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Differentiable solutions of a polynomial-like iterative equation with variable coefficients

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Abstract. This paper is concerned with a polynomial-like iterative equation with variable coefficients $\sum_{i=1}^{n} \lambda_i(x)\varphi^{[i]}(x) = F(x)$, where $\varphi^{[i]}(x)$ is the i^{th} iterate of the function $\varphi(x)$. Using the fixed point theorems of Schauder and Banach we discuss the existence, uniqueness and stability of $\operatorname{Lip} C^1$-solutions of the equation.

1. Introduction

Let $\varphi^{[k]}$ denote the k-th iterate of a function φ, and $\varphi^{[0]}$ the identify function. To find a function φ such that its k-th iterate $\varphi^{[k]}$ is equal to a give function F plays an important role in the theory of dynamical systems [1], [2]. As a natural generalization, the polynomial-like iterative functional equations in the following form

$$(*) \qquad \lambda_1\varphi(x) + \lambda_2\varphi^{[2]}(x) + \cdots + \lambda_n\varphi^{[n]}(x) = F(x)$$

for $x \in R$, $\lambda_i \in R$, $i = 1, 2, \ldots, n$, or some special cases were considered recently [3–10]. In particular, W. ZHANG [6] considered the existence, uniqueness and stability of differentiable solutions of equation $(*)$. However, conditions for the existence of differentiable solutions are not known in the case of variable coefficients. In this paper, we will consider a polynomial-like iterative equation with variable coefficients:

$$(1) \qquad \lambda_1(x)\varphi(x) + \lambda_2(x)\varphi^{[2]}(x) + \cdots + \lambda_n(x)\varphi^{[n]}(x) = F(x),$$

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where n is a positive integer greater than or equal to 2. By means of the fixed point theorems of Banach and Schauder, we discuss the existence, uniqueness and stability of $\operatorname{Lip} C^1$-solutions of equation (1).

We write $\varphi \in C^1$ if φ, φ' are continuous. The set of all C^1 function each of which maps a closed interval I into I will be denoted by $C^1(I,I)$. It is well known that when endowed with the norm $\|\cdot\|_{C^1}$, where

$$\|\varphi\|_{C^1} = \|\varphi\|_{C^0} + \|\varphi'\|_{C^0}, \quad \|\varphi\|_{C^0} = \max_{x\in I}\{|\varphi(x)|\},$$

$C^1(I,I)$ is a Banach space (see also [6]).

We write $\varphi \in \operatorname{Lip} C^1$ if $\varphi \in C^1(I,I)$ and φ' is Lipschitzian. Let $I = [a,b] \subset R$, for given constants $M > 0$, $M^* > 0$, we will denote by $\Omega(M, M^*; I)$ the subset of all $\varphi \in \operatorname{Lip} C^1$ each of which satisfies

$$\varphi(a) = a, \quad \varphi(b) = b, \qquad 0 \le \varphi'(x) \le M,$$
$$|\varphi'(x_1) - \varphi'(x_2)| \le M^*|x_1 - x_2|, \quad \forall x, x_1, x_2 \in I.$$

2. Preparatory lemmas

Our discussion depends on the following several preparatory lemmas the proof of which can be found in [6].

Lemma 1. *Suppose that* $\varphi \in \Omega(M, M^*; I)$. *Then*

$$|(\varphi^{[i]})'(x_1) - (\varphi^{[i]})'(x_2)| \le M^*\left(\sum_{j=i-1}^{2i-2} M^j\right)|x_1 - x_2|. \tag{2}$$

Lemma 2. *Suppose that* $\varphi_1, \varphi_2 \in \Omega(M, M^*; I)$. *Then*

$$\left\|\varphi_1^{[i]} - \varphi_2^{[i]}\right\|_{C^0} \le \left(\sum_{j=1}^{i} M^{j-1}\right)\|\varphi_1 - \varphi_2\|_{C^0}. \tag{3}$$

Lemma 3. *Suppose that* $\varphi_1, \varphi_2 \in \Omega(M, M^*; I)$. *Then*

$$\begin{aligned} \left\|\left(\varphi_1^{[k+1]}\right)' - \left(\varphi_2^{[k+1]}\right)'\right\|_{C^0} &\le (k+1)M^k\|\varphi_1' - \varphi_2'\|_{C^0} \\ &+ Q(k+1)M^*\left(\sum_{i=1}^{k}(k-i+1)M^{k+i-1}\right)\|\varphi_1 - \varphi_2\|_{C^0}, \end{aligned} \tag{4}$$

for $k = 0, 1, 2, \ldots,$ *where* $Q(s) = 0$ *when* $s = 1$ *and* $Q(s) = 1$ *when* $s = 2, 3, \ldots.$

Lemma 4. *If* φ_1, φ_2 *are homeomorphisms from* I *onto itself and*

$$|\varphi_i(x_1) - \varphi_i(x_2)| \leq L|x_1 - x_2|, \quad \forall x_1, x_2 \in I,$$

where L *is a positive constant and* $i = 1, 2$, *then*

$$\|\varphi_1 - \varphi_2\|_{C^0} \leq L\|\varphi_1^{-1} - \varphi_2^{-1}\|_{C^0}. \tag{5}$$

3. The existence and uniqueness of Lip C^1-solutions for equation (1)

In this section we give the existence and uniqueness theorems of Lip C^1-solutions for equation (1).

