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# Convergence of evolution operator families and its applications to functional limit theorems

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**Abstract.** The present paper deals with the convergence of triangular systems $\{X_{n,\ell} : n, \ell \in \mathbb{N}\}$ of rowwise independent random variables taking their values in an arbitrary locally compact group $G$. More precisely, sufficient conditions are given in terms of the expectations $\mathbb{E}(U \circ X_{n,\ell})$ for all irreducible (continuous, unitary) representations $U$ of $G$ such that the partial products $X_{n,1} \cdot \ldots \cdot X_{n,k_n(\cdot)}$ (for $\mathbb{Z}_+$-valued scaling functions $k_n$ on $\mathbb{R}_+$) converge as $n \to \infty$ towards a stochastically continuous càdlàg process $\{X(t) : t \in \mathbb{R}_+\}$ with independent increments in $G$. The conditions are established under the hypotheses that the limiting process is specified or that it remains unspecified. The approach is measure- and Fourier-theoretic and employs efficiently convolution hemigroups of finite variation on $G$ and their Fourier transforms. In the case of unspecified limits the validity of Lévy's continuity theorem for groups $G$ is required.

## 1. Introduction

A major matter of concern in central limit theory is the problem of convergence of triangular systems of random variables towards a limit which can be either specified or unspecified. There are good reasons for studying the problem for random variables taking values in an arbitrary locally compact group $G$. Following a natural hierarchy one deals with the problem by first considering Lie groups and then passes to Lie projective locally compact groups $G$. Among the Lie projective groups we find

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the Moore groups which by definition admit only finite dimensional irreducible representations. To reach the most general framework of arbitrary locally compact groups the full harmonic analysis of infinite dimensional representations can be employed.

In their previous work on the subject [11]–[13] the authors followed the hierarchic path in discussing the problem of convergence of triangular systems and successfully emphasized the measure-theoretic approach. While in [11] and [13] the conditions securing the convergence were given in terms of the characteristics of associated generating families, the phrasing of the conditions in [12] appeared to be in terms of Fourier transforms involving finite dimensional representations. In the present work the machinery developed in [12] will be extended to infinite dimensional representations of an arbitrary locally compact group $G$ with the aim of solving the problem in a somewhat ultimate way.

The problem of convergence of triangular systems of $G$-valued random variables will be treated in a four-step procedure: There is the problem $(\alpha)$ of convergence of a sequence of hemigroups $\{\mu_n(s,t) : 0 \leqslant s \leqslant t\}$ of probability measures on $G$ towards a limiting hemigroup $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$, and as a consequence, the problem $(\beta)$ of convergence of a sequence of partial convolution products $\mu_{n,k_n(s)+1} * \ldots * \mu_{n,k_n(t)}$ towards $\mu(s,t)$. Next, the problem $(\alpha)$ leads directly to the problem $(\gamma)$ of convergence of $G$-valued processes $\{X_n(t) : t \in \mathbb{R}_+\}$ with independent left increments towards a limiting process $\{X(t) : t \in \mathbb{R}_+\}$ (necessarily having again independent left increments) associated with a hemigroup $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$, and the problem $(\beta)$ yields the problem $(\delta)$ of convergence of a sequence of partial products $X_{n,1} \cdot \ldots \cdot X_{n,k_n(\cdot)}$ towards the process $X(\cdot)$.

All hemigroups and processes involved are assumed or deduced to be of finite variation. The conditions found to be sufficient for the respective convergence are in terms of related integrating families.

A special feature of our approach is the generalization of the steps $(\alpha)$ to $(\delta)$ to the situation for which the limiting hemigroup or process remains unspecified. In order to arrive at useful results the validity of the Lévy continuity theorem is assumed. For Moore groups $G$ this Lévy continuity property is readily available (see [10]). For more general classes of locally compact groups it still requires an appropriate investigation (see [6]).

The layout of our presentation can be described as follows. Section 2 is devoted to basics on càdlàg functions and functions of finite variation taking their values in the space $\mathcal{L}(B)$ of bounded linear operators on a Banach space $B$. In Section 3 we introduce the (bilinear) Lebesgue–Bochner–Stieltjes integral for operator-valued functions, following the exposition in [9]. Section 4 contains our important tool of integrating functions of evolution families in $\mathcal{L}(B)$, and in Section 5 we describe the convergence of evolution families in terms of their integrating functions. The last Section 6 contains the solution to problems $(\alpha)$ to $(\delta)$, at first for specified limits, and subsequently for unspecified ones. The final result in Section 6 is a martingale difference version of the theorem contributing to problem $(\delta)$ in the unspecified limit case. Its proof relies on results established in Section 5.

Although some of the arguments needed in proving the results are borrowed from the authors' previous publications on the subject, the idea of an integrating family related to a hemigroup on an arbitrary locally compact group could only be fruitfully employed by applying the Lebesgue–Bochner–Stieltjes integral. The attentive reader will recognize the spirit of two valuable sources that we profited from: Hucke's approach to the convergence problem via associated stochastic differential equations [14] and Schmidt's notion of evolution families of finite variation [16].

## 2. Càdlàg functions and functions of finite variation

Let $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geqslant 0\}$ and $\mathbb{S} := \{(s,t) \in \mathbb{R}^2 : 0 \leqslant s \leqslant t\}$. Let $B$ be a real or complex Banach space. Let $\mathcal{L}(B)$ denote the Banach space of bounded linear operators on $B$. A function $f : \mathbb{R}_+ \to \mathcal{L}(B)$ is called *càdlàg* if it is right continuous with left limits. (The limit is always meant in the norm topology on $\mathcal{L}(B)$.) Let $D(\mathbb{R}_+, \mathcal{L}(B))$ denote the space of all càdlàg functions from $\mathbb{R}_+$ into $\mathcal{L}(B)$. The spaces $D([0,T], \mathcal{L}(B))$, $T > 0$, are defined similarly. Let $D(\mathbb{S}, \mathcal{L}(B))$ denote the space of all functions from $\mathbb{S}$ into $\mathcal{L}(B)$ which are càdlàg separately in each variable. Any function $f \in D(\mathbb{R}_+, \mathcal{L}(B))$ is locally bounded, i.e.,

$$\|f\|_T := \sup_{t \in [0,T]} \|f(t)\| < \infty \quad \text{for all } T > 0.$$

The space $D([0,T], \mathcal{L}(B))$ is a (non-separable) Banach space with the above norm. The *local uniform topology* on $D(\mathbb{R}_+, \mathcal{L}(B))$ is associated with the metric

$$\varrho_{\mathrm{lu}}(f,g) := \sum_{n=1}^{\infty} 2^{-n} \min\{1, \|f-g\|_n\} \quad \text{for } f, g \in D(\mathbb{R}_+, \mathcal{L}(B)).$$

The space $D(\mathbb{R}_+, \mathcal{L}(B))$ is a complete (but not separable) metric space under $\varrho_{\mathrm{lu}}$. There is a metrizable topology on $D(\mathbb{R}_+, \mathcal{L}(B))$, called the *Skorokhod topology*, for which this space is complete (but again not necessarily separable; separability holds only if $B$ is finite dimensional; see Ethier and Kurtz [8, Proposition 3.5.6].). The Skorokhod topology is weaker than the local uniform topology. (See Ethier and Kurtz [8, Proposition 3.5.3].)

