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On c-normal subgroups of finite groups

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Abstract. A subgroup H of a finite group G is said to be c-normal in G if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N \leq H_G = \mathrm{core}_G(H)$. In this paper, we further investigate the influence of c-normality of some subgroups in finite groups. Some recent results are generalized.

1. Introduction

There has been much interest in the past in investigating the relationship between the properties of maximal subgroups of a finit group G and the structure of G [for example 1, 2, 3]. In this aspect, the concept of a c-normal subgroup in a finite group was introduced by WANG in [4] and he proved that a finite group G is solvable if and only if M is c-normal in G for every maximal subgroup M of G. As an application of the above result, some known theorems were generalized by using the concept "c-normality". Thus, c-normality provides a useful tool for the investigation of the structure of finite groups, which is further shown in [5], [6].

In this paper, we shall continue to study the c-normality of some subgroups in a finite group G. Some theorems of solvable groups, p-nilpotent groups and supersolvable groups are obtained by considering their c-normal subgroups. Some results in [3]–[6] are extended and generalized. Throughout this paper, all groups are finite groups. Our terminology and notation are standard, see e.g. ROBINSON [7]. We write $M < \cdot G$ to indicate that M is a maximal subgroup of G.

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2. Basic definitions and preliminary results

A subgroup H of a group G is said to be c-normal in G if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N \leq H_G = \mathrm{core}_G(H)$.

If $M < \cdot G$ and $H < \cdot M$, then H is called a 2-maximal subgroup of G.

Consider the following families of subgroups:

$$\mathcal{F}_c = \{M \mid M < \cdot G \text{ with } |G : M| \text{ is composite}\}$$

$$\mathcal{F}^p = \{M \mid M < \cdot G, N_G(P) \leq M \text{ for a } P \in \mathrm{Syl}_p(G)\}$$

$$\mathcal{F}^s = \bigcup_{p \in \pi(G)} \mathcal{F}^p$$

$$\mathcal{F}^{sc} = \mathcal{F}^s \cap \mathcal{F}_c$$

and define

$$S^s(G) = \bigcap \{M \mid M \in \mathcal{F}^{sc}\}$$

if $\mathcal{F}^{sc}$ is non-empty; otherwise $S^s(G) = G$.

For the sake of convenience, we list here some known results which will be useful in the sequel.

Lemma 2.1 [4, Lemma 2.1]**.** *Let G be a group. Then the following statements hold.*

1) *If H is c-normal in G such that $H \leq K \leq G$, then H is c-normal in K.*

2) *Let $K \trianglelefteq G$ and $K \leq H$, then H is c-normal in G if and only if H/K is c-normal in G/K.*

Lemma 2.2 [4, Lemma 2.4(b)]**.** *A group G is supersolvable if and only if $G = S^s(G)$.*

Lemma 2.3 [6,Lemma 2.4]**.** *Let H be a subgroup of G, then H is c-normal in G if and only if there exists a normal subgroup N of G such that $G = HN$ and $H \cap N = H_G = \mathrm{core}_G(H)$.*

Lemma 2.4 [5, Lemma 2.4]. *Let π be a set of primes, H a normal π'-subgroup of G and T a π-subgroup of G. If T is c-normal in G, then TH/H is c-normal in G/H.*

A class of groups $\mathcal{F}$ is called a formation provided that the following conditions are satisfed:

(1) $\mathcal{F}$ contains all homomorphic images of a group G in $\mathcal{F}$,

(2) if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is also in $\mathcal{F}$ for normal subgroups M and N of G.

Now let P be the set of all prime numbers. By a formation function f, we mean a function f defined on P such that $f(P)$, possibly empty, is a formation. A principal factor H/K of a group G is called f-central in G if $G/C_G(H/K) \in f(p)$ for all primes p dividing $|H/K|$. A formation $\mathcal{F}$ is said to be a local formation if there exists a formation function f such that $\mathcal{F}$ is the class of all groups G for which every principal factor of G is f-central in G. If $\mathcal{F}$ is a local formation defined by a formation function f, then we write $\mathcal{F} = LF(f)$ and call f a local definition of $\mathcal{F}$.

Among all possible local definitions for a local formation $\mathcal{F}$, there exists exactly one of them, denoted it by F, such that F is integrated (i.e. $F(p) \subseteq \mathcal{F}$ for all $p \in P$) as well as full (i.e. $S_pF(p) = F(p)$ for all $p \in P$).

A formation $\mathcal{F}$ is called saturated if $G/\Phi(G) \in \mathcal{F}$ implies that G belongs to $\mathcal{F}$. It is well known that a formation $\mathcal{F}$ is saturated if and only if $\mathcal{F}$ is a local formation.