Theorem 1. *Let* $I = [a, b]$, $\lambda_1, \lambda_2, \ldots, \lambda_n : I \to [0, 1]$ *be continuous,* $\lambda_1(x) \geq \alpha$, $\sum_{i=1}^{n} \lambda_i(x) = 1$ *for all* $x \in I$, *and*

$$|\lambda_k(x_1) - \lambda_k(x_2)| \leq \beta_k|x_1 - x_2|, \quad \forall x_1, x_2 \in I, \quad k = 1, 2, \ldots, n,$$

where $\alpha > (1 - \alpha)(M + \frac{\kappa}{\alpha}\sum_{i=1}^{n}\beta_i)\sum_{j=0}^{2n-4} M^{j+1}$, β_k $(k = 1, 2, \ldots, n)$ *are positive constants. Suppose that* $F \in \Omega(\alpha M, M'; I)$. *Then* (1) *has a soluton in* $\Omega(M, M^*; I)$. *Here*

$$M^* \geq \frac{M' + \sum_{i=1}^{n}\beta_i M^i}{\alpha - (1 - \alpha)\left(M + \frac{\kappa}{\alpha}\sum_{i=1}^{n}\beta_i\right)\sum_{j=0}^{2n-4} M^{j+1}}, \quad \kappa = \max\{|a|, |b|\}.$$

Proof. We will seek of a solution (1) in $\Omega(M, M^*; I)$. To this end, for each $\varphi \in \Omega(M, M^*; I)$, let us define

$$\varphi_x(t) = \sum_{i=1}^{n} \lambda_i(x)\varphi^{[i-1]}(t), \quad \forall t \in I. \tag{6}$$

It is easy to see that $\varphi_x(a) = a$, $\varphi_x(b) = b$, and $\varphi_x \in C^1(I, I)$. Since

$$\varphi_x'(t) = \sum_{i=1}^{n} \lambda_i(x)(\varphi^{[i-1]})'(t), \tag{7}$$

$$0 < \alpha \leq \lambda_1(x) \leq \varphi_x'(t) \leq \sum_{i=1}^{n} M^{i-1} := K_1. \tag{8}$$

So

$$0 < \frac{1}{K_1} \leq (\varphi_x^{-1})'(t) = \frac{1}{\varphi_x'(\varphi_x^{-1}(t))} \leq \frac{1}{\alpha}. \tag{9}$$

Thus $\varphi_x : I \to I$ is a self-diffeomorphism. □

First, we prove the following lemma.

Lemma 5. *Let $\varphi, g, h \in \Omega(M, M^*; I)$, and $x_1, x_2, t_1, t_2, t \in I$. Then*

$$|\varphi_x'(t_1) - \varphi_x'(t_2)| \leq K_2|t_1 - t_2|, \tag{10}$$

where $K_2 = (1-\alpha)M^* \sum_{j=0}^{2n-4} M^j$.

$$|(\varphi_x^{-1})'(t_1) - (\varphi_x^{-1})'(t_2)| \leq \frac{K_2}{\alpha^3}|t_1 - t_2|. \tag{11}$$

$$|\varphi_{x_1}'(t) - \varphi_{x_2}'(t)| \leq \left(\sum_{i=1}^{n} \beta_i M^{i-1}\right)|x_1 - x_2|. \tag{12}$$

$$|\varphi_{x_1}(t) - \varphi_{x_2}(t)| \leq \left(\kappa \sum_{i=1}^{n} \beta_i\right)|x_1 - x_2|. \tag{13}$$

$$|\varphi_{x_1}^{-1}(t) - \varphi_{x_2}^{-1}(t)| \leq \frac{\kappa}{\alpha}\left(\sum_{i=1}^{n} \beta_i\right)|x_1 - x_2|. \tag{14}$$

$$|\varphi_{x_1}^{-1}(t_1) - \varphi_{x_2}^{-1}(t_2)| \leq \frac{1}{\alpha}|t_1 - t_2| + \frac{\kappa}{\alpha}\left(\sum_{i=1}^{n} \beta_i\right)|x_1 - x_2|. \tag{15}$$

$$\begin{aligned} &|(\varphi_{x_1}^{-1})'(t_1) - (\varphi_{x_2}^{-1})'(t_2)| \\ &\leq \frac{1}{\alpha^2}\left(\frac{K_2\kappa}{\alpha}\sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n} \beta_i M^{i-1}\right)|x_1 - x_2| + \frac{K_2}{\alpha^3}|t_1 - t_2|. \end{aligned} \tag{16}$$

$$\begin{aligned} &|(\varphi_{x_1}^{-1})'(t) - (\varphi_{x_2}^{-1})'(t)| \\ &\leq \frac{1}{\alpha^2}\left(\frac{K_2\kappa}{\alpha}\sum_{i=1}^{n} \beta_i + \sum_{i=1}^{n} \beta_i M^{i-1}\right)|x_1 - x_2|. \end{aligned} \tag{17}$$

$$\|g_x - h_x\|_{C^0} \leq \left(\sum_{i=2}^{n}\sum_{j=1}^{i-1} M^{j-1}\right)\|g - h\|_{C^0}. \tag{18}$$

$$\|g_x' - h_x'\|_{C^0} \le \sum_{i=2}^{n}(i-1)M^{i-2}\|g'-h'\|_{C^0} + M^* \sum_{i=2}^{n}\sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3}\|g-h\|_{C^0}. \tag{19}$$