Let $C(\mathbb{R}_+, \mathcal{L}(B))$ and $C(\mathbb{S}, \mathcal{L}(B))$ denote the spaces of all continuous functions from $\mathbb{R}_+$ and from $\mathbb{S}$ into $\mathcal{L}(B)$, respectively. If $f \in C(\mathbb{R}_+, \mathcal{L}(B))$ and $(f_n)_{n\geqslant 1}$ is a sequence in $D(\mathbb{R}_+, \mathcal{L}(B))$ such that $f_n \to f$ for the Skorokhod topology, then $f_n \to f$ locally uniformly. (See Ethier and Kurtz [8, Lemma 3.10.1].)

A function $f : \mathbb{S} \to \mathcal{L}(B)$ is said to be of *finite variation* (*continuous finite variation*) if for all $t \in \mathbb{R}_+$

$$V_f(t) := \sup\left\{\sum_{i=1}^{m} \|f(\tau_{i-1}, \tau_i)\| : m \in \mathbb{N},\, 0 \leqslant \tau_0 < \tau_1 < \cdots < \tau_m \leqslant t\right\} < \infty$$

(and $V_f \in C(\mathbb{R}_+, \mathbb{R})$). A function $f : \mathbb{S} \to \mathcal{L}(B)$ is of (continuous) finite variation if and only if for all $T > 0$, there exists a (continuous) function $v_T : [0, T] \to \mathbb{R}$ such that $\|f(s,t)\| \leqslant v_T(t) - v_T(s)$ for all $(s,t) \in \mathbb{S}_T$. (If $f : \mathbb{S} \to \mathcal{L}(B)$ is of (continuous) finite variation then the function $V_f$ is a suitable choice for $v_T$ for each $T > 0$.) Any function $f : \mathbb{S} \to \mathcal{L}(B)$ of continuous finite variation is necessarily continuous.

A function $g : \mathbb{R}_+ \to \mathcal{L}(B)$ is said to be of (*continuous*) *finite variation* if the function $(s,t) \mapsto g(t) - g(s)$ from $\mathbb{S}$ into $\mathcal{L}(B)$ enjoys the corresponding property. It is easy to see that a function $g : \mathbb{R}_+ \to \mathcal{L}(B)$ of finite variation is continuous if and only if it is of continuous finite variation.

Let $FV(\mathbb{R}_+, \mathcal{L}(B))$ and $FV(\mathbb{S}, \mathcal{L}(B))$ denote the spaces of all functions of finite variation in $D(\mathbb{R}_+, \mathcal{L}(B))$ and in $D(\mathbb{S}, \mathcal{L}(B))$, respectively.

**Lemma 2.1.** *If $f \in FV(\mathbb{R}_+, \mathcal{L}(B))$ then $V_f$ is càdlàg.*
*If in addition, $f \in C(\mathbb{R}_+, \mathcal{L}(B))$, then $V_f$ is continuous.*

Proof. Let $t \in \mathbb{R}_+$. For each $\varepsilon > 0$, let us choose $\delta \in (0,1)$ such that $\|f(t+h) - f(t)\| < \varepsilon/2$ for all $0 < h < \delta$, and choose $t =: t_0 < t_1 < \cdots < t_n := t+1$ such that $t_1 - t < \delta$ and

$$V_f(t+1) - V_f(t) - \sum_{j=1}^{n} \|f(t_j) - f(t_{j-1})\| < \frac{\varepsilon}{2}.$$

Then

$$V_f(t+1) - V_f(t) < \varepsilon + \sum_{j=2}^{n} \|f(t_j) - f(t_{j-1})\| \leqslant \varepsilon + V_f(t+1) - V_f(t_1),$$

hence for all $0 < h < t_1 - t$,

$$V_f(t+h) - V_f(t) \leqslant V_f(t_1) - V_f(t) < \varepsilon$$

implies right continuity of $V_f$. Monotonicity of $V_f$ implies existence of left limits. The second statement can be proved similarly. □

## 3. Lebesgue–Bochner–Stieltjes integral

In what follows we shall apply a generalization of the (bilinear) Lebesgue–Bochner–Stieltjes integral for operator-valued functions introduced by Bogdanowicz (see [2]–[5]).

Let us consider the semiring

$$\mathcal{A} := \{\, ]a,b] : a,b \in \mathbb{R}_+,\ a \leqslant b\}\,.$$

Let $g \in FV(\mathbb{R}_+, \mathcal{L}(B))$. The right continuity of $V_g$ implies that the set function $\beta_g : \mathcal{A} \to \mathcal{L}(B)$, defined by

$$\beta_g(]a,b]) := g(b) - g(a), \qquad ]a,b] \in \mathcal{A}$$

is a $\sigma$-additive set function of finite variation. (See Günzler [9, A 1.62].)

A function $h : \mathbb{R}_+ \to \mathcal{L}(B)$ is said to be *simple* (with respect to $\mathcal{A}$) if it has the form

$$h = \sum_{k=1}^{n} A_k \mathbb{1}_{]a_k,b_k]},$$

where $]a_k, b_k] \in \mathcal{A}$, $k = 1,\dots,n$ are pairwise disjoint intervals and $A_1,\dots$ $\dots, A_n \in \mathcal{L}(B)$. For a simple function $h : \mathbb{R}_+ \to \mathcal{L}(B)$ of the above form, let

$$\int_{]0,\infty[} h\,dg := \sum_{k=1}^{n} A_k(g(b_k) - g(a_k)) \in \mathcal{L}(B).$$

Let $L^1(\mathbb{R}_+, \mathcal{A}, g, \mathcal{L}(B))$ denote the set of functions $f : \mathbb{R}_+ \to \mathcal{L}(B)$ such that there exists a sequence $(h_m)_{m\geqslant 1}$ of simple functions with

$$\sup_{m\geqslant 2} 4^m \int_{]0,\infty[} \|h_m - h_{m-1}\| \, dV_g < \infty$$

and with $h_m \to f$ $\lambda_{V_g}$-a.s., where $\lambda_{V_g}$ denotes the Lebesgue–Stieltjes measure on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ generated by $V_g$. For $f \in L^1(\mathbb{R}_+, \mathcal{A}, g, \mathcal{L}(B))$, let

$$\int_{]0,\infty[} f \, dg := \lim_{m\to\infty} \int_{]0,\infty[} h_m \, dg,$$

where $(h_m)_{m\geqslant 1}$ is a sequence of simple functions with the above properties. For a set $A \subset \mathbb{R}_+$ and for a function $f : \mathbb{R}_+ \to \mathcal{L}(B)$ with $f \cdot \mathbb{1}_A \in L^1(\mathbb{R}_+, \mathcal{A}, g, \mathcal{L}(B))$, we write

$$\int_A f \, dg := \int_{]0,\infty[} f \cdot \mathbb{1}_A \, dg.$$

**Lemma 3.1.** *Let $g \in FV(\mathbb{R}_+, \mathcal{L}(B))$.*

(i) *The mapping $f \mapsto \int_{]0,\infty[} f \, dg$ from $L^1(\mathbb{R}_+, \mathcal{A}, g, \mathcal{L}(B))$ into $\mathcal{L}(B)$ is well-defined, linear, independent of a $\lambda_{V_g}$-a.s. modification of $f$, and*

$$\left\| \int_{]0,\infty[} f(\tau) \, dg(\tau) \right\| \leqslant \int_{]0,\infty[} \|f(\tau)\| \, dV_g(\tau).$$

