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On the Daróczy–Kátai-Birthday expansion

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Abstract. The recently introduced Daróczy–Kátai-Birthday (DKB) expansion is a special Oppenheim expansion with fundamental inequality $d_{n+1} \geq d_n^3 - d_n^2 + 1$ for the denominators. We refer to expansions in which ultimately equation holds in the fundamental inequality as series with minimal growth rate. It is shown that the minimum growth rate is the same magnitude as the growth rate valid with probability one. Furthermore, we establish the surprising result that, for almost all x in $(0,1)$, $d_{n+1} \geq d_n^3 + \frac{d_n^3}{(\log d_n)^c}$ with some constant $c > 0$.

Introduction

The Oppenheim series expansion of real numbers $0 < x \leq 1$ is defined by the algorithm: d_n, $n \geq 1$ are integers

$$x = x_1, \quad \frac{1}{d_n} < x_n \leq \frac{1}{d_n - 1} \tag{1}$$

and

$$x_{n+1} = \left(x_n - \frac{1}{d_n} \right) \frac{1}{r_n(d_n)}, \tag{2}$$

where $r_n(j) = \frac{h_n(j)}{j(j-1)}$, and $h_n(j) \geq 1$ is an integer valued function, $n \geq 1$, $j \geq 1$. The integers $d_n = d_n(x)$ are called the digits of the expansion, and they satisfy the fundamental inequality:

$$d_{n+1} \geq h_n(d_n) + 1, \quad n \geq 1. \tag{3}$$

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The theory of Oppenheim series was developed by J. Galambos in a series of papers, which is presented in a unified way in the book Galambos [1]; see particularly sections 1.3, 2.2, 5.3, and all sections of Chapter VI.

In a recent paper, Galambos [2] introduced and studied the special case $h_n(j) = j^2(j-1)$, and called the resulting algorithm and series expansion Daróczy–Kátai-Birthday expansion, or DKB-expansion. Repeated applications of (1) and (2) lead to the series representation.

$$x = \frac{1}{d_1} + \frac{d_1}{d_2} + \frac{d_1 d_2}{d_3} + \cdots + \frac{d_1 d_2 \cdots d_n}{d_{n+1}} + \cdots \tag{4}$$

in which (3) takes the form

$$d_{n+1} \geq d_n^2(d_n - 1) + 1 = d_n^3 - d_n^2 + 1. \tag{3a}$$

It also follows from the cited general theory of Oppenheim series that if x is given by a series of the form of (4) satisfying (3a) then (4) is produced by the algorithm (1) and (2). In particular, if at (4) $d_n = V_n$ with V_{n_0} fixed,

$$V_{n+1} = V_n^3 - V_n^2 + 1 \quad \text{for all } n \geq n_0 \tag{5}$$

and (3a) holds for $n < n_0$, then (4) is a DKB-expansion.

The sequence generated in (5) will be referred to as a DKB-series with minimum growth rate. In the present paper I study the sequence V_n of minimal growth rate as defined at (5) and compare with the result

$$\lim_{n\to\infty} \frac{\log d_n(x)}{3^n} = G(x), \quad G(x) > 0 \text{ finite}, \tag{6}$$

(see Galambos [1], Theorem 6.13).

In contrast to the minimal growth for d_n, one can consider the extent to which the fundamental inequality (3a) governs the growth rate for almost all x. In this direction I shall show that, for almost all x,

$$d_{n+1} \geq d_n^3 + \frac{d_n^3}{(\log d_n)^c} \quad \text{for all } n \geq n_0, \tag{7}$$

with some constant $c = c(x) > 0$. For proving (7), a general representation of d_{n+1}, as quoted in Galambos [2] at (12), p. 380, will be the major tool, together with a Borel–Cantelli-type of argument. In all statements, probability is Lebesgue measure on the Borel sets of $(0, 1]$.

The results and proofs

Theorem 1. *For the sequence V_n of integers, defined at (5) with $n_0 = 1$ and $V_1 = 2$, we have, as $n \to +\infty$*

$$0 < \lim_{n\to\infty} 3^{-n} \log V_n < +\infty.$$

Proof of Theorem 1. From

$$3^{-(n+1)} \log V_{n+1} - 3^{-n} \log V_n = 3^{-(n+1)} \log \left(1 - \frac{V_n^2 - 1}{V_n^3}\right) < 0$$

it follows that $3^{-n} \log V_n$ decreases for increasing n and converges to a positive limit V. Furthermore

$$V_{n+1} - \frac{2}{3} > \left(V_n - \frac{2}{3}\right)^3$$

so

$$\log V_n > \log \left(V_n - \frac{2}{3}\right) > 3^{n-k} \log \left(V_k - \frac{2}{3}\right) \quad \text{for } k = 1, 2, \dots, n.$$

Applying this formula $k = 1$ we obtain

$$\log V_n > 3^n \frac{1}{3} \log \frac{4}{3}$$

so

$$V = \lim_{n\to\infty} 3^{-n} \log V_n \ge \frac{1}{3} \log \frac{4}{3} > 0.$$

This completes the proof. □

For the next result I quote a representation of d_n from Galambos [2], p. 380. The sequence $d_n = d_n(x)$ has the same distributional properties as the stochastic sequence D_n, $n \ge 1$, defined as $D_1 = [\exp(x_1)] + 1$, and

$$D_{n+1} = \left[(D_n^3 - D_n^2) \exp(x_{n+1})\right] + 1, \quad n \ge 1, \tag{8}$$

where $x_1, x_2, \dots$ are independent unit exponential variables, that is, $F(x) = P(x_j \le x) = 1 - e^{-x}$, $x \ge 0$, where P is any probability measure on an arbitrary abstract probability space. The space, of course, can be chosen as Borel sets of $(0, 1]$, P as Lebesgue measure and $x_j = -\log u_j$, where $0 < u_j \le 1$ are independent uniformly chosen points in $(0, 1]$. So, we can use the same underlying probability space for (8) as for (1) and (2).