$$\|g_x^{-1} - h_x^{-1}\|_{C^0} \le \frac{1}{\alpha}\left(\sum_{i=2}^{n}\sum_{j=1}^{i-1} M^{j-1}\right)\|g-h\|_{C^0}. \tag{20}$$

$$\begin{aligned}&\|(g_x^{-1})' - (h_x^{-1})'\|_{C^0}\\ &\quad\le \left[\frac{K_2}{\alpha^3}\sum_{i=2}^{n}\sum_{j=1}^{i-1} M^{j-1} + \frac{M^*}{\alpha^2}\sum_{i=2}^{n}\sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3}\right]\\ &\qquad \times \|g-h\|_{C^0} + \frac{1}{\alpha^2}\sum_{i=2}^{n}(i-1)M^{i-2}\|g'-h'\|_{C^0}.\end{aligned} \tag{21}$$

Proof of Lemma 5. By Lemma 1 we see that

$$|\varphi_x'(t_1) - \varphi_x'(t_2)| = \left|\sum_{i=1}^{n}\lambda_i(x)\left[\left(\varphi^{[i-1]}\right)'(t_1) - \left(\varphi^{[i-1]}\right)'(t_2)\right]\right|$$

$$\le \sum_{i=2}^{n}\lambda_i(x)\left(M^*\sum_{j=i-2}^{2i-4} M^j\right)|t_1-t_2| \le \sum_{i=2}^{n}\lambda_i(x)\left(M^*\sum_{j=0}^{2n-4} M^j\right)|t_1-t_2|$$

$$= (1-\lambda_1(x))M^*\sum_{j=0}^{2n-4} M^j|t_1-t_2| \le (1-\alpha)M^*\sum_{j=0}^{2n-4} M^j|t_1-t_2| = K_2|t_1-t_2|.$$

This proves (10).

From (8)–(10) we have

$$|(\varphi_x^{-1})'(t_1) - (\varphi_x^{-1})'(t_2)| = \left|\frac{1}{\varphi_x'(\varphi_x^{-1}(t_1))} - \frac{1}{\varphi_x'(\varphi_x^{-1}(t_2))}\right|$$

$$= \left|\frac{\varphi_x'(\varphi_x^{-1}(t_1)) - \varphi_x'(\varphi_x^{-1}(t_2))}{\varphi_x'(\varphi_x^{-1}(t_1))\varphi_x'(\varphi_x^{-1}(t_2))}\right| \le \frac{K_2}{\alpha^2}|\varphi_x^{-1}(t_1) - \varphi_x^{-1}(t_2)| \le \frac{K_2}{\alpha^3}|t_1-t_2|.$$

This proves (11).

From (7) it follows that

$$
\begin{aligned}
|\varphi'_{x_1}(t)-\varphi'_{x_2}(t)| &= \left|\sum_{i=1}^{n}(\lambda_i(x_1)-\lambda_i(x_2))\big(\varphi^{[i-1]}\big)'(t)\right| \\
&\le \sum_{i=1}^{n}|\lambda_i(x_1)-\lambda_i(x_2)|\,\left|\big(\varphi^{[i-1]}\big)'(t)\right| \le \left(\sum_{i=1}^{n}\beta_i M^{i-1}\right)|x_1-x_2|.
\end{aligned}
$$

This proves (12).

By (6) we have

$$
\begin{aligned}
|\varphi_{x_1}(t)-\varphi_{x_2}(t)| &= \left|\sum_{i=1}^{n}(\lambda_i(x_1)-\lambda_i(x_2))\,\varphi^{[i-1]}(t)\right| \\
&\le \sum_{i=1}^{n}|\lambda_i(x_1)-\lambda_i(x_2)|\,\left|\varphi^{[i-1]}(t)\right| \le \kappa\sum_{i=1}^{n}\beta_i|x_1-x_2|.
\end{aligned}
$$

This proves (13).

(14) follows from

$$
\begin{aligned}
\left|\varphi_{x_1}^{-1}(t)-\varphi_{x_2}^{-1}(t)\right| &= \left|\varphi_{x_1}^{-1}\left(\varphi_{x_2}(\varphi_{x_2}^{-1}(t))\right)-\varphi_{x_1}^{-1}\left(\varphi_{x_1}(\varphi_{x_2}^{-1}(t))\right)\right| \\
&\overset{(9)}{\le} \frac{1}{\alpha}\left|\varphi_{x_2}(\varphi_{x_2}^{-1}(t))-\varphi_{x_1}(\varphi_{x_2}^{-1}(t))\right| \overset{(13)}{\le} \left(\frac{\kappa}{\alpha}\sum_{i=1}^{n}\beta_i\right)|x_1-x_2|.
\end{aligned}
$$

From (9) and (14) we have

$$
\begin{aligned}
|\varphi_{x_1}^{-1}(t_1)-\varphi_{x_2}^{-1}(t_2)| &\le |\varphi_{x_1}^{-1}(t_1)-\varphi_{x_1}^{-1}(t_2)|+|\varphi_{x_1}^{-1}(t_2)-\varphi_{x_2}^{-1}(t_2)| \\
&\le \frac{1}{\alpha}|t_1-t_2|+\left(\frac{\kappa}{\alpha}\sum_{i=1}^{n}\beta_i\right)|x_1-x_2|.
\end{aligned}
$$

This proves (15).