(ii) *If $f \in D(\mathbb{R}_+, \mathcal{L}(B))$ then for all $0 \leqslant s < t$, $(f_-) \cdot \mathbb{1}_{]s,t]} \in L^1(\mathbb{R}_+, \mathcal{A}, g, \mathcal{L}(B))$, where $f_- : \mathbb{R}_+ \to \mathcal{L}(B)$ is defined by $f_-(0) := 0$ and $f_-(u) := f(u-) := \lim_{v\uparrow u} f(v)$ for $u > 0$. Moreover,*

$$\left\| \int_{]s,t]} f(\tau-) \, dg(\tau) \right\| \leqslant \int_{]s,t]} \|f(\tau-)\| \, dV_g(\tau).$$

*Particularly, for all $0 \leqslant s < t \leqslant T$,*

$$\left\| \int_{]s,t]} f(\tau-) \, dg(\tau) \right\| \leqslant \|f\|_T (V_g(t) - V_g(s)).$$

(iii) *If $f \in FV(\mathbb{R}_+, \mathcal{L}(B))$ then for all $0 \leqslant s < t \leqslant T$,*

$$\left\| \int_{]s,t]} f(\tau-) \, dg(\tau) \right\| \leqslant \|g\|_T (2\|f\|_T + V_f(t) - V_f(s)).$$

(iv) *If* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *is an increasing càdlàg function with* $h(0) = 0$ *then*

$$\int_{]0,t]} h(\tau-)^{m-1}\, dh(\tau) \leqslant \frac{h(t)^m}{m} \quad \text{for all } t > 0,\ m \in \mathbb{N}.$$

Proof. (i). See Günzler [9, A 2.57] or Bogdanowicz [2].

(ii) If $f : \mathbb{R}_+ \to \mathcal{L}(B)$ is left continuous then for all $T > 0$ and for all $\varepsilon > 0$ there exists a simple function $h_{T,\varepsilon} : \mathbb{R}_+ \to \mathcal{L}(B)$ such that $\sup_{t\in[0,T]} \|f(t) - h_{T,\varepsilon}(t)\| \leqslant \varepsilon$. (See Billingsley [1, Lemma 14.1].) Obviously for all $\varepsilon_1, \varepsilon_2 > 0$ and for all $0 \leqslant s < t \leqslant T$,

$$\int_{]s,t]} \|h_{T,\varepsilon_1} - h_{T,\varepsilon_2}\|\, dV_g \leqslant (\varepsilon_1 + \varepsilon_2) V_g(T),$$

hence we have

$$\sup_{m\geqslant 2} 4^m \int_{]s,t]} \|h_{T,4^{-m}} - h_{T,4^{-m+1}}\|\, dV_g < \infty$$

and $h_{T,4^{-m}}(t) \to f(t)$ for all $t \in [0,T]$. Consequently,
$f \cdot \mathbb{1}_{]s,t]} \in L^1(\mathbb{R}_+, \mathcal{A}, g, \mathcal{L}(B))$ for all $0 \leqslant s < t \leqslant T$.

(iii) This follows from (ii) using the subsequent partial integration formula:

$$\int_{]s,t]} f(\tau-)\, dg(\tau) + \int_{[s,t[} g(\tau+)\, df(\tau) = f(t-)g(t+) - f(s-)g(s+).$$

(Here the second integral is defined starting with the semiring $\widetilde{\mathcal{A}} := \{[a,b[ : a,b \in \mathbb{R}_+,\ a \leqslant b\}$ and the left continuous function $f_- \in FV(\mathbb{R}_+, \mathcal{L}(B))$, hence $g \cdot \mathbb{1}_{[s,t[} \in L^1(\mathbb{R}_+, \widetilde{\mathcal{A}}, f_-, \mathcal{L}(B))$, as in (ii).) Denoting $S_{s,t} := \{(u,v) \in \mathbb{R}^2 : s \leqslant u < v \leqslant t\}$, we have

$$\begin{aligned}\int_{[s,t]} \int_{[s,t]} \mathbb{1}_{S_{s,t}}(u,v)\, df(u)\, dg(v) &= \int_{]s,t]} (f(v-) - f(s-))\, dg(v)\\ &= \int_{]s,t]} f(v-)\, dg(v) - f(s-)(g(t+) - g(s+)),\end{aligned}$$

and

$$\int_{[s,t]} \int_{[s,t]} \mathbb{1}_{S_{s,t}}(u,v)\, dg(u)\, df(v) = \int_{[s,t[} (g(t+) - f(u+))\, df(u)$$

$$= g(t+)(f(t-) - f(s-)) - \int_{[s,t[} g(u+)\, df(u).$$

Hence Fubini's theorem (see Günzler [9, A 4.64] or Bogdanowicz [5]) implies the above partial integration formula.

(iv) It can be proved by Itô's formula in Jacod and Shiryayev [15, I 4.57]. (See Heyer and Pap [12], Lemma 2.1 (iii).) □

## 4. Generation of evolution families

*Definition 4.1.* A function $f : \mathbb{S} \to \mathcal{L}(B)$ is called *multiplicative* if $f(s,t)f(t,u) = f(s,u)$ for all $(s,t), (t,u) \in \mathbb{S}$, and $f(t,t) = I$ for all $t \in \mathbb{R}_+$.

A family $\{f(s,t) : 0 \leqslant s \leqslant t\}$ in $\mathcal{L}(B)$ is called an *evolution family* if the function $(s,t) \mapsto f(s,t)$ from $\mathbb{S}$ into $\mathcal{L}(B)$ is multiplicative and $f \in D(\mathbb{S}, \mathcal{L}(B))$.

An evolution family $\{f(s,t) : 0 \leqslant s \leqslant t\}$ in $\mathcal{L}(B)$ is said to be of (*continuous*) *finite variation* or *continuous* if the function $(s,t) \mapsto f(s,t) - I$ from $\mathbb{S}$ into $\mathcal{L}(B)$ enjoys the corresponding property.

*Remark 4.2.* If a function $f : \mathbb{S} \to \mathcal{L}(B)$ is multiplicative and $f - I$ is of finite variation then the function $f$ is of finite variation separately in each variable (see Schmidt [16, Proposition 8 (a)]).

If a function $f : \mathbb{S} \to \mathcal{L}(B)$ is multiplicative and $f - I$ is of continuous finite variation then the function $f$ is continuous; especially, $\{f(s,t) : 0 \leqslant s \leqslant t\}$ is a continuous evolution family in $\mathcal{L}(B)$. (This can be proved as Lemma 3.1 in Heyer and Pap [11].)

