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# On the mean value formula for the non-symmetric form of the approximate functional equation of $\zeta^2(s)$ in the critical strip

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**Abstract.** The object of this paper is to derive the mean value formula of the error term $R^*\left(s;\frac{l}{k}\right)$ of the non-symmetric form in the approximate functional equation for $\zeta^2(s)$ in the critical strip $0 \le \sigma \le 1$.

## 1. Introduction

Let $s = \sigma + it$ $(0 \le \sigma \le 1,\ t \ge 1)$ be a complex variable, $\zeta(s)$ the Riemann zeta-function, $d(n)$ the number of positive divisors of $n$, $\gamma$ the Euler constant, $k$ and $l$ co-prime integers with $1 \le l \le k$. The error term $R^*(s; l/k)$ in the approximate functional equation for $\zeta^2(s)$ is defined by

$$\zeta^2(s) = \sum_{n \le \frac{lt}{2\pi k}}{}' \frac{d(n)}{n^s} + \chi^2(s) \sum_{n \le \frac{kt}{2\pi l}}{}' \frac{d(n)}{n^{1-s}} + R^*\left(s; \frac{l}{k}\right)$$

where

$$\chi(s) = 2^s \pi^{s-1} \sin\left(\frac{1}{2}\pi s\right)\Gamma(1-s), \tag{1.1}$$

and $\sum'_{n \le y}$ indicates that the last term is to be halved if $y$ is an integer. For $k \ne l$, this is called the "non-symmetric form" of the approximate functional equation for $\zeta^2(s)$. By using the method of Meurman [6] and

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the Motohashi formula (2.2) given below, the mean value formula of the function $|R^*(1/2+it;l/k)|$ was first studied by Kiuchi [4], who obtained, for $kl \le T(\log T)^{-20}$, the asymptotic formula

$$\int_1^T \left|R^*\Big(\frac{1}{2}+it;\frac{l}{k}\Big)\right|^2 dt = \sqrt{2\pi}C_{k,l}T^{1/2}+K_{k,l}(T) \tag{1.2}$$

with

$$K_{k,l}(T) = O\Big((kl)^{3/4}T^{1/4}\log^3 T\Big) \tag{1.3}$$

where

$$C_{k,l} = \sum_{n=1}^{\infty}\frac{d^2(n)H_{k,l}^2(n)}{\sqrt{n}}$$

and

$$\begin{aligned} H_{k,l}(n) = (kl)^{-1/4}\sqrt{\frac{2}{\pi}}\int_0^\infty \Big(y+\frac{n\pi}{kl}\Big)^{-1/2} \\ \times\cos\Big(y+\Big(\frac{\bar{k}}{l}+\frac{\bar{l}}{k}\Big)n\pi+\frac{\pi}{4}\Big)dy^{\dagger} \ll \frac{(kl)^{1/4}}{\sqrt{n}}. \end{aligned} \tag{1.4}$$

Here the residue classes $\bar{k} \pmod{l}$ and $\bar{l} \pmod{k}$ are defined by $k\bar{k}\equiv 1 \pmod{l}$ and $l\bar{l}\equiv 1 \pmod{k}$, respectively. The purpose of this paper is to derive the mean value formula of the function $|R^*(s;l/k)|$ in the critical strip $0\le\sigma\le 1$, and the basic tool is the non-symmetric form of the Motohashi formula (2.2) given below. The principle of the proof is the same as in Kiuchi [5], and the main result is

