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# On the characterization of additive functions on Gaussian integers

By KATALIN KOVÁCS (Budapest)

**Abstract.** Let $f$ denote an additive function on the Gaussian integers. We prove some theorems of characterization with linear and quadratic arguments. If e.g. for a completely additive function $f(a\alpha+b)-tf(\alpha)=c$ or $f(\alpha^2+1)+f(\alpha^2-1)=c$, then $f(\alpha)=0$ for all nonzero Gaussian integers $\alpha$.

In 1946 Erdős [2] proved the following theorems:

**Theorem 1** (Erdős). *Let $f$ be a real valued additive function. If $f(n+1)-f(n)\to 0$, then $f(n)=c\log n$ for all $n\in\mathbb{N}$.*

**Theorem 2** (Erdős). *If a real valued additive function $f$ is monotonically increasing, then $f(n)=c\log n$ for all $n\in\mathbb{N}$.*

I. Kátai [3] generalized Theorem 1 for completely additive functions using a result of E. Wirsing [6]:

**Theorem 3** (Kátai). *Let $f$ be a completely additive function. If $\sum_{i=1}^{m} c_i f(n+a_i)=o(\log n)$, then $f(n)=c\log n$ for all $n\in\mathbb{N}$ or $f=0$.*

The following generalizations are due to P.D.T.A. Elliott [1] and myself ([4], [5]):

**Theorem 4** (Elliott, [1]). *Let $f$ be an additive function, $A>0$, $C>0$, $B$, $D$ integers and $\Delta_1=AC(AD-BC)\neq 0$. If $f(An+B)-f(Cn+D)\to c$, then $f(n)=c'\log n$ for all $(n,\Delta_1)$.*

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**Theorem 5** [5]. *Let $f$ be a completely additive function. If $f(2n+A)-f(n)$ is monotonic from some number on, then $f(n) = c\log n$ for all $n \in \mathbb{N}$.*

**Theorem 6** [4]. *Let $f$ denote a completely additive function. If $f(n^2+1) = s_1 f(n) + s_2 f(n-1) + o(\log n)$ ($s_1$, $s_2$ are not both zero), then $f(n) = c\log n$.*

In this article we intend to prove some similar results on the set of the Gaussian integers. Let $G^*$ denote the set of the nonzero Gaussian integers. Let $\alpha$, $\beta$ be the elements of this set and $N(\alpha) := \alpha\overline{\alpha}$.

*Definition 1.* The function $f$ is G-additive, if $f(\alpha\beta) = f(\alpha) + f(\beta)$ for all relatively prime $\alpha, \beta \in G^*$.

*Definition 2.* The function $f$ is completely G-additive, if $f(\alpha\beta) = f(\alpha) + f(\beta)$ for all $\alpha, \beta \in G^*$.

*Remark.* We can prove easily that $f(\epsilon) = 0$ for any additive $f$ and arbitrary Gaussian unit $\epsilon$.

We prove the following results:

**Theorem 7.** *Let $a$, $b$ denote some fixed elements of $G^*$ and let $t \in \mathbb{C} \setminus \{0\}$.*

(i) *If for a G-additive function*

$$f(a\alpha + b) - tf(\alpha) \to c, \tag{1}$$

*then $f(n) = c' \log n$ for all $n \in \mathbb{N}^+$ coprime to $2N(ab)$.*

(ii) *If for a completely G-additive function*

$$f(a\alpha + b) - tf(\alpha) = c, \tag{1'}$$

*then $f(\alpha) = 0$ for all $\alpha \in G^*$.*

**Theorem 8.** *If for a completely G-additive function*

$$f(\alpha^2+1) + f(\alpha^2-1) = c,$$

*then $f(\alpha) = 0$ for all $\alpha \in G^*$.*

**Theorem 9.** (i) *If for a completely G-additive function*

$$f(\alpha^2+1) = s_1 f(\alpha) + s_2 f(\alpha-1) + o(\log N(\alpha)) \quad (s_1, s_2 \text{ are not both zero}),$$

*then* $f(z) = 0$ *for all* $z \in \mathbb{Z} \setminus \{0\}$.