**Theorem 4.3.** *Let $g \in FV(\mathbb{R}_+, \mathcal{L}(B))$.*

(i) *There exists a unique $f \in D(\mathbb{R}_+, \mathcal{L}(B))$ such that*

$$f(t) = I + \int_{]0,t]} f(\tau-)\, dg(\tau) \quad \text{for all } t \in \mathbb{R}_+.$$

*In fact,*

$$f(t) = I + \sum_{k=1}^{\infty} \int_{0<\tau_k<\ldots<\tau_1\leqslant t} \!\!\cdots\!\! \int dg(\tau_k)\ldots dg(\tau_1) \quad \text{for all } t \in \mathbb{R}_+,$$

*and the series is locally uniformly convergent on* $\mathbb{R}_+$*. Moreover,* $f \in FV(\mathbb{R}_+, \mathcal{L}(B))$,

$$\|f\|_T \leqslant \exp\{V_g(T)\} \quad \textit{for all } T > 0, \quad \textit{and}$$
$$V_f(t) - V_f(s) \leqslant (V_g(t) - V_g(s)) \exp\{V_g(T)\} \quad \textit{for all } 0 \leqslant s \leqslant t \leqslant T.$$

*If in addition,* $g \in C(\mathbb{R}_+, \mathcal{L}(B))$*, then* $f \in C(\mathbb{R}_+, \mathcal{L}(B))$.

(ii) *There exists a unique* $f \in D(\mathbb{S}, \mathcal{L}(B))$ *such that*

$$f(s,t) = I + \int_{]s,t]} f(s,\tau-)\, dg(\tau) \quad \textit{for all } 0 \leqslant s \leqslant t.$$

*In fact,*

$$f(s,t) = I + \sum_{k=1}^{\infty} \int \cdots \int_{s<\tau_k<\cdots<\tau_1\leqslant t} dg(\tau_k) \dots dg(\tau_1) \quad \textit{for all } 0 \leqslant s \leqslant t,$$

*and the series is locally uniformly convergent on* $\mathbb{S}$*. Moreover,* $\{f(s,t) : 0 \leqslant s \leqslant t\}$ *is an evolution family of finite variation in* $\mathcal{L}(B)$.
*If in addition,* $g \in C(\mathbb{R}_+, \mathcal{L}(B))$*, then the evolution family* $\{f(s,t) : 0 \leqslant s \leqslant t\}$ *is of continuous finite variation.*

Proof. For $h \in D(\mathbb{R}_+, \mathcal{L}(B))$, let

$$Ah(t) := I + \int_{]0,t]} h(\tau-)\, dg(\tau) \quad \text{for all } t \in \mathbb{R}_+.$$

The inequalities in Lemma 3.1 (ii) imply that $Ah \in D(\mathbb{R}_+, \mathcal{L}(B))$.

Applying Lemma 3.1 (ii) and (iv) we obtain

$$\|A^m h_1 - A^m h_2\|_T \leqslant \|h_1 - h_2\|_T \frac{(V_g(T))^m}{m!}$$

for all $h_1, h_2 \in D(\mathbb{R}_+, \mathcal{L}(B))$ and for all $m \in \mathbb{N}$, and we can follow the line of argument used in the proof of Theorem 3.4 in Heyer and Pap [12] (based on usual fixed point method). □

*Definition 4.4.* Let $\{f(s,t) : 0 \leqslant s \leqslant t\}$ be an evolution family in $\mathcal{L}(B)$. A function $g \in FV(\mathbb{R}_+, \mathcal{L}(B))$ is called an *integrating function* of $\{f(s,t) : 0 \leqslant s \leqslant t\}$ if $g(0) = 0$ and

$$f(s,t) = I + \int_{]s,t]} f(s,\tau-)\, dg(\tau) \quad \text{for all } 0 \leqslant s \leqslant t.$$

*Remark 4.5.* If an evolution family $\{f(s,t) : 0 \leqslant s \leqslant t\}$ in $\mathcal{L}(B)$ has an integrating function then in view of Theorem 4.3 (ii), $\{f(s,t) : 0 \leqslant s \leqslant t\}$ is necessarily of finite variation.

**Theorem 4.6.** *Let $\{f(s,t) : 0 \leqslant s \leqslant t\}$ be an evolution family of finite variation in $\mathcal{L}(B)$. Then it has an integrating function. Moreover, $f$ is locally bounded, more precisely,*

$$\|f(s,t)\| \leqslant 1 + V_{f-I}(T) \quad \text{for all } 0 \leqslant s \leqslant t \leqslant T.$$

*If $g \in FV(\mathbb{R}_+, \mathcal{L}(B))$ is an integrating function of $f$ then*

$$\|f(s,t) - I - (g(t) - g(s))\| \leqslant (1 + V_{f-I}(T))(V_g(t) - V_g(s))^2 \quad \text{for all } 0 \leqslant s \leqslant t \leqslant T.$$

*If in addition, $f$ is continuous, then it has a continuous integrating function.*

*If in addition, $f$ is of continuous finite variation, then its integrating function is uniquely determined (and continuous).*

Proof. The existence of an integrating function can be proved exactly as in Theorem 3.10 in Heyer and Pap [12] (using Proposition 8 (a) (ii) in Schmidt [16] for explicit construction). Local boundedness follows from

$$\|f(s,t)\| \leqslant \|f(s,t) - I\| + \|I\| \leqslant 1 + V_{f-I}(T).$$

Using Lemma 3.1 (ii) we obtain for all $0 \leqslant s \leqslant t \leqslant T$,

$$\begin{aligned}\|f(s,t) - I\| &= \left\| \int_{]s,t]} f(s,\tau-)\, dg(\tau) \right\| \leqslant \int_{]s,t]} \|f(s,\tau-)\|\, dV_g(\tau) \\ &\leqslant (1 + V_{f-I}(T))(V_g(t) - V_g(s)).\end{aligned}$$

Consequently,

$$\begin{aligned}\|f(s,t) - I - (g(t) - g(s))\| &= \left\| \int_{]s,t]} (f(s,\tau-) - I)\, dg(\tau) \right\| \\ &\leqslant \int_{]s,t]} \|f(s,\tau-) - I\|\, dV_g(\tau) \leqslant (1 + V_{f-I}(T))(V_g(t) - V_g(s))^2.\end{aligned}$$

If $f$ is continuous, then the existence of a continuous integrating function can be proved exactly as in Theorem 3.10 in Heyer and Pap [12].

If $f$ is of continuous finite variation in $\mathcal{L}(B)$ and $g_1, g_2 \in FV(\mathbb{R}_+, \mathcal{L}(B))$ are integrating functions of $f$ then for all $0 \leqslant s \leqslant t \leqslant T$,

$$\begin{aligned} &\|(g_1(t) - g_1(s)) - (g_2(t) - g_2(s))\| \\ &\leqslant (1 + V_{f-I}(T)) \left( (V_{g_1}(t) - V_{g_1}(s))^2 + (V_{g_2}(t) - V_{g_2}(s))^2 \right). \end{aligned}$$