**Theorem.** *For $1\le l\le k$, $(k,l)=1$, $kl\le T(\log T)^{-20}$ and $T\ge 1$, we have*

$$\int_1^T \left|R^*\Big(s;\frac{l}{k}\Big)\right|^2 dt \tag{1.5–6–7}$$

$$= \begin{cases} A_{k,l}(\sigma)T^{3/2-2\sigma}+O\Big(\Big(\frac{l}{k}\Big)^{1-2\sigma}(kl)^{3/4}T^{5/4-2\sigma}\log^3 T\Big) & \text{if } 0\le\sigma\le 5/8, \\ A_{k,l}(\sigma)T^{3/2-2\sigma}+B_{k,l}(\sigma)+O\Big(\Big(\frac{l}{k}\Big)^{1-2\sigma}(kl)^{3/4}T^{5/4-2\sigma}\log^3 T\Big) & \text{if } 5/8<\sigma\le 1, \\ & \sigma\ne 3/4, \\ \pi\sqrt{\frac{k}{l}}C_{k,l}\log T + B_{k,l}\Big(\frac{3}{4}\Big)+O(k^{5/4}l^{1/4}T^{-1/4}\log^3 T) & \text{if } \sigma = 3/4, \end{cases}$$

†The author corrects a misprint of the function $H_{k,l}(n)$ in [4].

*where*

$$A_{k,l}(\sigma) = \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma}\Big(\frac{l}{k}\Big)^{1-2\sigma} C_{k,l}$$

*and $B_{k,l}(\sigma)$ is a certain constant.*

**Corollary.** *For $1 \le l \le k$, $(k,l)=1$, $kl \le t(\log t)^{-20}$ and $t \ge 2$, this theorem includes the fact that*

$$\Big|R^*\Big(s;\frac{l}{k}\Big)\Big| = \begin{cases} \Omega\Big(\Big(\frac{l}{k}\Big)^{1/2-\sigma} C_{k,l}^{1/2} t^{1/4-\sigma}\Big) & \text{if } 0 \le \sigma < 3/4, \\ \Omega\Big(\Big(\frac{k}{l}\Big)^{1/4} C_{k,l}^{1/2} t^{-1/2}\sqrt{\log t}\Big) & \text{if } \sigma = 3/4. \end{cases}$$

Comparing (1.6) and (1.7), we observe that the line $\sigma = 3/4$ has a kind of critical property. This is a situation similar to the case of the error term $R(s;t/(2\pi))$ in the "symmetric form" of the approximate functional equation for $\zeta^2(s)$, which is defined by

$$\zeta^2(s) = \sum_{n\le\frac{t}{2\pi}}{}' \frac{d(n)}{n^s} + \chi^2(s)\sum_{n\le\frac{t}{2\pi}}{}' \frac{d(n)}{n^{1-s}} + R\Big(s;\frac{t}{2\pi}\Big)$$

for a fixed number $\sigma$ $(0 \le \sigma \le 1)$. Kiuchi and Matsumoto [3] first showed that

$$\int_1^T \Big|R\Big(\frac{1}{2}+it;\frac{t}{2\pi}\Big)\Big|^2 dt = \sqrt{2\pi}CT^{1/2} + K(T) \tag{1.8}$$

with $K(T)=O(T^{1/4}\log T)$, and in [5], the improvement $K(T)=O(\log^4 T)$ has recently proved by Kiuchi, where

$$C = \sum_{n=1}^{\infty}\frac{d^2(n)h^2(n)}{\sqrt{n}}$$

and

$$h(n) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}(y+n\pi)^{-1/2}\cos\Big(y+\frac{\pi}{4}\Big)dy.$$

Further, a simple argument to deduce sharp results on the mean square of $|R(s;t/(2\pi))|$ was discovered by KIUCHI [5], who proved, for $0 \le \sigma \le 1$, the asymptotic formula

$$\tag{1.9} \int_1^T \left| R\Big(s;\frac{t}{2\pi}\Big)\right|^2 dt$$

$$= \begin{cases} A_1(\sigma)T^{3/2-2\sigma} + O(T^{1-2\sigma}\log^4 T) & \text{if } 0 \le \sigma \le 1/2, \\ A_1(\sigma)T^{3/2-2\sigma} + A_2(\sigma) + O(T^{1-2\sigma}\log^4 T) & \text{if } 1/2 < \sigma \le 1,\ \sigma \ne 3/4, \\ \pi C \log T + A_2\Big(\dfrac{3}{4}\Big) + O(T^{-1/2}\log^4 T) & \text{if } \sigma = 3/4, \end{cases}$$

with a certain constant $A_2(\sigma)$, where

$$A_1(\sigma) = \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} C.$$

From (1.9), KIUCHI has observed, as already pointed out in [5], that the line $\sigma = 3/4$ is a kind of "critical line" in the theory of the Riemann zeta-function, or at least for the function $R(s;t/(2\pi))$. Our theorem indicates that the similar critical property on the line $\sigma = 3/4$ appears in the mean value formulas of more generalized quantity $R^*(s;l/k)$. Comparing (1.5)–(1.7) and (1.9), one may formulate the following