In view of (9), (10), (12) and (14), (16) follows from

$$
\begin{aligned}
|(\varphi_{x_1}^{-1})'(t_1)-(\varphi_{x_2}^{-1})'(t_2)| &= \left|\frac{1}{\varphi'_{x_1}(\varphi_{x_1}^{-1}(t_1))}-\frac{1}{\varphi'_{x_2}(\varphi_{x_2}^{-1}(t_2))}\right| \\
&\le \frac{1}{\alpha^2}|\varphi'_{x_1}(\varphi_{x_1}^{-1}(t_1))-\varphi'_{x_2}(\varphi_{x_2}^{-1}(t_2))|
\end{aligned}
$$

$$\le \frac{1}{\alpha^2}\left[|\varphi'_{x_1}(\varphi_{x_1}^{-1}(t_1)) - \varphi'_{x_1}(\varphi_{x_2}^{-1}(t_1))| + |\varphi'_{x_1}(\varphi_{x_2}^{-1}(t_1)) - \varphi'_{x_1}(\varphi_{x_2}^{-1}(t_2))|\right.$$

$$\left. + |\varphi'_{x_1}(\varphi_{x_2}^{-1}(t_2)) - \varphi'_{x_2}(\varphi_{x_2}^{-1}(t_2))|\right] \le \frac{1}{\alpha^2}$$

$$\times\left[K_2|\varphi_{x_1}^{-1}(t_1) - \varphi_{x_2}^{-1}(t_1)| + K_2|\varphi_{x_2}^{-1}(t_1) - \varphi_{x_2}^{-1}(t_2)| + \left(\sum_{i=1}^{n}\beta_i M^{i-1}\right)|x_1 - x_2|\right]$$

$$\le \frac{1}{\alpha^2}\left[\frac{K_2\kappa}{\alpha}\sum_{i=1}^{n}\beta_i|x_1 - x_2| + \frac{K_2}{\alpha}|t_1 - t_2| + \left(\sum_{i=1}^{n}\beta_i M^{i-1}\right)|x_1 - x_2|\right]$$

$$= \frac{1}{\alpha^2}\left(\frac{K_2\kappa}{\alpha}\sum_{i=1}^{n}\beta_i + \sum_{i=1}^{n}\beta_i M^{i-1}\right)|x_1 - x_2| + \frac{K_2}{\alpha^3}|t_1 - t_2|.$$

(17) follows from

$$|(\varphi_{x_1}^{-1})'(t) - (\varphi_{x_2}^{-1})'(t)| = \left|\frac{1}{\varphi'_{x_1}(\varphi_{x_1}^{-1}(t))} - \frac{1}{\varphi'_{x_2}(\varphi_{x_2}^{-1}(t))}\right|$$

$$\le \frac{1}{\alpha^2}|\varphi'_{x_1}(\varphi_{x_1}^{-1}(t)) - \varphi'_{x_2}(\varphi_{x_2}^{-1}(t))|$$

$$\le \frac{1}{\alpha^2}\left[|\varphi'_{x_1}(\varphi_{x_1}^{-1}(t)) - \varphi'_{x_1}(\varphi_{x_2}^{-1}(t))| + |\varphi'_{x_1}(\varphi_{x_2}^{-1}(t)) - \varphi'_{x_2}(\varphi_{x_2}^{-1}(t))|\right]$$

$$\overset{(10),(12)}{\le} \frac{1}{\alpha^2}\left[K_2|\varphi_{x_1}^{-1}(t) - \varphi_{x_2}^{-1}(t)| + \left(\sum_{i=1}^{n}\beta_i M^{i-1}\right)|x_1 - x_2|\right]$$

$$\overset{(14)}{\le} \frac{1}{\alpha^2}\left[\frac{K_2\kappa}{\alpha}\sum_{i=1}^{n}\beta_i|x_1 - x_2| + \sum_{i=1}^{n}\beta_i M^{i-1}|x_1 - x_2|\right]$$

$$= \frac{1}{\alpha^2}\left(\frac{K_2\kappa}{\alpha}\sum_{i=1}^{n}\beta_i + \sum_{i=1}^{n}\beta_i M^{i-1}\right)|x_1 - x_2|.$$

(18) follows from

$$\|g_x - h_x\|_{C^0} = \max_{t\in I}|g_x(t) - h_x(t)|$$

$$= \max_{t\in I}\left|\sum_{i=1}^{n}\lambda_i(x)(g^{[i-1]}(t) - h^{[i-1]}(t))\right| \le \sum_{i=1}^{n}\|g^{[i-1]} - h^{[i-1]}\|_{C^0}$$

$$\overset{(3)}{\leq} \left(\sum_{i=2}^{n} \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0}.$$

(19) follows from

$$\begin{aligned} \|g'_x - h'_x\|_{C^0} &= \max_{t \in I} \left| \sum_{i=1}^{n} \lambda_i(x) \big((g^{[i-1]})'(t) - (h^{[i-1]})'(t) \big) \right| \\ &\leq \sum_{i=1}^{n} \lambda_i(x) \big\| \big(g^{[i-1]}\big)' - \big(h^{[i-1]}\big)' \big\|_{C^0} \overset{(4)}{\leq} \sum_{i=1}^{n} \Bigg[(i-1) M^{i-2} \|g'_x - h'_x\|_{C^0} \\ &\quad + Q(i-1) M^* \left(\sum_{j=1}^{i-2} (i-j-1) M^{i+j-3} \right) \|g_x - h_x\|_{C^0} \Bigg] \\ &\leq \sum_{i=2}^{n} (i-1) M^{i-2} \|g'_x - h'_x\|_{C^0} \\ &\quad + M^* \sum_{i=2}^{n} \sum_{j=1}^{i-2} Q(i-1)(i-j-1) M^{i+j-3} \|g - h\|_{C^0}. \end{aligned}$$