Hence for all $n \in \mathbb{N}$,

$$\begin{aligned} &\|g_1(t) - g_2(t)\| \\ &= \left\| \sum_{k=1}^{n} \left( \left( g_1\left(\frac{kt}{n}\right) - g_1\left(\frac{(k-1)t}{n}\right) \right) - \left( g_2\left(\frac{kt}{n}\right) - g_2\left(\frac{(k-1)t}{n}\right) \right) \right) \right\| \\ &\leqslant (1 + V_{f-I}(T)) \sum_{k=1}^{n} \left( \left( V_{g_1}\left(\frac{kt}{n}\right) - V_{g_1}\left(\frac{(k-1)t}{n}\right) \right)^2 \right. \\ &\quad \left. + \left( V_{g_2}\left(\frac{kt}{n}\right) - V_{g_2}\left(\frac{(k-1)t}{n}\right) \right)^2 \right) \\ &\leqslant (1 + V_{f-I}(T)) \left( V_{g_1}(T) \max_{1 \leqslant k \leqslant n} \left( V_{g_1}\left(\frac{kt}{n}\right) - V_{g_1}\left(\frac{(k-1)t}{n}\right) \right) \right. \\ &\quad \left. + V_{g_2}(T) \max_{1 \leqslant k \leqslant n} \left( V_{g_2}\left(\frac{kt}{n}\right) - V_{g_2}\left(\frac{(k-1)t}{n}\right) \right) \right). \end{aligned}$$

Since $n \in \mathbb{N}$ is arbitrary and, in view of Lemma 2.1, $V_{g_1}$ and $V_{g_2}$ are continuous we obtain $g_1(t) = g_2(t)$ for all $t \geqslant 0$. □

## 5. Convergence of evolution families

*Definition 5.1.* A sequence $(f_n)_{n \geqslant 1}$ in $D(\mathbb{R}_+, \mathcal{L}(B))$ is called *C-relatively compact* if it is relatively compact in $D(\mathbb{R}_+, \mathcal{L}(B))$ and if all limit points of the sequence $(f_n)_{n \geqslant 1}$ (with respect to the Skorokhod topology) are in $C(\mathbb{R}_+, \mathcal{L}(B))$.

For $f \in D(\mathbb{R}_+, \mathcal{L}(B))$, $T > 0$ and $\delta > 0$, let

$$\omega_T(f;\delta) := \sup\{\|f(t) - f(s)\| : 0 \leqslant s < t \leqslant T,\, t - s \leqslant \delta\},$$

$$\omega'_T(f;\delta) := \inf\left\{\max_{1\leqslant i\leqslant r} \sup_{t_{i-1}<s<t\leqslant t_i} \|f(t) - f(s)\| : \right.$$
$$\left. 0 = t_0 < \ldots < t_r = T,\, \min_{1\leqslant i\leqslant r}(t_i - t_{i-1}) \geqslant \delta\right\}.$$

The following lemma is well known (see Ethier and Kurtz [8, Theorem 3.6.3]).

**Lemma 5.2.** *Let $(f_n)_{n\geqslant 1}$ be a sequence in $D(\mathbb{R}_+, \mathcal{L}(B))$. Then the following statements are equivalent:*

(i) *$(f_n)_{n\geqslant 1}$ is relatively compact.*

(i) (a) *There is a dense subset $D$ of $\mathbb{R}_+$ and there are compact sets $K_t \subset \mathcal{L}(B)$, $t \in D$, such that $\{f_n(t) : n \in \mathbb{N}\} \subset K_t$ for all $t \in D$,*

(b) $\lim\limits_{\delta\to 0} \sup\limits_{n\geqslant 1} \omega'_T(f_n;\delta) = 0$ *for all $T > 0$.*

**Theorem 5.3.** *Let $(f_n)_{n\geqslant 1}$ be a sequence in $D(\mathbb{R}_+, \mathcal{L}(B))$. Then the following statements are equivalent:*

(i) *$(f_n)_{n\geqslant 1}$ is $C$-relatively compact.*

(ii) (a) *There is a dense subset $D$ of $\mathbb{R}_+$ and there are compact sets $K_t \subset \mathcal{L}(B)$, $t \in D$, such that $\{f_n(t) : n \in \mathbb{N}\} \subset K_t$ for all $t \in D$,*

(b) $\lim\limits_{\delta\to 0} \limsup\limits_{n\to\infty} \omega'_T(f_n;\delta) = 0$ *for all $T > 0$,*

(c) $\lim\limits_{n\to\infty} \sup\limits_{t\in[0,T]} \|f_n(t) - f_n(t-)\| = 0$ *for all $T > 0$.*

(iii) (a) *There is a dense subset $D$ of $\mathbb{R}_+$ and there are compact sets $K_t \subset \mathcal{L}(B)$, $t \in D$, such that $\{f_n(t) : n \in \mathbb{N}\} \subset K_t$ for all $t \in D$,*

(b) $\lim\limits_{\delta\to 0} \limsup\limits_{n\to\infty} \omega_T(f_n;\delta) = 0$ *for all $T > 0$.*

Proof. (i) $\Longrightarrow$ (ii) In view of Lemma 5.2, relative compactness of the sequence $(f_n)_{n\geqslant 1}$ implies (ii) (a) and (b). Moreover, (ii)(c) follows from Lemma 3.10.1 in Ethier and Kurtz [8].

(ii) $\Longrightarrow$ (iii) follows from the inequality

$$\omega_T(f;\delta) \leqslant 2\omega'_T(f;\delta) + \sup_{t\in[0,T]} \|f(t) - f(t-)\|$$

valid for all $T > 0$, $f \in D(\mathbb{R}_+, \mathcal{L}(B))$ and $\delta > 0$.

(iii) $\Longrightarrow$ (i) We have the inequality

$$\omega'_T(f;\delta) \leqslant 2\omega_T(f;\delta)$$

for all $T > 0$, $f \in D(\mathbb{R}_+, \mathcal{L}(B))$ and $\delta > 0$. (See BILLINGSLEY [1, (14.9)].) Hence, in view of Lemma 5.2, (iii) (a) and (b) imply relative compactness of the sequence $(f_n)_{n\geqslant 1}$. We also have the inequality

$$J_f(T) := \sup_{t\in[0,T]} \|f(t) - f(t-)\| \leqslant \omega_T(f;\delta)$$

valid for all $T > 0$, $f \in D(\mathbb{R}_+, \mathcal{L}(B))$ and $\delta > 0$, thus (iii) (b) implies

$$\lim_{n\to\infty} J_{f_n}(T) = 0.$$

If $f_{n'} \to f$ in the Skorokhod topology then $J_{f_{n'}} \to J_f$ in the Skorokhod topology (see Proposition 3.5.3 and the beginning of Section 3.10 in ETHIER and KURTZ [8]), hence $J_f(T) = 0$ for all continuity point of $J_f$. Since $J_f$ is nondecreasing, we conclude $J_f(T) = 0$ for all $T \in \mathbb{R}_+$, hence $f(t) = f(t-)$ for all $t > 0$. Consequently, the sequence $(f_n)_{n\geqslant 1}$ is $C$-relatively compact. □

**Theorem 5.4.** *Let $(g_n)_{n\geqslant 1}$ be a sequence in $FV(\mathbb{R}_+, \mathcal{L}(B))$. Suppose that*

(a) *there is a dense subset $D$ of $\mathbb{R}_+$ such that for all $t \in D$, the sequence $(g_n(t))_{n\geqslant 1}$ in $\mathcal{L}(B)$ is convergent,*

(b) $\lim_{\delta\to 0} \limsup_{n\to\infty} \omega_T(V_{g_n};\delta) = 0$ *for all $T > 0$.*

*Then the following assertions are valid:*

(i) *There is a function $g \in FV(\mathbb{R}_+, \mathcal{L}(B)) \bigcap C(\mathbb{R}_+, \mathcal{L}(B))$ such that $g_n \to g$ locally uniformly.*