**Conjecture.** *For $0 \le \sigma \le 1$, $1 \le l \le k$, $(k,l) = 1$, and $kl \le T(\log T)^{-20}$, the error term $O\Big(\Big(\frac{l}{k}\Big)^{1-2\sigma}(kl)^{3/4}T^{5/4-2\sigma}\log^3 T\Big)$ in Theorem can be replaced by*

$$O\Big(\Big(\frac{l}{k}\Big)^{1-2\sigma}(kl)^{3/4}T^{1-2\sigma}\log^4 T\Big).$$

This work was done while the author was staying at the Graduate School of Polymathematics, Nagoya University.

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## 2. Application of the Motohashi formula

Let $a$ and $b$ be integers with $a \geq 1$ and $(a, b) = 1$. For $x \geq 1$, we put

$$\Delta\Big(x; \frac{b}{a}\Big) = {\sum_{n \leq x}}' d(n) e\Big(\frac{b}{a} n\Big) - \frac{x}{a}\Big(\log \frac{x}{a^2} + 2\gamma - 1\Big) - E\Big(0; \frac{b}{a}\Big) \tag{2.1}$$

where $e(\alpha) = \exp(2\pi i \alpha)$, and $E(0; b/a)$ is the value at $s = 0$ of the analytic continuation of

$$E\Big(s; \frac{b}{a}\Big) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} e\Big(\frac{b}{a} n\Big)$$

which is first defined for Re $s > 1$. Our starting point is the following "non-symmetric form" of the Riemann–Siegel formula for $\zeta^2(s)$, which was proved by Motohashi [8; Theorem 7] (see also [7]):

For $t \geq 2$ and $0 \leq \sigma \leq 1$, we have, uniformly for $kl \leq t(\log t)^{-20}$,

$$\begin{aligned}\chi(1-s) R^*\left(s; \frac{l}{k}\right) &= M\Big(s; \frac{l}{k}\Big) + \overline{M\Big(1 - \bar{s}; \frac{k}{l}\Big)} \\ &\quad + O\Big(\Big(\frac{l}{k}\Big)^{1/2-\sigma} \Big(\frac{kl}{t}\Big)^{1/2} \log^3 t\Big),\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}M\Big(s; \frac{l}{k}\Big) &= -e\Big(-\frac{1}{8}\Big) \Big(\frac{t}{2\pi}\Big)^{-1/2} \Big(\frac{l}{k}\Big)^{-s} \Delta\Big(\frac{lt}{2\pi k}; -\frac{k}{l}\Big) \\ &\quad + \frac{1}{2} e\Big(-\frac{1}{8}\Big) \Big(\frac{kl}{2\pi t}\Big)^{1/4} \Big(\frac{l}{k}\Big)^{1/2-s} \sum_{n=1}^{\infty} \frac{d(n)}{n^{1/4}} e\Big(\frac{\bar{k}}{l} n\Big) \\ &\quad \times \sin\Big(2\sqrt{\frac{2\pi t n}{kl}} + \frac{\pi}{4}\Big) \int_0^{\infty} (\xi + n\pi)^{-3/2} \exp\Big(\frac{i\xi}{kl}\Big) d\xi.\end{aligned} \tag{2.3}$$