(20) follows from

$$\|g_x^{-1} - h_x^{-1}\|_{C^0} \overset{(5)}{\leq} \frac{1}{\alpha} \|g_x - h_x\|_{C^0} \overset{(18)}{\leq} \left(\frac{1}{\alpha} \sum_{i=2}^{n} \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0}.$$

Finally, by (9), (10), (19) and (20), we have

$$\begin{aligned} \|(g_x^{-1})' - (h_x^{-1})'\|_{C^0} &= \max_{t \in I} \left| \frac{1}{g'_x(g_x^{-1}(t))} - \frac{1}{h'_x(h_x^{-1}(t))} \right| \\ &\leq \frac{1}{\alpha^2} \max_{t \in I} |g'_x(g_x^{-1}(t)) - h'_x(h_x^{-1}(t))| \\ &\leq \frac{1}{\alpha^2} \max_{t \in I} [|g'_x(g_x^{-1}(t)) - g'_x(h_x^{-1}(t))| + |g'_x(h_x^{-1}(t)) - h'_x(h_x^{-1}(t))|] \\ &\leq \frac{1}{\alpha^2} [K_2 \|g_x^{-1} - h_x^{-1}\|_{C^0} + \|g'_x - h'_x\|_{C^0}] \\ &\leq \left(\frac{K_2}{\alpha^3} \sum_{i=2}^{n} \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0} + \frac{1}{\alpha^2} \sum_{i=2}^{n} (i-1) M^{i-2} \|g' - h'\|_{C^0} \end{aligned}$$

$$+\frac{M^*}{\alpha^2}\sum_{i=2}^{n}\sum_{j=1}^{i-2}Q(i-1)(i-j-1)M^{i+j-3}\|g-h\|_{C^0}$$

$$=\left[\frac{K_2}{\alpha^3}\sum_{i=2}^{n}\sum_{j=1}^{i-1}M^{j-1}+\frac{M^*}{\alpha^2}\sum_{i=2}^{n}\sum_{j=1}^{i-2}Q(i-1)(i-j-1)M^{i+j-3}\right]\|g-h\|_{C^0}$$

$$+\frac{1}{\alpha^2}\sum_{i=2}^{n}(i-1)M^{i-2}\|g'-h'\|_{C^0}.$$

This proves (21).

Now we continue to prove Theorem 1. Define an operator T from $\Omega(M,M^*;I)$ into $C^1(I,I)$ by

$$(T\varphi)(x)=\varphi_x^{-1}(F(x)),\quad \varphi\in\Omega(M,M^*;I). \tag{22}$$

Clearly $(T\varphi)(a)=a$, $T(\varphi)(b)=b$, $T\varphi\in C^1(I,I)$, and (9) yields that

$$0\le\frac{F'(x)}{K_1}\le(T\varphi)'(x)=(\varphi_x^{-1})'(F(x))F'(x)\le\frac{1}{\alpha}\cdot\alpha M=M.$$

Furthermore, by (11), (17), we have

$$\begin{aligned}
&|(T\varphi)'(x_1)-(T\varphi)'(x_2)|=|(\varphi_{x_1}^{-1})'(F(x_1))F'(x_1)-(\varphi_{x_2}^{-1})'(F(x_2))F'(x_2)|\\
&\le|(\varphi_{x_1}^{-1})'(F(x_1))-(\varphi_{x_1}^{-1})'(F(x_2))|F'(x_1)+(\varphi_{x_1}^{-1})'(F(x_2))|F'(x_1)-F'(x_2)|\\
&\quad+|(\varphi_{x_1}^{-1})'(F(x_2))-(\varphi_{x_2}^{-1})'(F(x_2))|F'(x_2)\\
&\le\frac{K_2}{\alpha^3}|F(x_1)-F(x_2)|\cdot\alpha M+\frac{1}{\alpha}M'|x_1-x_2|\\
&\quad+\frac{1}{\alpha^2}\left(\frac{K_2\kappa}{\alpha}\sum_{i=1}^{n}\beta_i+\sum_{i=1}^{n}\beta_iM^{i-1}\right)|x_1-x_2|\cdot\alpha M\\
&\le\left[\frac{K_2M^2}{\alpha}+\frac{M'}{\alpha}+\frac{M}{\alpha}\left(\frac{K_2\kappa}{\alpha}\sum_{i=1}^{n}\beta_i+\sum_{i=1}^{n}\beta_iM^{i-1}\right)\right]|x_1-x_2|\\
&\le M^*|x_1-x_2|,
\end{aligned}$$

so $(T\varphi)(x)\in\Omega(M,M^*;I)$, that is, T is a operator from $\Omega(M,M^*;I)$ into itself.