(ii) *Let* $f \in FV(\mathbb{R}_+, \mathcal{L}(B))$ *and* $f_n \in FV(\mathbb{R}_+, \mathcal{L}(B))$, $n \in \mathbb{N}$, *such that*

$$f(t) = I + \int_{]0,t]} f(\tau-)\, dg(\tau) \qquad \text{for all } t \in \mathbb{R}_+,$$

$$f_n(t) = I + \int_{]0,t]} f_n(\tau-)\, dg_n(\tau) \qquad \text{for all } t \in \mathbb{R}_+.$$

*Then* $f \in C(\mathbb{R}_+, \mathcal{L}(B))$ *and* $f_n \to f$ *locally uniformly. More precisely, there exist* $c > 0$ *and* $c' > 0$ *such that*

$$\|f_n - f\|_T \leqslant c\|g_n - g\|_T \exp\{c' V_g(T)\} \qquad \text{for all } T > 0,\ n \in \mathbb{N}.$$

(iii) *Let* $h \in D(\mathbb{S}, \mathcal{L}(B))$ *and* $h_n \in D(\mathbb{S}, \mathcal{L}(B))$, $n \in \mathbb{N}$, *such that*

$$h(s,t) = I + \int_{]s,t]} h(s,\tau-)\, dg(\tau) \qquad \text{for all } 0 \leqslant s \leqslant t,$$

$$h_n(s,t) = I + \int_{]s,t]} h_n(s,\tau-)\, dg_n(\tau) \qquad \text{for all } 0 \leqslant s \leqslant t.$$

*Then* $h \in C(\mathbb{S}, \mathcal{L}(B))$ *and*

$$\lim_{n\to\infty} \sup_{0\leqslant s\leqslant t\leqslant T} \|h_n(s,t) - h(s,t)\| = 0 \qquad \text{for all } T > 0.$$

*More precisely,*

$$\|h_n(s,t) - h(s,t)\| \leqslant c\|g_n - g\|_T \exp\{c' V_g(T)\}$$
$$\text{for all } 0 \leqslant s \leqslant t \leqslant T,\ n \in \mathbb{N}.$$

Proof. It is performed in the same way as the proof of Theorem 4.4 in Heyer and Pap [12] (using again fixed point method). We remark that in [12] $g_n(0) = 0$ was supposed for all $n \in \mathbb{N}$, but it is not needed. □

**Theorem 5.5.** *Let* $\Gamma$ *be a topological space, and let* $\{g_\gamma : \gamma \in \Gamma\} \subset FV(\mathbb{R}_+, \mathcal{L}(B))$. *Suppose that*

(a) *the mapping* $\gamma \mapsto g_\gamma$ *from* $\Gamma$ *into* $D(\mathbb{R}_+, \mathcal{L}(B))$ *is continuous with respect to the local uniform topology (i.e., for all* $T > 0$, $\gamma \in \Gamma$ *and* $\delta > 0$, *there exists a neighborhood* $W \subset \Gamma$ *of* $\gamma$ *such that* $\|g_\gamma - g_{\gamma'}\|_T < \delta$ *for* $\gamma' \in W$*),*

(b) *the mapping* $\gamma \mapsto V_{g_\gamma}$ *from* $\Gamma$ *into* $D(\mathbb{R}_+, \mathbb{R}_+)$ *is locally bounded (i.e., for all* $T > 0$ *and* $\gamma \in \Gamma$*, there exists a neighborhood* $W \subset \Gamma$ *of* $\gamma$ *such that* $\sup_{\gamma' \in W} V_{g_{\gamma'}}(T) < \infty$*).*

*Then the following assertions are valid:*

(i) *Let* $f_\gamma \in FV(\mathbb{R}_+, \mathcal{L}(B))$, $\gamma \in \Gamma$, *such that*

$$f_\gamma(t) = I + \int_{]0,t]} f_\gamma(\tau-)\, dg_\gamma(\tau) \qquad \text{for all } t \in \mathbb{R}_+,\ \gamma \in \Gamma.$$

*Then for all* $t \geqslant 0$*, the mapping* $\gamma \mapsto f_\gamma(t)$ *from* $\Gamma$ *into* $\mathcal{L}(B)$ *is continuous. More precisely, for all* $T > 0$, $\gamma, \gamma' \in \Gamma$,

$$\|f_\gamma - f_{\gamma'}\|_T \leqslant 4\|g_\gamma - g_{\gamma'}\|_T \exp\left\{2V_{g_{\gamma'}}(T)\right\}.$$

(ii) *Let* $h_\gamma \in D(\mathbb{S}, \mathcal{L}(B))$, $\gamma \in \Gamma$, *such that*

$$h_\gamma(s,t) = I + \int_{]s,t]} h_\gamma(s, \tau-)\, dg_\gamma(\tau) \qquad \text{for all } 0 \leqslant s \leqslant t,\ \gamma \in \Gamma.$$

*Then for all* $0 \leqslant s \leqslant t$*, the mapping* $\gamma \mapsto h_\gamma(s,t)$ *from* $\Gamma$ *into* $\mathcal{L}(B)$ *is continuous. More precisely, for all* $T > 0$, $\gamma, \gamma' \in \Gamma$,

$$\sup_{0 \leqslant s \leqslant t \leqslant T} \|h_\gamma(s,t) - h_{\gamma'}(s,t)\| \leqslant 4\|g_\gamma - g_{\gamma'}\|_T \exp\left\{2V_{g_{\gamma'}}(T)\right\}.$$

Proof. For $\gamma \in \Gamma$, $f \in D(\mathbb{R}_+, \mathcal{L}(B))$ and $t \in \mathbb{R}_+$ let

$$A_\gamma f(t) := I + \int_{]0,t]} f(\tau-)\, dg_\gamma(\tau).$$

Then $A_\gamma f_\gamma = f_\gamma$ for all $\gamma \in \Gamma$.

Let $T > 0$ and $\gamma \in \Gamma$. The inequalities in Lemma 3.1 (ii) imply (as in the proof of Theorem 3.4 in Heyer and Pap [12]) that for all $f, \widetilde{f} \in D(\mathbb{R}_+, \mathcal{L}(B))$ and $m \in \mathbb{N}$,

$$\|A_\gamma^m f - A_\gamma^m \widetilde{f}\|_T \leqslant \|f - \widetilde{f}\|_T \frac{(V_{g_\gamma}(T))^m}{m!}.$$

We can choose $k \in \mathbb{N}$ such that

$$\frac{(V_{g_\gamma}(T))^k}{k!} \leqslant \frac{1}{2}.$$

We have for all $\gamma' \in \Gamma$,

$$\|f_\gamma - f_{\gamma'}\|_T \leqslant \|f_\gamma - A_\gamma^k f_{\gamma'}\|_T + \|A_\gamma^k f_{\gamma'} - f_{\gamma'}\|_T.$$

Clearly

$$\|f_\gamma - A_\gamma^k f_{\gamma'}\|_T = \|A_\gamma^k f_\gamma - A_\gamma^k f_{\gamma'}\|_T \leqslant \frac{1}{2}\|f_\gamma - f_{\gamma'}\|_T,$$

hence

$$\|f_\gamma - f_{\gamma'}\|_T \leqslant 2\|A_\gamma^k f_{\gamma'} - f_{\gamma'}\|_T.$$