(ii) *If for a completely G-additive function*

$$f(\alpha^2+1) = s_1 f(\alpha) + s_2 f(\alpha-1) + c \quad (s_1, s_2 \text{ are not both zero}),$$

*then* $f(\alpha) = 0$ *for all* $\alpha \in G^*$.

## Proofs

Proof of Theorem 7. (i) If $f$ is G-additive, then by substituting $\overline{a}bN(b)\alpha$ into (1) we have

$$f(N(ab)\alpha+1) - tf(\alpha) \to C', \tag{2}$$

for all $\alpha$ coprime to $N(ab)$ with $C' = c + tf(\overline{a}bN(b)) - f(b)$. By substituting $2\alpha$ into (2) we have

$$f(2N(ab)\alpha+1) - tf(\alpha) \to C'' \tag{3}$$

for all $\alpha$ coprime to $2N(ab)$ with $C'' = c + tf(2\overline{a}bN(b)) - f(b)$. The difference of (3) and (2) shows that

$$f(2N(ab)\alpha+1) - f(N(ab)\alpha+1) \to C''' \tag{4}$$

for all $\alpha$ coprime to $2N(ab)$. By substituting $2N(ab)\alpha+1$ into (4) we have that

$$f(4N^2(ab)\alpha + 2N(ab) + 1) - f(2N^2(ab)\alpha + N(ab) + 1) \to C'''$$

for all $\alpha \in G^*$. Applying Theorem 4 we get that $f(n) = c' \log n$ for all $n \in \mathbb{N}$ coprime to $2N(ab)$.

(ii) Let $f$ be a completely G-additive function.

By substituting $b\alpha$ into (1') we have

$$f(a\alpha+1) = tf(\alpha) + c_1 \tag{5}$$

with $c_1 = c + (t-1)f(b)$. Therefore

$$f((a\alpha+1)^2) = f(a[\alpha(a\alpha+2)]+1) = tf(\alpha) + tf(a\alpha+2) + c_1$$

and

$$f((a\alpha+1)^2) = 2f(a\alpha+1) = 2tf(\alpha) + 2c_1,$$

i.e.

$$f(a\alpha+2) = f(\alpha) + c_1/t. \tag{6}$$

By substituting $2\alpha$ into (6) we get

$$f(a\alpha+1) = f(\alpha) + c_1/t. \tag{7}$$

By the comparison of (5) and (7) $f$ is constant, i.e. $f(\alpha) = 0$ for all $\alpha \in G^*$ or $t = 1$. If $t = 1$, then we also prove , that $f(\alpha) = 0$ for all $\alpha \in G^*$. First we prove by induction, that

$$f(a\alpha+s) = f(\alpha) + c_1 \tag{8}$$

for all $s \in \mathbb{N}^+$. For $s = 1$ it is true by (7). By the assumption of induction for $s \leq z$ we have

$$f((a\alpha+1)(a\alpha+z)) = f(a\alpha+1) + f(a\alpha+z) = 2f(\alpha) + 2c_1$$

and

$$f(a[\alpha(a\alpha+z+1)]+z) = f(\alpha) + f(a\alpha+z+1) + c_1,$$

which follow $f(a\alpha+z+1) = f(\alpha) + c$. By substituting $s = a$ into (8) we have $f(\alpha+1) = f(\alpha) + c_1 - f(a)$. By restricting $\alpha$ to the natural numbers $f(n) - f(n-1) = c_1 - f(a)$ for all $n \in \mathbb{N}$. By substituting $n^2$ here we have $c_1 - f(a) = f(n^2) - f(n^2-1) = [f(n) - f(n-1)] - [f(n+1) - f(n)] = 0$, i.e. $c_1 = f(a)$. Applying Theorem 1 $f(n+1) = f(n)$ follows $f(n) = 0$ for all $n \in \mathbb{N}$ . Using that $f(\alpha+1) = f(\alpha)$ for all $\alpha \in G^*$, we prove by induction that $f(\delta) = 0$ also for all not real Gaussian primes $\delta$. By the Remark it is enough to consider the Gaussian primes of form $\pi = 1 + i$ and $\pi = x + yi$ with even number $x$ and odd number $y$ as $f(\pi) + f(\overline{\pi}) = f(N(\pi)) = 0$ and $f(i\pi) = f(-i\pi) = f(-\pi) = f(\pi)$. We have $0 = f(2) = 2f(1+i)$. For any other $\pi$ the Gaussian integer $\pi - 1$ is divisible by $1 + i$, i.e. it is not a Gaussian prime. We assume that $f(\gamma) = 0$ for all Gaussian-primes which norm is less then $f(\pi)$. As $f(\pi) = f(\pi - 1)$ and $f(\beta) = 0$ for all prime divisors $\beta$ of $\pi - 1$ by the hypotesis of the induction, therefore $f(\pi) = 0$ is also satisfied as $f$ is a completely G-additive function. □