Lemma 2.5 [8, Proposition IV. 3.11]. *Let $\mathcal{F}_1 = LF(F_1)$ and $\mathcal{F}_2 = LF(F_2)$, where F_i is both an integrated and full formation function of $\mathcal{F}_i$ $(i = 1, 2)$. Then the following statements are equivalent.*

(1) $\mathcal{F}_1 \subseteq \mathcal{F}_2$,

(2) $F_1(p) \subseteq F_2(p)$ *for all* $p \in P$.

3. Main results

In this section, we concentrate on the structure of a finite group under the influence of its c-normal subgroups. Some theorems for solvable groups, p-nilpotent groups and supersolable groups are obtained.

Our first theorem is to give a characterization theorem for solvable groups. This result generalizes Theorem 3.5 in [4] and Theorem 3.1 in [5] by minimizing the number of the the restricted maximal subgroups of a finite groups.

Theorem 3.1. *A group G is solvable if and only if every non-nilpotent maximal subgroup M in $\mathcal{F}^{sc}$ is c-normal in G.*

Proof. Since the necessity part is straightforward from [4, Theorem 3.5], we only need to prove the sufficient part. For this purpose, we suppose that the theorem is not true and let G be a minimal counterexample.

If $\mathcal{F}^{sc} = \emptyset$, then $G = S^s(G)$ and so G is supersolvable by Lemma 2.2. Now assume that $\mathcal{F}^{sc} \neq \emptyset$ and $M \in \mathcal{F}^{sc}$. If M itself is nilpotent, then, by the well known Tompson's Theorem [7, Theorem 10.4.2], M must be a group of even order. By [2, Theorem 1], $M_{2'}$ (the $2'$-Hall subgroup of M) is a normal subgroup in G. Trivially, the hypotheses is quotient closed. If $M_{2'} \neq 1$, then by the minimality of G and Lemma 2.1, we have that $G/M_{2'}$ is solvable and consequently G is solvable since $M_{2'}$ is nilpotent. Hence we may assume that $M_{2'} = 1$ and therefore M is a Sylow 2-subgroup of G if $M \in \mathcal{F}^{sc}$ and M is nilpotent.

Let p be the largest prime in $\pi(G)$ and $P \in \mathrm{Syl}_p(G)$. The choice of G implies that $N_G(P) < G$. Hence there exists a maximal subgroup L of G such that $N_G(P) \leq L$. If $L_G \neq 1$, then G is of course not simple. Now assume that $L_G = 1$. If $[G : L] = q$ is a prime number, then, since $L_G = 1$, $G = G/L_G$ is a homomorphic image of S_q, the symmetric group of degree q. Thus $|G| \mid q!$ and q is the largest prime of $\pi(G)$. It follows that $q = p$. Hence $p \nmid [G : L]$ since $P \leq L$, which is a contradiction. If $[G : L]$ is composite, then since L is not nilpotent by the above proof, we have, by our hypotheses, there exists a normal subgroup K of G such that $G = LK$ and $L \cap K \leq L_G$. This implies that G is not simple. By using induction and in virtue of the fact that if there were two minimal normal subgroups N_1 and N_2 of G, then G can be embedded in $G/N_1 \times G/N_2$. We can easily see that N is the unique minimal normal subgroup of G and G/N is solvable.

Let q be the largest prime of $\pi(N)$ and $Q_1 \in \mathrm{Syl}_q(N)$. By our choice of G, we have $Q_1 < N$ and Q_1 is not normal in G. Hence there exists a maximal subgroup L of G such that $N_G(Q_1) \leq L$. By using the Frattini argument, we have $G = NN_G(Q_1) = NL$. Now, consider $Q \in \mathrm{Syl}_q(G)$ with $Q_1 \leq Q$ so that $Q_1 = Q \cap N$. Then, for any $x \in N_G(Q)$, we have $Q_1^x = (N \cap Q)^x = N \cap Q = Q_1$. It follows that $N_G(Q) \leq N_G(Q_1) \leq L$. If $[G : L] = r$ is a prime, then, since $L_G = 1$, we have $G = G/L_G$ is a homomorphic image of S_r, the symmetric group of degree r. This shows that $|G| \mid r!$ and r is the largest prime of $\pi(G)$, thereby we obtain $r = p$. As $[G : L] = [N : N \cap L]$, it leads to p is a prime factor of $|N|$, and hence $p = q$. By $N_G(Q) \leq N_G(Q_1) \leq L$ together with $Q \in \mathrm{Syl}_q(G)$, we infer that q is not a factor of $[G : L]$, in contradiction to that $[G : L] = q$. On the other hand, if $[G : L]$ is composite, then $L \in \mathcal{F}^{sc}$. If L is nilpotent, then, by the above proof, L is a Sylow 2-subgroup of G. Hence $q = 2$ and thereby N must be a 2-group, contradicts to $Q_1 < N$. This shows that L must be a non-nilpotent group and so $L \in \mathcal{F}^{sc}$. However, by our hypotheses, there exists a normal subgroup K of G such that $G = LK$ and $K \cap L \leq L_G = 1$. By noticing that $K \neq 1$, we have $N \leq K$ and so it gives $N \cap L = 1$ and $[G : L] = |N|$. As q is a factor of $|N|$ and q does not divide $[G : L]$, we obtain a contradiction. Hence, after all, we have shown that N is a q-group and therefore G is solvable. The proof is completed. $\square$