Moreover,

$$\begin{aligned}\|A_\gamma^k f_{\gamma'} - f_{\gamma'}\|_T &\leqslant \sum_{j=0}^{k-1} \|A_\gamma^{j+1} f_{\gamma'} - A_\gamma^j f_{\gamma'}\|_T \leqslant \|A_\gamma f_{\gamma'} - f_{\gamma'}\| \sum_{j=0}^{k-1} \frac{(V_{g_\gamma}(T))^j}{j!}\\ &\leqslant \|A_\gamma f_{\gamma'} - f_{\gamma'}\| \exp\{V_{g_\gamma}(T)\}.\end{aligned}$$

Using Lemma 3.1 (iii) we obtain for all $t \in [0,T]$,

$$\begin{aligned}\|A_\gamma f_{\gamma'} - f_{\gamma'}\| = \|A_\gamma f_{\gamma'} - A_{\gamma'} f_{\gamma'}\| &= \left\| \int_{]0,t]} f_{\gamma'}(\tau) d(g_\gamma(\tau) - g_{\gamma'}(\tau)) \right\|\\ &\leqslant \|g_\gamma - g_{\gamma'}\|_T (2\|f_{\gamma'}\|_T + V_{f_{\gamma'}}(T)).\end{aligned}$$

By Theorem 4.3 (i),

$$\|f_{\gamma'}\|_T \leqslant \exp\{V_{g_{\gamma'}}(T)\} \quad \text{and} \quad V_{f_{\gamma'}}(T) \leqslant V_{g_{\gamma'}}(T) \exp\{V_{g_{\gamma'}}(T)\},$$

hence

$$\begin{aligned}\|A_\gamma f_{\gamma'} - f_{\gamma'}\|_T &\leqslant \|g_\gamma - g_{\gamma'}\|_T (2 + V_{g_{\gamma'}}(T)) \exp\{V_{g_{\gamma'}}(T)\}\\ &\leqslant 2\|g_\gamma - g_{\gamma'}\|_T \exp\{2V_{g_{\gamma'}}(T)\}.\end{aligned}$$

Collecting the estimates we conclude that

$$\|f_\gamma - f_{\gamma'}\|_T \leqslant 4\|g_\gamma - g_{\gamma'}\|_T \exp\{2V_{g_{\gamma'}}(T)\}.$$

Applying (i) for the functions $g_\gamma^{(s)}(t) := g_\gamma(s+t)$, $\gamma \in \Gamma$, $t \in \mathbb{R}_+$, $s \in \mathbb{R}_+$, we obtain

$$\begin{aligned}&\sup_{t\in[s,T]} \|h_\gamma(s,t) - h_{\gamma'}(s,t)\|\\ &\leqslant 4 \sup_{t\in[s,T]} \|g_\gamma(t) - g_{\gamma'}(t)\| \exp\{2(V_{g_{\gamma'}}(T) - V_{g_{\gamma'}}(s))\}\end{aligned}$$

for all $s \in \mathbb{R}_+$ and $T \in [s, \infty[$, which implies

$$\sup_{0 \leqslant s \leqslant t \leqslant T} \|h_\gamma(s,t) - h_{\gamma'}(s,t)\| \leqslant 4\|g_\gamma - g_{\gamma'}\|_T \exp\{2V_{g_{\gamma'}}(T)\},$$

hence the assertion in (ii) has been proved. □

## 6. Convergence of convolution hemigroups

Let $G$ be a locally compact group. A *representation* of $G$ is in our context always a continuous homomorphism $U$ from $G$ into the group $\mathcal{U}(\mathcal{H}_U)$ of unitary operators on a (complex) Hilbert space $\mathcal{H}_U$. More precisely one speaks of a continuous, unitary representation of $G$ with representing Hilbert space $\mathcal{H}_U$. The totality of representations of $G$ will be abbreviated by $\mathrm{Rep}(G)$.

$U \in \mathrm{Rep}(G)$ is said to be *irreducible* if there exists no nontrivial closed $U$-invariant subspace of $\mathcal{H}_U$. By the Gelfand–Raikov theorem the set $\mathrm{Irr}(G)$ of irreducible representations of $G$ separates the points of $G$.

For a locally compact space $E$, let $\mathfrak{M}^b(E)$ denote the Banach space of (real bounded Radon) measures on $E$ considered as continuous linear functionals on the space of continuous real-valued functions with compact supports on $E$. Subsets of $\mathfrak{M}^b(E)$ of particular interest are the cone $\mathfrak{M}^b_+(E)$ of nonnegative measures and the convex set $\mathfrak{M}^1(E)$ of probability measures on $E$. The set $\mathfrak{M}^b_+(E)$ will be equipped with the weak topology $\tau_w$ (of measures). For every $x \in E$, $\varepsilon_x$ denotes the Dirac measure in $x$.

Once $E$ is assumed to be a locally compact group $G$, the space $\mathfrak{M}^b(G)$ becomes a Banach algebra with respect to convolution (of measures), and $\mathfrak{M}^1(G)$ becomes a semigroup with unit element $\varepsilon_e$ (where $e$ denotes the neutral element of the multiplicatively written group $G$).

Finally we note that $\mathfrak{M}^1(G)$ is metrizable and separable iff $G$ admits a countable basis of its topology.

The *Fourier transform* of a measure $\mu \in \mathfrak{M}^b(G)$ is given by

$$\langle \widehat{\mu}(U)u, v\rangle = \int_G \langle U(x)u, v\rangle \, \mu(dx)$$

whenever $U \in \mathrm{Rep}(G)$, $u, v \in \mathcal{H}_U$.

Clearly, for given $U \in \mathrm{Rep}(G)$, $\widehat{\mu}(U)$ belongs to the space $\mathcal{L}(\mathcal{H}_U)$ of bounded linear operators on $\mathcal{H}_U$. In fact, one has

$$\|\widehat{\mu}(U)\| \leqslant \|\mu\|.$$

Moreover, the mapping $\mu \mapsto \widehat{\mu}$ from $\mathfrak{M}^b(G)$ into the set of mappings $\mathrm{Rep}(G) \to \bigcup\{\mathcal{L}(\mathcal{H}_U) : U \in \mathrm{Rep}(G)\}$ is injective (even on $\mathrm{Irr}(G)$ ), linear, multiplicative, and sequentially bicontinuous in the sense of the following equivalences expressed for sequences $(\mu_n)_{n\geqslant 0}$ of measures in $\mathfrak{M}^1(G)$:

(i) $\mu_n \to \mu_0$.

(ii) $\widehat{\mu}_n(U)u \to \widehat{\mu}_0(U)u$ for all $U \in \mathrm{Irr}(G)$, $u \in \mathcal{H}_U$.

(iii) $\langle\widehat{\mu}_n(U)u, v\rangle \to \langle\widehat{\mu}_0(U)u, v\rangle$ for all $U \in \mathrm{Irr}(G)$, $u, v \in \mathcal{H}_U$.

(For the proof see, for example, Siebert [17].)

*Definition 6.1.* A mapping $(s,t) \mapsto \mu(s,t)$ from $\mathbb{S}$ into $\mathfrak{M}^1(G)$ is said to be *multiplicative* if $\mu(s,r) * \mu(r,t) = \mu(s,t)$ for all $0 \leqslant s \leqslant r \leqslant t$, and $\mu(t,t) = \varepsilon_e$ for all $t \in \mathbb{R}_+$.