Now we will show that T is continuous. Let $g, h \in \Omega(M, M^*; I)$, $(Tg)(x) = g_x^{-1}(F(x))$, $(Th)(x) = h_x^{-1}(F(x))$, then

$$
\begin{aligned}
(23)\quad \|Tg - Th\|_{C^1} &= \|Tg - Th\|_{C^0} + \|(Tg)' - (Th)'\|_{C^0} \\
&= \max_{x\in I}\left\{|g_x^{-1}(F(x)) - h_x^{-1}(F(x))|\right\} \\
&\quad + \max_{x\in I}\left\{|(g_x^{-1})'(F(x))F'(x) - (h_x^{-1})'(F(x))F'(x)|\right\} \\
&\le \|g_x^{-1} - h_x^{-1}\|_{C^0} + \alpha M\|(g_x^{-1})' - (h_x^{-1})'\|_{C^0} \\
&\le \left(\frac{1}{\alpha}\sum_{i=2}^{n}\sum_{j=1}^{i-1} M^{j-1}\right)\|g - h\|_{C^0} + \alpha M \\
&\quad \times \left\{\left[\frac{K_2}{\alpha^3}\sum_{i=2}^{n}\sum_{j=1}^{i-1} M^{j-1} + \frac{M^*}{\alpha^2}\sum_{i=2}^{n}\sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3}\right]\right. \\
&\quad \left.\times \|g - h\|_{C^0} + \frac{1}{\alpha^2}\sum_{i=2}^{n}(i-1)M^{i-2}\|g' - h'\|_{C^0}\right\} \\
&= \left[\frac{1}{\alpha}\sum_{i=2}^{n}\sum_{j=1}^{i-1} M^{j-1} + \frac{K_2 M}{\alpha^2}\sum_{i=2}^{n}\sum_{j=1}^{i-1} M^{j-1}\right. \\
&\quad \left. + \frac{MM^*}{\alpha}\sum_{i=2}^{n}\sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3}\right]\|g - h\|_{C^0} \\
&\quad + \frac{M}{\alpha}\sum_{i=2}^{n}(i-1)M^{i-2}\|g' - h'\|_{C^0} \le \Theta\|g - h\|_{C^1},
\end{aligned}
$$

where

$$
\begin{aligned}
(24)\quad \Theta = \max&\left\{\left(\frac{1}{\alpha} + \frac{K_2 M}{\alpha^2}\right)\sum_{i=2}^{n}\sum_{j=1}^{i-1} M^{j-1}\right. \\
&\left. + \frac{MM^*}{\alpha}\sum_{i=2}^{n}\sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3};\ \frac{M}{\alpha}\sum_{i=2}^{n}(i-1)M^{i-2}\right\}.
\end{aligned}
$$

This shows that T is continuous.

It is easy to see that $\Omega(M, M^*; I)$ is closed and convex. We now show that $\Omega(M, M^*; I)$ is a relatively compact subset of $C^1(I, I)$. For any

$\varphi = \varphi(x)$ in $\Omega(M, M^*; I)$,

$$\|\varphi\|_{C^1} = \|\varphi\|_{C^0} + \|\varphi'\|_{C^0} \le \kappa + M.$$

Hence $\Omega(M, M^*; I)$ is bounded in $C^1(I, I)$. Next, for any $\varphi = \varphi(x)$ in $\Omega(M, M^*; I)$ and any $x_1, x_2 \in I$, we have

$$|\varphi(x_1) - \varphi(x_2)| \le M|x_1 - x_2|.$$

This shows that $\Omega(M, M^*; I)$ is equicontinuous on I. By means of the Arzela–Ascoli theorem, we see that $\Omega(M, M^*; I)$ is relatively compact in $C^1(I, I)$. By Schauder's fixed point theorem we assert that there is a function $\varphi \in \Omega(M, M^*; I)$ such that

$$\varphi(x) = (T\varphi)(x) = \varphi_x^{-1}(F(x))$$

or

$$\varphi_x(\varphi(x)) = F(x),$$

that is, φ is a solution of equation (1) in $\Omega(M, M^*; I)$. This completes the proof. □

Theorem 2. *Under the hypothses of Theorem 1,* (1) *has a unique solution in* $\Omega(M, M^*; I)$ *if* $\Theta < 1$ *in* (24).

Proof. Since $\Theta < 1$, we see that T defined by (22) is contraction mapping on the closed subset $\Omega(M, M^*; I)$ of $C^1(I, I)$. Thus the fixed point φ in the proof of Theorem 1 must be unique. This completes the proof. □

Remark 1. A referee of this paper proposes the following interesting question: Whether it is possible that (under the weaker assumptions) of Theorem 1 there exist indeed more than one differential solution of (1) in the sense that T has at least two fixed points. We do not know how to solve this question.

4. The stability of $\mathrm{Lip}\, C^1$-solutions for equation (1)

In this section we consider the problem of the continuous dependence of $\mathrm{Lip}\, C^1$-solutions of equation (1) on the given functions. We have the following

Theorem 3. *The unique solution obtained in Theorem 2 depends continuously on the given functions F and λ_i $(i = 1, 2, \dots, n)$.*