The following theorem gives the conditions for a finite group to be p-nilpotent.

Theorem 3.2. *Let G be a group and p the smallest prime number dividing the order of G. If all 2-maximal subgroups of every Sylow p-subgroup of G are c-normal in G and G is A_4 free, then G is p-nilpotent.*

Proof. We use induction on the order of G. Let p^α be the order of a Sylow p-subgroup P of G. We consider the following two cases:

Case 1: $\alpha \leq 2$.

In this case, the Sylow p-subgroup P of G is abelian. If P is cyclic, then G is p-nilpotent by [7, 10.1.9]. So we can assume that $\alpha = 2$ and P is an elementary abelian p-group. Let L be a maximal subgroup of G. If the Sylow p-subgroups of L are all cyclic, then L is p-nilpotent and if p^2 divides $|L|$, then L is p-nilpotent by induction. So in this case we may assume that G is a minimal non-p-nilpotent group (that is, G is not p-nilpotent but every maximal subgroup of G is p-nilpotent). Then by [7, 10.3.3 and 9.1.9], we have $G = PQ$, where P is normal in G and Q is a cyclic Sylow q-subgroup of G ($p \neq q$). In particular, we now know that $1 \neq G/C_G(P)$, which is isomorphic to a subgroup of $\mathrm{Aut}(P)$, is a q-group. Observe that $|\mathrm{Aut}(P)| = (p^2 - 1)(p^2 - p)$, we therefore have $q \mid p + 1$ and consequently $p = 2$ and $q = 3$. It is now clear that $G/\Phi(Q)$ is isomorphic to A_4, a contradiction. So, G is a p-nilpotent group in this case.

Case 2: $\alpha \geq 3$.

Let P_1 be a 2-maximal subgroup of P. Then $P_1 \neq 1$. By our hypotheses and by Lemma 2.3, we know that there exists a normal subgroup M of G such that $G = P_1M$ and $P_1 \cap M = (P_1)_G$. It follows that $P = P_1(P \cap M)$ and $P \cap M$ is a Sylow p-subgroup of M. It is clear that $|P \cap M/(P_1)_G| = p^2$. By applying Case 1, we know that $M/(P_1)_G$ is p-nilpotent. Let $H/(P_1)_G$ be the normal Hall p'-subgroup of $M/(P_1)_G$. Then, we have $H \trianglelefteq M$ and $(P_1)_G$ is a Sylow p-subgroup of H. Also by Schure-Zassenhaus theorem there exists a Hall p'-subgroup K of H. It is clear that K is a Hall p'-subgroup of G as well. By using Frattini argument again, we arrive that $M = HN_M(K) = (P_1)_G N_M(K)$ and it follows that $G = P_1N_G(K)$. Therefore, $N_P(K)$ is a Sylow p-subgroup of $N_G(K)$. If $[G : N_G(K)] = |P : N_P(K)| \geq p^2$, then we can let P_2 be a 2-maximal subgroup of P such that $N_P(K) \leq P_2$. By using the above proof once again we obtain a normal subgroup M_1 of G such that $G = P_2M_1$, $P_2 \cap M_1 = (P_2)_G$ and $M_1 = (P_2)_G N_{M_1}(K_1)$, where K_1 is a Hall p'-subgroup of G. Observe that the following group series

$$1 \leq (P_1)_G < H < M < G.$$

It is clear that the above series is a normal series and every factor in the series is either a p-group or a p'-group, hence G is p-slovable. Thus, there exists $g \in P$ such that $K_1^g = K$ and consequently $N_G(K_1)^g = N_G(K)$. Let P^* be a maximal subgroup of P such that $P_2 < P^*$. Then, we have $G = P^*N_G(K_1) = P^*N_G(K_1)^g = P^*N_G(K)$ since P^* is normal in P. It follows that $P = P^*(P \cap N_G(K)) = P^*N_P(K)$. But $N_P(K) \leq P_2 < P^*$ and therefore $P = P^*$, a contradiction. Thus, we obtain that $|G : N_G(K)| = |P : N_P(K)| \leq p$. Suppose that $|G : N_G(K)| = p$. Then, $N_G(K)$ must be normal in G because p is the smallest prime number dividing $|G|$. It follows that $K \trianglelefteq G$ and therefore $[G : N_G(K)] = 1$, a contradiction. This shows that $K \trianglelefteq G$ and G is p-nilpotent. The proof is complete. □

Corollary 3.3. *Let G be a group. If G is A_4-free and all 2-maximal subgroups of every Sylow subgroup of G are c-normal in G, then G has a Sylow tower of supersolvable type.*

By using similar arguments as the proof of Theorem 3.2, we can also prove the following theorem.

