}\right\}$$

is dense in the interval $[f(x_0), f(y_0)]$ and, in view of equation (10), the set

$$f\left([x_0, y_0]\right) = \{f(x) : x_0 \leq x \leq y_0\}$$

is a fortiori dense in $[f(x_0), f(y_0)]$. Hence, the monotone function f has no gaps; i.e., it is continuous. $\square$

For a pair M, N of continuous means on I, we put $M \sim N$ if and only if there exist a surjective and continuous solution to equation (1). As it was asserted in the Introduction, '$\sim$' is an equivalence

Theorem 6. *The relation '$\sim$' is an equivalence on the set of continuous means on a real interval I.*

Proof. Since the identity map id_I is continuous and surjective, relation '$\sim$' is reflexive. The transitivity of '$\sim$' is a consequence of the fact that $f \circ g$ is continuous and surjective when so f and g are. Now, it is a simple matter to extend Proposition 4 to the case in which the means M and N are defined on an arbitrary real interval I. To this end, let us consider a non-constant continuous solution f to equation (1). Assuming that f is not injective, we can find $x_0, y_0 \in I$, $x_0 < y_0$, such that $f(x_0) = f(y_0) = C$. In view of Proposition 4, we realize that $f(x) = C$ for every $x \in [x_0, y_0]$. Since f is not constant on I, there exists $z_0 \in I \setminus [x_0, y_0]$ such that $f(z_0) \neq C$. Taking, for instance, $z_0 < x_0$ and $f(z_0) < f(x_0) = f(y_0)$, a new application of Proposition 4 shows that f should be strictly increasing in $[z_0, y_0]$, in contradiction with the assumption $f(x_0) = f(y_0) = C$. The remaining cases can be analogously treated, thus concluding that a non-constant continuous solution f to (1) must be injective. With this result at hand, the symmetry of '$\sim$' is easily proved. In fact, if $M \sim N$ then a surjective and continuous solution f to equation (1) exists. But f turns out to be injective so that f^{-1}, which is also surjective and continuous, satisfies

$$f^{-1}\left(N(x,y)\right) = M\left(f^{-1}(x), f^{-1}(y)\right), \quad x, y \in I;$$

and therefore $N \sim M$. $\square$

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