Proof. Under the assumptions of Theorem 2, if $G = G(x)$ and $H = H(x)$ are any two functions in $\Omega(\alpha M, M'; I); \alpha_j(x)$ and $\mu_j(x)$ $(j = 1, 2, \dots, n)$ are any functions which satisfy the same conditions as $\lambda_j(x)$ $(j = 1, 2, \dots, n)$ in Theorem 1. Then there correspond two unique functions $g = g(x)$ and $h = h(x)$ in $\Omega(M, M^*; I)$ such that

$$g(x) = g_x^{-1}(G(x))$$

and

$$h(x) = h_x^{-1}(H(x)),$$

where

$$g_x(t) = \sum_{i=1}^{n} \alpha_i(x) g^{[i-1]}(t),$$

$$h_x(t) = \sum_{i=1}^{n} \mu_i(x) h^{[i-1]}(t).$$

First of all, it is easy to see that

$$\begin{aligned}
|g_x(t) - h_x(t)| &\le \sum_{i=1}^{n} |\alpha_i(x) - \mu_i(x)| \left|g^{[i-1]}(t)\right| \\
&\quad + \sum_{i=1}^{n} |\mu_i(x)| \left|g^{[i-1]}(t) - h^{[i-1]}(t)\right| \\
&\le \kappa \sum_{i=1}^{n} \|\alpha_i - \mu_i\|_{C^0} + \left(\sum_{i=2}^{n} \sum_{j=1}^{i-1} M^{j-1} \right) \|g - h\|_{C^0}, \\
|g_x'(t) - h_x'(t)| &= \left| \sum_{i=1}^{n} \left(\alpha_i(x) \big(g^{[i-1]}\big)'(t) - \mu_i(x) (h^{[i-1]})'(t) \right) \right| \\
&\le \sum_{i=1}^{n} \left[|\alpha_i(x) - \mu_i(x)| \big(g^{[i-1]}\big)'(t) + \mu_i(x) \left| \big(g^{[i-1]}\big)'(t) - \big(h^{[i-1]}\big)'(t) \right| \right] \\
&\le \sum_{i=1}^{n} M^{i-1} \|\alpha_i - \mu_i\|_{C^0} + \sum_{i=2}^{n} \left\| \big(g^{[i-1]}\big)' - \big(h^{[i-1]}\big)' \right\|_{C^0}
\end{aligned}$$

$$\leq \sum_{i=1}^{n} M^{i-1}\|\alpha_i-\mu_i\|_{C^0}+\sum_{i=2}^{n}(i-1)M^{i-2}\|g'-h'\|_{C^0}$$

$$+ M^*\sum_{i=2}^{n}\sum_{j=1}^{i-2} Q(i-1)(i-j-1)M^{i+j-3}\|g-h\|_{C^0}.$$

$$\|(g_x^{-1})'-(h_x^{-1})'\|_{C^0}=\max_{t\in I}\left|\frac{1}{g_x'(g_x^{-1}(t))}-\frac{1}{h_x'(h_x^{-1}(t))}\right|$$

$$\leq \frac{1}{\alpha^2}\max_{t\in I}\left|g_x'(g_x^{-1}(t))-h_x'(h_x^{-1}(t))\right|$$

$$\leq \frac{1}{\alpha^2}\max_{t\in I}\left[|g_x'(g_x^{-1}(t))-g_x'(h_x^{-1}(t))|+|g_x'(h_x^{-1}(t))-h_x'(h_x^{-1}(t))|\right]$$

$$\leq \frac{1}{\alpha^2}[K_2\|g_x^{-1}-h_x^{-1}\|_{C^0}+\|g_x'-h_x'\|_{C^0}]$$

$$\leq \frac{1}{\alpha^2}\left[\frac{K_2}{\alpha}\|g_x-h_x\|_{C^0}+\|g_x'-h_x'\|_{C^0}\right]$$

$$\leq \frac{K_2\kappa}{\alpha^3}\sum_{i=1}^{n}\|\alpha_i-\mu_i\|_{C^0}+\frac{1}{\alpha^2}\sum_{i=1}^{n}M^{i-1}\|\alpha_i-\mu_i\|_{C^0}$$

$$+\left[\frac{K_2}{\alpha^3}\sum_{i=2}^{n}\sum_{j=1}^{i-1}M^{j-1}+\frac{M^*}{\alpha^2}\sum_{i=2}^{n}\sum_{j=1}^{i-2}Q(i-1)(i-j-1)M^{i+j-3}\right]\|g-h\|_{C^0}$$

$$+\frac{1}{\alpha^2}\sum_{i=2}^{n}(i-1)M^{i-2}\|g'-h'\|_{C^0}.$$

Thus

$$\|g-h\|_{C^1}=\|g-h\|_{C^0}+\|g'-h'\|_{C^0}=\max_{x\in I}\left\{|g_x^{-1}(G(x))-h_x^{-1}(H(x))|\right\}$$

$$+\max_{x\in I}\left\{|(g_x^{-1})'(G(x))G'(x)-(h_x^{-1})'(H(x))H'(x)|\right\}$$

$$\leq \max_{x\in I}\left\{|g_x^{-1}(G(x))-h_x^{-1}(G(x))|+|h_x^{-1}(G(x))-h_x^{-1}(H(x))|\right\}$$

$$+\max_{x\in I}\Big\{|(g_x^{-1})'(G(x))-(h_x^{-1})'(G(x))|G'(x)$$

$$+|(h_x^{-1})'(G(x))-(h_x^{-1})'(H(x))|G'(x)+(h_x^{-1})'(H(x))|G'(x)-H'(x)|\Big\}$$

