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On weakly symmetric Riemannian spaces

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Dedicated to Professor Lajos Tamássy on his 70th birthday

1. Introduction

The notion of a weakly symmetric space, $(WS)_n$, was introduced by L. Tamássy and the present author [5]. This is a non-flat Riemannian space V_n whose curvature tensor R_{hijk} satisfies the condition

$$R_{hijk,l} = \alpha_l R_{hijk} + \beta_h R_{lijk} + \gamma_i R_{hljk} + \sigma_j R_{hilk} + \mu_k R_{hijl}\,, \tag{1}$$

Where $\alpha, \beta, \gamma, \sigma, \mu$ are 1-forms which are nonzero simultaneously, and the comma denotes covariant differentiation with respect to the metric tensor of the space. In the case of $\beta = \gamma = \sigma = \mu = \frac{1}{2}\alpha$ a $(WS)_n$ is just a pseudo-symmetric space $(PS)_n$, which was introduced and investigated by Chaki [1], so the notion of a $(WS)_n$ is a natural generalization of that of a $(PS)_n$.

M. C. Chaki and U. C. De [3] showed that: *i) If a $(PS)_n$ is a decomposable space $V_r \times V_{n-r} (r \geq 2, n-r \geq 2)$, then one of the composition spaces is flat and the other is a pseudo symmetric space. ii) If the metric tensor of $(PS)_n$ $(n \geq 3)$ is positive definite and has cyclic Ricci tensor, then the space is an Einstein space of zero scalar curvature.*

In the present paper the above two results of Chaki and De are transplanted and generalized to a weakly symmetric Riemannian space. Using the method of Chaki and De [3] we prove the following two theorems:

Theorem 1. *If a $(WS)_n$ with $\alpha \neq 0$ is a decomposable space $V_r \times V_{n-r}$ $(r, n-r \geq 2)$, then one of the composition spaces is flat and the other is weakly symmetric; and conversely, if in a product space $V_n = V_r \times V_{n-r}$ one of the composition spaces is flat and the other is weakly symmetric with $\alpha \neq 0$, then V_n is a $(WS)_n$ with $\alpha \neq 0$.*

Theorem 2. *If a $(WS)_n$ has cyclic Ricci tensor, moreover*

$$\Omega = \beta + \gamma + \sigma + \mu \tag{2}$$

is not orthogonal to

$$\Theta = \alpha + \gamma + \sigma \tag{3}$$

and the cyclic sum $\sum_{(X,Y,Z)} \alpha(X)\Theta(Y)\Theta(Z)$ *is not zero for* $\forall X, Y, Z$ *vector fields, then the space is an Einstein space of zero scalar curvature.*

In the special case of $\beta = \gamma = \sigma = \mu = \frac{1}{2}\alpha(\neq 0)$ our $(WS)_n$ is a $(PS)_n$. In this case our theorems yield the ones of Chaki and De, yet more, namely our Theorem 1 contains also the conversed statement, and Theorem 2 does not use the positive definitness of the Riemannian metric.

2. Proof of Theorem 1

If a $(WS)_n$ is a product $V_r \times V_{n-r}$, then local coordinates can be found so that the metric takes the form (see also [3])

$$ds^2 = \sum_{a,b=1}^{r} g_{ab}dx^a dx^b + \sum_{a,'b'=r+1}^{n} g_{a'b'}dx^{a'}dx^{b'} = \sum_{i,j=1}^{n} g_{ij}dx^i dx^j, \tag{4}$$

where g_{ab} are functions of $x^1, x^2, ..., x^r$ and $g_{a'b'}$ are functions of $x^{r+1}, ..., x^n$ only; $a, b, c,$ ... range from 1 to r and a', b', c', ...range from $r+1$ to n. From (1) we get

$$R_{abcd,a'} = \alpha_{a'}R_{abcd} + \beta_a R_{a'bcd} + \gamma_b R_{aa'cd} + \sigma_c R_{aba'd} + p_d R_{abca'}. \tag{5}$$

In view of (4) in this product space all Γ_{ij}^k must vanish, except if $1 \leq i, j, k \leq r$, or else $r + 1 \leq i, j, k \leq n$. So so it follows that

$$R_{abcd,a'} = R_{a'bcd} = R_{aa'cd} = R_{aba'd} = R_{abca'} = 0\,.$$

Hence equation (5) takes the form

$$\alpha_{a'}R_{abcd} = 0\,. \tag{6}$$

Similarly we get

$$\alpha_a R_{a'b'c'd'} = 0\,. \tag{7}$$

Since $\alpha \neq 0$, all its components cannot vanish. Suppose $\alpha_{a'} \neq 0$ for some a'. Then from (6) it follows that $R_{abcd} = 0 \quad \forall a, b, c, d$ which means that the decomposition factor V_r is flat. Similarly if α_a is not zero for some a, then $R_{a'b'c'd'} = 0$ which implies the flatness of V_{n-r}.