Theorem 3.4. *Let G be a group and p the smallest prime dividing the order of G. If all the maximal subgroups of every Sylow p-subgroup of G are c-normal in G, then G is p-nilpotent.*

Finally, we consider the influence of some c-normal subgroups of the Sylow subgroups in a finite group G. The following interesting theorem is formulated.

Theorem 3.5. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of supersolvable groups. Let H be a normal subgroup of a group G such that $G/H \in \mathcal{F}$ and all maximal subgroups of every Sylow subgroup of H are c-normal in G. Then G belongs to $\mathcal{F}$.*

Proof. Let F_i $(i = 1, 2)$ be the full and integrated formation function such that $\mathcal{U} = LF(F_1)$ and $\mathcal{F} = LF(F_2)$. If the theorem is false, then we can let G be a minimal counterexample. By applying Lemma 2.1 and Theorem 3.4, we know that H has a Sylow tower of supersolvable type. Let p be the largest prime number in $\pi(H)$ and $P \in \mathrm{Syl}_p(H)$. Then P must be a normal subgroup of G. Clearly, $(G/P)/(H/P) \simeq G/H \in \mathcal{F}$ and all maximal subgroups of every Sylow subgroup of H/P are c-normal in G/P by Lemma 2.4. Thus, by the minimality of G, we know that $G/P \in \mathcal{F}$ and every maximal subgroup of P is c-normal in G.

Let N be a minimal normal subgroup of G and $N \leq P$. It is easy to see that the quotient group G/N satisfies the hypotheses of our theorem. By our choice of G, we have $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, we know that N is the unique minimal normal subgroup of G contained in P and also $\Phi(P) = 1$.

Since P is an elementary abelian p-group, there exists $N_1 \leq P$ such that $P = N \times N_1$. Let P_1 be a maximal subgroup of P such that $N_1 \leq P_1$. Then, by our hypotheses, there exists a normal subgroup M of G such that $G = MP_1$ and $M \cap P_1 \leq (P_1)_G$.

If $N < P$, then $N \not\leq (P_1)_G$. Again, as N is the unique minimal normal subgroup of G contained in P, we have $(P_1)_G = 1$. If $N = P$, then $N \not\leq P_1$ and so we still have $(P_1)_G = 1$. It follows that $|P \cap M| = p$ and $P \cap M \lhd G$, and therefore $N = P \cap M$ is a cyclic group of order p. This leads to $\mathrm{Aut}(N)$ is a cyclic group of order $p - 1$. Since $G/C_G(N) \leq \mathrm{Aut}(N)$, by Lemma 2.5, we have $G/C_G(N) \in F_1(p) \subseteq F_2(p)$ and therefore $G \in \mathcal{F}$, a contradiction. The proof is now completed. □

Remark 1. Let $\mathcal{F}$ be the class of groups G with G' nilpotent. It is easy to see that $\mathcal{F}$ is a saturated formation containing the class $\mathcal{U}$. Now, by Theorem 3.5, we can see that $G \in \mathcal{F}$ if $G/H \in \mathcal{F}$ and all maximal subgroups of the Sylow subgroups of H are c-normal in G.

Remark 2. It is noted that Theorem 3.5 is not true if the saturated formations $\mathcal{F}$ does not contain $\mathcal{U}$ (the class of supersolvable groups). For example, if $\mathcal{F}$ is the saturated formation of all niplotent groups, then the symmetric group of degree three is a counterexample.

Remark 3. It is also noted that Theorem 3.5 is generally not true for non-saturated formation. To see this remark, we let $\mathcal{F}$ be a formation composed by all groups G such that $G^{\mathcal{U}}$, the supersolvable residual, is elementary abelian. Clearly, $\mathcal{F} \geq \mathcal{U}$, but $\mathcal{F}$ is not saturated. Let $G = SL(2,3)$ and $H = Z(G)$. Then G/H is isomorphic to the alternative group of degree four and thereby $G/H \in \mathcal{F}$. But G does not belong to $\mathcal{F}$. This illustrates the situation.

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