$$
\begin{aligned}
&\leq \left\| g_x^{-1} - h_x^{-1} \right\|_{C^0} + \frac{1}{\alpha} \| G - H \|_{C^0} + \alpha M \left\| (g_x^{-1})' - (h_x^{-1})' \right\|_{C^0} \\
&\quad + \alpha M \cdot \frac{K_2}{\alpha^3} \| G - H \|_{C^0} + \frac{1}{\alpha} \| G' - H' \|_{C^0} \\
&\leq \frac{1}{\alpha} \| g_x - h_x \|_{C^0} + \alpha M \left\{ \frac{K_2 \kappa}{\alpha^3} \sum_{i=1}^{n} \| \alpha_i - \mu_i \|_{C^0} + \frac{1}{\alpha^2} \sum_{i=1}^{n} M^{i-1} \| \alpha_i - \mu_i \|_{C^0} \right. \\
&\quad + \left[\frac{K_2}{\alpha^3} \sum_{i=2}^{n} \sum_{j=1}^{i-1} M^{j-1} + \frac{M^*}{\alpha^2} \sum_{i=2}^{n} \sum_{j=1}^{i-2} Q(i-1)(i-j-1) M^{i+j-3} \right] \| g - h \|_{C^0} \\
&\quad \left. + \frac{1}{\alpha^2} \sum_{i=1}^{n} (i-1) M^{i-2} \| g' - h' \|_{C^0} \right\} + \frac{\alpha + K_2 M}{\alpha^2} \| G - H \|_{C^1} \\
&\leq \left(\frac{\kappa}{\alpha} + \frac{K_2 \kappa M}{\alpha^2} \right) \sum_{i=1}^{n} \| \alpha_i - \mu_i \|_{C^0} + \frac{M}{\alpha} \sum_{i=1}^{n} M^{i-1} \| \alpha_i - \mu_i \|_{C^0} \\
&\quad + \Theta \| g - h \|_{C^1} + \frac{\alpha + K_2 M}{\alpha^2} \| G - H \|_{C^1}.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\| g - h \|_{C^1} \leq{}& \frac{\kappa \alpha + K_2 \kappa M}{\alpha^2 (1 - \Theta)} \sum_{i=1}^{n} \| \alpha_i - \mu_i \|_{C^0} + \frac{1}{\alpha (1 - \Theta)} \sum_{i=1}^{n} M^i \| \alpha_i - \mu_i \|_{C^0} \\
&+ \frac{\alpha + K_2 M}{\alpha^2 (1 - \Theta)} \| G - H \|_{C^1}.
\end{aligned}
$$

We now may conclude that the solution of (1) depends continuously on the function F and $\lambda_j (j = 1, 2, \ldots, n)$. This completes the proof. □

Remark 2. During the proof of Theorem 1–Theorem 3, the differentiability of $\lambda_i(x) (i = 1, 2, \ldots, n)$ are not required.

5. Example

In this section we show the conditions in Theorem 1 do not self-contradict by means of an example. Consider the following equation

$$
\lambda_1(x) \varphi(x) + \lambda_2(x) \varphi(\varphi(x)) = F(x), \quad x \in I = [0, 1], \tag{25}
$$

where

$$\lambda_1(x)=\frac{2-(2-\alpha)x}{2-x},\quad \lambda_2(x)=\frac{(1-\alpha)x}{2-x},$$
$$0<\alpha<1,\quad F(x)=\ln(1+x)-x\ln\frac{2}{e}.$$

Obviously, $\lambda_1(x)+\lambda_2(x)=1$. Since

$$0\le\lambda_2(x)=\frac{(1-\alpha)x}{2-x}\le 1-\alpha<1,$$
$$\alpha\le\lambda_1(x)=1-\lambda_2(x)\le 1.$$

Moreover, we also have

$$F(0)=0,F(1)=1,$$

and

$$0<F'(x)=\frac{1}{x+1}-\ln\frac{2}{e}$$
$$\le 2-\ln 2=\alpha M\Big(M=\frac{1}{\alpha}(2-\ln 2)\Big),$$
$$|F'(x_1)-F'(x_2)|=\left|\frac{1}{x_1+1}-\frac{1}{x_2+1}\right|\le|x_1-x_2|,$$
$$|(\lambda_1(x))'|=|(1-\lambda_2(x))'|=|\lambda_2(x))'|=\frac{2(1-\alpha)}{(2-x)^2}\le 2(1-\alpha).$$

Choose $M'=1$, $\beta_1=2(1-\alpha)$, $\beta_2=2(1-\alpha)$.

On the other hand, since

$$\lim_{\alpha\to 1}\left[(1-\alpha)\Big(M+\frac{\kappa}{\alpha}\sum_{i=1}^{2}\beta_i\Big)M\right]=\lim_{\alpha\to 1}\left[(1-\alpha)\Big(M+\frac{4(1-\alpha)}{\alpha}\Big)M\right]=0,$$

there is a positive constant $\Lambda<1$ such that

$$\alpha>(1-\alpha)\Big(M+\frac{\kappa}{\alpha}\sum_{i=1}^{2}\beta_i\Big)M$$

for any $\alpha \in (\Lambda, 1)$. Namely, the conditions of Theorem 1 are satisfied for any $\alpha \in (\Lambda, 1)$.

We have thus shown that there will be a solution of (25) in $\Omega(M, M^*; I)$ $\left(M^* \geq \frac{1+2(1-\alpha)M(1+M)}{\alpha-(1-\alpha)M\left[M+\frac{4}{\alpha}(1-\alpha)\right]}\right)$.

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