Proof of Theorem 8. As $\alpha^2 + 1 = (\alpha + i)(\alpha - i)$, we have

$$f(\alpha + i) + f(\alpha - i) + f(\alpha + 1) + f(\alpha - 1) = c. \tag{9}$$

By substituting $\alpha - i$ into (9), $(1 + i)\alpha$ into (10), $2\alpha + i$ and $(1 - i)\alpha + i$ into (11) we have that

$$f(\alpha) + f(\alpha - 2i) + f(\alpha + 1 - i) + f(\alpha - 1 - i) = c, \tag{10}$$

$$f(\alpha) + f(\alpha - 1 - i) + f(\alpha - i) + f(\alpha - 1) = c_1, \tag{11}$$

$$f(2\alpha + i) + f(2\alpha - 1) + f(\alpha) + f(2\alpha - 1 + i) = c_2 \tag{12}$$

and

$$f(2\alpha - 1 + i) + f(2\alpha - 1 - i) + f(\alpha) + f(\alpha - 1) = c_3. \tag{13}$$

The difference of (12) and (13) shows that

$$f(2\alpha + i) - f(2\alpha - 1 - i) + f(2\alpha - 1) - f(\alpha - 1) = c_4. \tag{14}$$

By substituting $2\alpha$ into (11) and (9) we have

$$f(\alpha) + f(2\alpha - 1 - i) + f(2\alpha - i) + f(2\alpha - 1) = c_1 \tag{15}$$

and

$$f(2\alpha + i) + f(2\alpha - i) + f(2\alpha + 1) + f(2\alpha - 1) = c. \tag{16}$$

The difference of (15) and (16) shows that

$$f(2\alpha + i) - f(2\alpha - 1 - i) + f(2\alpha + 1) - f(\alpha) = c_5. \tag{17}$$

By the comparison of (14) and (17) we get

$$f(2\alpha + 1) - f(\alpha) = f(2\alpha - 1) - f(\alpha - 1) + c_6. \tag{18}$$

By restricting $\alpha$ to natural numbers (18) follows that $f(2n + 1) - f(n)$ is monotonic. Applying Theorem 5 we obtain $f(n) = 0$ for all $n \in \mathbb{N}$ and also that $c_6 = 0$. We prove by induction that $f(\pi) = 0$ also for all not real Gaussian primes. As the norm of $2\alpha+1$ is greater than the norm of any other argument in (18) and the minimal value of $N(2\alpha+1)$ is 13, it is enough to verify that $f(1+i)=f(2+i)=f(2+3i)=0$ using the Remark. It is true as $0=f(2)=2f(1+i)$ and $\alpha=1+i$ and $\alpha=i$ in (18) imply $f(1+2i)=f(3+2i)$ and $f(1+2i)=f(1+i)$. □

Proof of Theorem 9. (i) is a direct consequence of Theorem 6 and the Remark.

(ii) By induction we prove that $f(x+yi)=0$ for all $x\in\mathbb{Z}$ and $y\in\mathbb{N}$. For $y=0$ it is true by (i) ($x$ can be arbitrarily choosen). Let us assume that it is true for all $0\le y\le s$. If we substitute $x+si$ in the condition of the theorem we get

$$f(x+(s+1)i)+f(x+(s-1)i)=sf(x+si)+tf(x-1+si),$$

i.e. by the assumption of the induction $f(x+(s+1)i)=0$. If $y<0$, then $f(x+iy)=f(-x-iy)=0$ as $-y\in\mathbb{N}$. □

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KATALIN KOVÁCS
EÖTVÖS UNIVERSITY
DEPT. OF ALGEBRA AND NUMBER THEORY
H–1053, BUDAPEST
KECSKEMÉTI U. 10–12
HUNGARY

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