We now suppose that V_r is flat, i.e. $R_{abcd} = 0$. Then $R_{a'b'c'd'} \neq 0$ for some a', b', c', d' because $(WS)_n$ is not flat. Hence form (7) we get $\alpha_a = 0, a = 1, ...r$ and then $\alpha_{f'} \neq 0$ for some f'. Therefore (1) implies

$$R_{a'b'c'd',f'} = \alpha_{f'} R_{a'b'c'd'} + \beta_{a'} R_{f'b'c'd'} + \gamma_{b'} R_{a'f'c'd'} + \sigma_{c'} R_{a'b'f'd'} + \mu_{d'} R_{a'b'c'f'}$$

which means that V_{n-r} is a $(WS)_{n-r}$.

Turning to the conversed part of the theorem, consider a product space $V_r \times V_{n-r}$ with ds^2 as in (4). In this V_n all R_{ijkh} and $R_{ijkh,l}$ vanish except if $1 \leq i, j, k, h, l \leq r$, or else $r + 1 \leq i, j, k, h, l \leq n$. Now assuming that V_r is flat and that $V_{n-r} = (WS)_{n-r}$, i.e. $R_{a'b'c'd',e'}$ satisfies (1) (with a nonvanishing α), then by extending $\alpha, \beta, \gamma, \sigma, \mu$ from V_{n-r} to $V_n = V_r \times V_{n-r}$ so that $\alpha_a = \beta_a = \gamma_a = \sigma_a = \mu_a = 0 \ \ \forall a = 1, ...r$, we can easily see that V_n is a $(WS)_n$.

3. Proof of Theorem 2

Transvecting (1) with g^{hk} we have

$$R_{ij,l} = \alpha_l R_{ij} + \beta^k R_{lijk} + \gamma_i R_{lj} + \sigma_j R_{il} + \mu^k R_{kilj}. \tag{8}$$

Transvecting again with g^{ij}, by the symmetry of the Ricci tensor and by (2) we obtain

$$R, l = \alpha_l R + (\beta^k + \gamma^k + \sigma^k + \mu^k) R_{kl} = \alpha_l R + \Omega^k R_{kl}. \tag{9}$$

Here $\beta^k, \gamma^k, ...$ denote the vector fields associated to $\beta, \gamma, ...$ i.e. $\beta^k = g^{ik}\beta_i$ and so on. Consider now the second Bianchi identity

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0.$$

By transvecting with g^{jk} we get

$$R_{il,m} + g^{jk} R_{ijlm,k} - R_{im,l} = 0,$$

and transvecting again with g^{im} we obtain

$$R_{,l} = 2g^{im} R_{il,m} \,. \tag{10}$$

A Riemannian space is said, by definition, to have a cyclic Ricci tensor if

$$R_{ij,k} + R_{jk,i} + R_{ki,j} = 0. \tag{11}$$

Transvecting this with g^{ij} and taking into account (10) we get

$$R_{,l} = 0. \tag{12}$$

Thus in the case of a cyclic Ricci tensor, from (9) and (12) we have

$$\alpha_l R + \Omega^k R_{kl} = 0. \tag{13}$$

Consider now the cyclic sum of $R_{ij,k}$. From (8) , (11) and the first Bianchi identity we have

$$\Theta_k R_{ij} + \Theta_i R_{jk} + \Theta_j R_{ik} = 0. \tag{14}$$

Multiplying (14) with Ω^k and summing for k we get

$$\Theta_k \Omega^k R_{ij} + \Omega^k \Theta_i R_{jk} + \Omega^k \Theta_j R_{ki} = 0. \tag{15}$$

Using (13), (15) takes the form

$$\Lambda R_{ij} + R(\Theta_i \alpha_j + \Theta_j \alpha_i) = 0, \tag{16}$$

where $\Lambda := \Omega^k \Theta_k$. Multiply now (16) with Θ_k and take the cyclic sum over i, j, k. From (14) it follows

$$R(\alpha_i \Theta_j \Theta_k + \alpha_j \Theta_k \Theta_i + \alpha_k \Theta_i \Theta_j) = 0. \tag{17}$$

Since condition $\sum\limits_{(X,Y,Z)} \alpha(X)\Theta(Y)\Theta(Z) \neq 0$ in Theorem 2 is nothing but

$$\alpha_i \Theta_j \Theta_k + \alpha_j \Theta_k \Theta_i + \alpha_k \Theta_i \Theta_j \neq 0,$$

(17) yields

$$R = 0. \tag{18}$$

Since Λ is the inner product of Θ and Ω, and Λ is not zero by our assumption, we get from (16) that

$$R_{ij} = 0.$$

(18) and (19) complete the proof.

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