Theorem 2. *For almost all x, there is an $n_0 = n_0(x)$ such that*

$$d_{n+1} \geq d_n^3 + \frac{d_n^3}{(\log d_n)^c}, \quad \text{for all } n \geq n_0$$

with some constant $c > 0$.

Proof of Theorem 2. In order to emphasize the role of the fundamental inequality (3a), we determine $c_k = c_k(x)$ such that c_k is of smaller magnitude than d_n^3 and

$$d_{k+1} \leq d_k^3 - d_k^2 + c_k + 1 \tag{9}$$

should not hold for infinitely many k. We turn to (8), and calculate the probability of

$$D_{k+1} = \left[(D_k^3 - D_k^2)\exp(x_{k+1})\right] + 1 \leq D_k^3 - D_k^2 + c_k + 1. \tag{10}$$

From the above inequality we obtain

$$x_{k+1} \leq \log\left(1 + \frac{c_k}{D_k^3 - D_k^2}\right).$$

Since D_k is defined through (8) by $x_1, x_2, \ldots, x_k$, the random variable D_k and x_{k+1} are independent. Consequently, upon conditioning on D_k, x_{k+1} remains exponential, and thus

$$\begin{aligned} P&\left(x_{k+1} \leq \log\left(1 + \frac{c_k}{D_k^3 - D_k^2}\right) \,\Big|\, D_k = y\right) \\ &= 1 - \exp\left\{-\log\left(1 + \frac{c_k}{y^3 - y^2}\right)\right\} = 1 - \frac{1}{1 + \frac{c_k}{y^3 - y^2}}. \end{aligned}$$

Let us take expectations; we get

$$P\left(x_{k+1} \leq \log\left(1 + \frac{c_k}{D_k^3 - D_k^2}\right)\right) = E\left(1 - \frac{1}{1 + \frac{c_k}{D_k^3 - D_k^2}}\right).$$

But D_k and d_k are identically distributed, which entails

$$P\left(x_{k+1} \leq \log\left(1 + \frac{c_k}{D_k^3 - D_k^2}\right)\right) = E\left(1 - \frac{1}{1 + \frac{c_k}{d_k^3 - d_k^2}}\right). \tag{11}$$

Let us choose $c_k = \frac{d_k^3}{(\log d_k)^c}$, with some $c > 0$. Then (11) implies

$$P\left(x_{k+1} \leq \log\left(1 + \frac{c_k}{D_k^3 - D_k^2}\right)\right) = E\left(\frac{c_k}{d_k^3 - d_k^2}(1 + o(1))\right),$$

and thus

$$\sum_{k=1}^{+\infty} P\left(x_{k+1} \leq \log\left(1 + \frac{c_k}{D_k^3 - D_k^2}\right)\right) = E\left\{\sum_{k=1}^{+\infty} \frac{c_k}{d_k^3 - d_k^2}(1 + o(1))\right\}, \tag{12}$$

where we interchanged summation and taking expectation on the right hand side (allowed because the terms are positive) and we note that the error terms $o(1)$ are irrelevant for our purposes since our only aim with (12) is to establish that the right hand side of (12) is finite (so, the boundedness of $o(1)$ suffices for us). Now, if we put

$$a_k = \frac{c_k}{d_k^3 - d_k^2} = \frac{d_k^3}{(d_k^3 - d_k^2)(\log d_k)^c},$$

we apply the ratio test. We have from (6),

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{c_{k+1}}{d_{k+1}^3 - d_{k+1}^2} \frac{d_k^3 - d_k^2}{c_k} \\
&= \frac{1 - \frac{1}{d_k}}{1 - \frac{1}{d_{k+1}}} \left(\frac{\frac{\log d_k}{3^k}}{\frac{\log d_{k+1}}{3^{k+1}}}\right)^c \left(\frac{1}{3}\right)^c \longrightarrow \left(\frac{1}{3}\right)^c < 1 \quad \text{as } k \to +\infty.
\end{aligned}$$

This rate of convergence guarantees not only that the sum on the right hand side of (12) converges but that its integral (expectation) is finite as well. We thus have that the left hand side of (12) is finite, implying via the Borel–Cantelli lemma that, with our choice of c_k, (10) can occur only a finite number of times with probability one. Since D_k and d_k are identically distributed, the theorem is established since

$$d_{k+1} \geq d_k^3 - d_k^2 + c_k + 1$$

with some $c > 0$ in $c_k = \frac{d_k^3}{(\log d_k)^c}$ implies that

$$d_{k+1} \geq d_k^3 + \frac{d_k^3}{(\log d_k)^c}$$

with another $c > 0$.

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