Jutila [2] (see also (2.6.7) of [8]) proved the following formula, which is an analogue of the Voronoï formula for (2.1):

$$\Delta\Big(x;\frac{b}{a}\Big)=\frac{a^{1/2}x^{1/4}}{\pi\sqrt{2}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{3/4}}e\Big(-\frac{\bar{b}}{a}n\Big)\cos\Big(4\pi\frac{\sqrt{nx}}{a}-\frac{\pi}{4}\Big) + O(a^{3/2}x^{-1/4}),$$

where $x \geq a^2(\log 2a)^3$, and the residue class $\bar{b} \pmod a$ is defined by $b\bar{b}\equiv 1 \pmod a$. Applying this formula to (2.3) and using integration by parts, we have

$$M\Big(s;\frac{l}{k}\Big)=\frac{i}{\sqrt{2\pi k}}\Big(\frac{l}{k}\Big)^{1/4-s}\Big(\frac{t}{2\pi}\Big)^{-1/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/4}}e\Big(\frac{\bar{k}}{l}n\Big)\sin\Big(2\sqrt{\frac{2\pi tn}{kl}}+\frac{\pi}{4}\Big)$$

$$\times\int_0^{\infty}\Big(\xi+\frac{n\pi}{kl}\Big)^{-1/2}\exp\Big(i\Big(\xi-\frac{\pi}{4}\Big)\Big)d\xi+O(k^{1/4+\sigma}l^{5/4-\sigma}t^{-3/4}), \tag{2.4}$$

and

$$\overline{M\Big(1-\bar{s};\frac{k}{l}\Big)}=\frac{-i}{\sqrt{2\pi l}}\Big(\frac{l}{k}\Big)^{3/4-s}\Big(\frac{t}{2\pi}\Big)^{-1/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/4}}e\Big(-\frac{\bar{l}}{k}n\Big) \tag{2.5}$$

$$\times\sin\Big(2\sqrt{\frac{2\pi tn}{kl}}+\frac{\pi}{4}\Big)\int_0^{\infty}\Big(\xi+\frac{n\pi}{kl}\Big)^{-1/2}\exp\Big(-i\Big(\xi-\frac{\pi}{4}\Big)\Big)d\xi$$

$$+O(k^{1/4+\sigma}l^{5/4-\sigma}t^{-3/4})$$

for $t\geq 2\pi kl(\log 2k)^3$. Substituting (2.4) and (2.5) into (2.2), we obtain, for $kl \leq t(\log t)^{-20}$ and $t \geq 2$,

$$\chi(1-s)R^*\Big(s;\frac{l}{k}\Big)=\Big(\frac{l}{k}\Big)^{1/2-\sigma}\Big\{\Big(\frac{k}{l}\Big)^{it}\Big(\frac{t}{2\pi}\Big)^{-1/4}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/4}}e\Big(\frac{1}{2}\Big(\frac{\bar{k}}{l}-\frac{\bar{l}}{k}\Big)n\Big)$$

$$\times\sin\Big(2\sqrt{\frac{2\pi tn}{kl}}+\frac{\pi}{4}\Big)H_{k,l}(n)+O\Big(\Big(\frac{kl}{t}\Big)^{1/2}\log^3 t\Big)\Big\}$$

where

$$H_{k,l}(n)=(kl)^{-1/4}\sqrt{\frac{2}{\pi}}\int_0^\infty\Big(y+\frac{n\pi}{kl}\Big)^{-1/2}\cos\Big(y+\Big(\frac{\bar{k}}{l}+\frac{\bar{l}}{k}\Big)\pi n+\frac{\pi}{4}\Big)dy.$$

Put $\sigma=1/2$ in this formula, and compare it with the above. Then, using the relation $\chi(1-s)\chi(s)=1$, we have

$$R^*\Big(s;\frac{l}{k}\Big)=\Big(\frac{l}{k}\Big)^{1/2-\sigma}\chi(s)\Big\{\chi\Big(\frac{1}{2}-it\Big)R^*\Big(\frac{1}{2}+it;\frac{l}{k}\Big)+E_{k,l}(t)\Big\}$$

with

$$E_{k,l}(t)=O\Big(\Big(\frac{kl}{t}\Big)^{1/2}\log^3 t\Big). \tag{2.6}$$

Therefore, we obtain the following

**Lemma.** *For $0\le\sigma\le1$, $1\le l\le k$, $(k,l)=1$ and $kl\le t(\log t)^{-20}$, we have*

$$\Big|R^*\Big(s;\frac{l}{k}\Big)\Big|^2=\Big(\frac{l}{k}\Big)^{1-2\sigma}|\chi(s)|^2\Big\{\Big|R^*\Big(\frac{1}{2}+it;\frac{l}{k}\Big)\Big|^2+F_{k,l}(t)\Big\} \tag{2.7}$$

*where*

$$F_{k,l}(t)\ll\Big|R^*\Big(\frac{1}{2}+it;\frac{l}{k}\Big)\Big||E_{k,l}(t)|+|E_{k,l}(t)|^2. \tag{2.8}$$

## 3. Proof of the Theorem

It follows from the asymptotic formula (see (1.25) of Ivić [1]) of (1.1) that

$$|\chi(s)|^2=\Big(\frac{t}{2\pi}\Big)^{1-2\sigma}+G_\sigma(t)\qquad(t\ge t_0>0) \tag{3.1}$$

with

$$G_\sigma(t)=O(t^{-2\sigma}). \tag{3.2}$$

From (2.7), we have

$$\int_1^T\Big|R^*\Big(s;\frac{l}{k}\Big)\Big|^2dt\quad=\quad\Big(\frac{l}{k}\Big)^{1-2\sigma}\Big\{I_1(1,T)+I_2(1,T)\Big\}, \tag{3.3}$$

where

$$I_1(T_1, T_2) = \int_{T_1}^{T_2} |\chi(s)|^2 \Big| R^*\Big(\frac{1}{2} + it; \frac{l}{k}\Big)\Big|^2 dt,$$

and

$$I_2(T_1, T_2) = \int_{T_1}^{T_2} |\chi(s)|^2 F_{k,l}(t) dt.$$

Hereafter we assume that $T_1 < T_2 \le 2T_1$. Applying (1.2), (3.1) and integrating by parts, we have, for $\sigma \neq 3/4$,

$$(3.4) \qquad I_1(T_1, T_2) = \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} C_{k,l} t^{3/2-2\sigma} + \Big(\frac{t}{2\pi}\Big)^{1-2\sigma} K_{k,l}(t) \Big|_{T_1}^{T_2}$$
$$+ (2\pi)^{2\sigma-1}(2\sigma - 1) \int_{T_1}^{T_2} t^{-2\sigma} K_{k,l}(t) dt$$
$$+ \int_{T_1}^{T_2} G_\sigma(t) \Big| R^*\Big(\frac{1}{2} + it; \frac{l}{k}\Big)\Big|^2 dt.$$

From (1.3), we have

$$\int_{T_1}^{T_2} t^{-2\sigma} K_{k,l}(t) dt = O((kl)^{3/4} T_1^{5/4-2\sigma} \log^3 T_1).$$

From (1.2), (1.3), (1.4), (3.2) and $kl \le t(\log t)^{-20}$, we obtain

$$\int_{T_1}^{T_2} G_\sigma(t) \Big| R^*\Big(\frac{1}{2} + it; \frac{l}{k}\Big)\Big|^2 dt \ll \max_{T_1 \le t \le T_2} |G_\sigma(t)| \int_{T_1}^{T_2} \Big| R^*\Big(\frac{1}{2} + it; \frac{l}{k}\Big)\Big|^2 dt$$
$$\ll (kl)^{1/2} T^{1/2-2\sigma}.$$

Hence we obtain, for $0 \le \sigma \le 5/8$,

$$(3.5) \qquad I_1(1, T) = \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} C_{k,l} T^{3/2-2\sigma} + O((kl)^{3/4} T^{5/4-2\sigma} \log^3 T),$$

and for $5/8 < \sigma \le 1$ ($\sigma \ne 3/4$),

$$I_1(1,T) = \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma} C_{k,l} T^{3/2-2\sigma} + (2\pi)^{2\sigma-1} T^{1-2\sigma} K_{k,l}(T) + (2\pi)^{2\sigma-1}(2\sigma-1)\int_1^\infty t^{-2\sigma} K_{k,l}(t)dt + \int_1^\infty G_\sigma(t)\Big|R^*\Big(\frac{1}{2}+it;\frac{l}{k}\Big)\Big|^2 dt + c_1(\sigma;k,l) + O((kl)^{3/4} T^{5/4-2\sigma}\log^3 T), \tag{3.6}$$

where the constant $c_1(\sigma;k,l)$ depends on $\sigma$, $k$ and $l$. Similarly in case $\sigma = 3/4$, we obtain, from (1.2), (3.1) and integration by parts,

$$I_1(1,T) = \pi C_{k,l}\log T + \sqrt{2\pi}T^{-1/2}K_{k,l}(T) + \sqrt{\frac{\pi}{2}}\int_1^\infty t^{-3/2}K_{k,l}(t)dt + \int_1^\infty G_{3/4}(t)\Big|R^*\Big(\frac{1}{2}+it;\frac{l}{k}\Big)\Big|^2 dt + c_1\Big(\frac{3}{4};k,l\Big) + O((kl)^{3/4}T^{-1/4}\log^3 T). \tag{3.7}$$

From (1.2), (1.3), (1.4), (2.6), (2.8), (3.1), (3.2) and Schwarz's inequality we have

$$I_2(T_1,T_2) \ll \Big(\int_{T_1}^{T_2}|\chi(s)|^2\Big|R^*\Big(\frac{1}{2}+it;\frac{l}{k}\Big)\Big|^2 dt\Big)^{1/2}\Big(\int_{T_1}^{T_2}|\chi(s)|^2|E_{k,l}(t)|^2 dt\Big)^{1/2} + \int_{T_1}^{T_2}|\chi(s)|^2|E_{k,l}(t)|^2 dt \ll (kl)^{3/4}T_1^{5/4-2\sigma}\log^3 T_1.$$

Hence we have

$$I_2(1,T) = \begin{cases} O((kl)^{3/4}T^{5/4-2\sigma}\log^3 T) & \text{if } 0 \le \sigma \le 5/8, \\ I_2(1,\infty) + O((kl)^{3/4}T^{5/4-2\sigma}\log^3 T) & \text{if } 5/8 < \sigma \le 1. \end{cases} \tag{3.8}$$

Substituting (3.5) and (3.8) into (3.3), we obtain, for $0 \le \sigma \le 5/8$,

$$\int_1^T \Big| R^*\Big(s; \frac{l}{k}\Big)\Big|^2 dt = \frac{(2\pi)^{2\sigma-1/2}}{3-4\sigma}\Big(\frac{l}{k}\Big)^{1-2\sigma} C_{k,l} T^{3/2-2\sigma} + O\Big(\Big(\frac{l}{k}\Big)^{1-2\sigma}(kl)^{3/4} T^{5/4-2\sigma}\log^3 T\Big).$$

Similarly in case $5/8 < \sigma \le 1$, we obtain

$$\int_1^T \Big| R^*\Big(s; \frac{l}{k}\Big)\Big|^2 dt$$

$$= \begin{cases} \dfrac{(2\pi)^{2\sigma-1/2}}{3-4\sigma}\Big(\dfrac{l}{k}\Big)^{1-2\sigma} C_{k,l} T^{3/2-2\sigma} + B_{k,l}(\sigma) \\ \qquad + O\Big(\Big(\dfrac{l}{k}\Big)^{1-2\sigma}(kl)^{3/4} T^{5/4-2\sigma}\log^3 T\Big) & \text{if } \sigma \ne 3/4, \\ \pi\sqrt{\frac{k}{l}}C_{k,l}\log T + B_{k,l}\Big(\frac{3}{4}\Big) + O(k^{5/4}l^{1/4}T^{-1/4}\log^3 T) & \text{if } \sigma = 3/4. \end{cases}$$

with a certain constant $B_{k,l}(\sigma)$. Therefore now we have the assertion of Theorem.

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