A family $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ in $\mathfrak{M}^1(G)$ is called a *convolution hemigroup* (briefly hemigroup) in $\mathfrak{M}^1(G)$ if the mapping $(s,t) \mapsto \mu(s,t)$ from $\mathbb{S}$ into $\mathfrak{M}^1(G)$ is multiplicative and càdlàg.

A convolution hemigroup $\{\mu(s,t) : (s,t) \in \mathbb{S}\}$ in $\mathfrak{M}^1(G)$ is said to be *continuous* if the mapping $(s,t) \mapsto \mu(s,t)$ from $\mathbb{S}$ into $\mathfrak{M}^1(G)$ is continuous.

Let $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ be a family in $\mathfrak{M}^1(G)$. Then the following statements are equivalent:

(i) $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ is a convolution hemigroup in $\mathfrak{M}^1(G)$,

(ii) $\{\widehat{\mu}(s,t)(U) : 0 \leqslant s \leqslant t\}$ is an evolution family in $\mathcal{L}(\mathcal{H}_U)$ for every $U \in \mathrm{Irr}(G)$.

Let $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ be a convolution hemigroup in $\mathfrak{M}^1(G)$. Then the continuity of $\{\widehat{\mu}(s,t)(U) : 0 \leqslant s \leqslant t\}$ for every $U \in \mathrm{Irr}(G)$ implies the continuity of $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$. (We note that the space $\mathcal{L}(\mathcal{H}_U)$ has been endowed with the norm topology.)

*Definition 6.2.* Let $\Gamma \subset \mathrm{Rep}(G)$. A mapping $(s,t) \mapsto \mu(s,t)$ from $\mathbb{S}$ into $\mathfrak{M}^1(G)$ is said to be of *(continuous) $\mathfrak{F}$-finite variation with respect to* $\Gamma$ if for all $U \in \Gamma$, the function $(s,t) \mapsto \widehat{\mu}(s,t)(U) - I$ from $\mathbb{S}$ into $\mathcal{L}(\mathcal{H}_U)$ is of (continuous) finite variation.

*Remark 6.3.* If $G$ is a locally compact group and a mapping $(s,t) \mapsto \mu(s,t)$ from $\mathbb{S}$ into $\mathfrak{M}^1(G)$ is multiplicative and of continuous $\mathfrak{F}$-finite variation in $\mathfrak{M}^1(G)$ with respect to $\mathrm{Irr}(G)$ then it is continuous; especially, $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ is a continuous convolution hemigroup in $\mathfrak{M}^1(G)$.

*Definition 6.4.* Let $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ be a convolution hemigroup in $\mathfrak{M}^1(G)$. Let $\Gamma \subset \mathrm{Rep}(G)$. A family of mappings $\varphi^U \in FV(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$, $U \in \Gamma$ is called an *integrating family* related to $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ if for all $U \in \Gamma$, we have $\varphi^U(0) = 0$ and

$$\widehat{\mu}(s,t)(U) = I + \int_{]s,t]} \widehat{\mu}(s,\tau-)(U)\, d\varphi^U(\tau) \qquad \text{for all } (s,t) \in \mathbb{S}.$$

If a convolution hemigroup $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ in $\mathfrak{M}^1(G)$ has an integrating family with $\Gamma \subset \mathrm{Rep}(G)$ then it is necessarily of $\mathfrak{F}$-finite variation with respect to $\Gamma$. Moreover, if $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ is a convolution hemigroup of $\mathfrak{F}$-finite variation in $\mathfrak{M}^1(G)$ with respect to $\Gamma \subset \mathrm{Rep}(G)$ then by Theorem 4.6 it has an integrating family with parameter set $\Gamma$. If $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ is a convolution hemigroup of continuous $\mathfrak{F}$-finite variation in $\mathfrak{M}^1(G)$ with respect to $\Gamma$ then the integrating family $\{\varphi^U : U \in \Gamma\}$ of $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ is uniquely determined and consists of continuous functions in view of Theorem 4.6.

**Theorem 6.5.** *Let* $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ *and* $\{\mu_n(s,t) : 0 \leqslant s \leqslant t\}$, $n \geqslant 1$, *be convolution hemigroups of* $\mathfrak{F}$*-finite variation in* $\mathfrak{M}^1(G)$ *with respect to* $\mathrm{Irr}(G)$. *Let* $\{\varphi^U : U \in \mathrm{Irr}(G)\}$ *and* $\{\varphi_n^U : U \in \mathrm{Irr}(G)\}$, $n \geqslant 1$, *be some related integrating families. Suppose that for all* $U \in \mathrm{Irr}(G)$,

(a) *there is a dense subset* $D$ *of* $\mathbb{R}_+$ *such that* $\lim_{n\to\infty} \varphi_n^U(t) = \varphi^U(t)$ *for all* $t \in D$,

(b) $\lim_{\delta\to 0} \limsup_{n\to\infty} \omega_T(V_{\varphi_n^U}; \delta) = 0$ *for all* $T > 0$.

*Then*

$$\mu_n(s,t) \to \mu(s,t) \qquad \text{for all } 0 \leqslant s \leqslant t,$$

*and* $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ *is a convolution hemigroup of continuous* $\mathfrak{F}$*-finite variation with respect to* $\mathrm{Irr}(G)$.

Proof. The statement follows from Theorem 5.4 (iii) applied for the sequence $(\varphi_n^U)_{n\geqslant 1}$ in $FV(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$ for all $U \in \mathrm{Irr}(G)$. □

**Corollary 6.6.** *Let* $\{\mu_{n,\ell} : n, \ell \in \mathbb{N}\}$ *be an array in* $\mathfrak{M}^1(G)$. *For all* $n \in \mathbb{N}$ *let* $k_n : \mathbb{R}_+ \to \mathbb{Z}_+$ *be an increasing, right continuous function with* $k_n(0) = 0$ *and* $k_n(\mathbb{R}_+) = \mathbb{Z}_+$.

*Let* $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ *be a convolution hemigroup of* $\mathfrak{F}$*-finite variation in* $\mathfrak{M}^1(G)$ *with respect to* $\operatorname{Irr}(G)$. *Let* $\{\varphi^U : U \in \operatorname{Irr}(G)\}$ *be some related integrating family. Suppose that for all* $U \in \operatorname{Irr}(G)$,

(a) *there is a dense subset* $D$ *of* $\mathbb{R}_+$ *such that* $\lim\limits_{n\to\infty} \sum\limits_{\ell=1}^{k_n(t)} (\widehat{\mu}_{n,\ell}(U) - I) = \varphi^U(t)$ *for all* $t \in D$,

(b) $\lim\limits_{\delta\to 0} \limsup\limits_{n\to\infty} \sup\limits_{\substack{0\leqslant s\leqslant t\leqslant T\\ t-s\leqslant\delta}} \sum\limits_{\ell=k_n(s)+1}^{k_n(t)} \|\widehat{\mu}_{n,\ell}(U) - I\| = 0$ *for all* $T > 0$.

*Then*

$$\mathop{\Huge *}_{\ell=k_n(s)+1}^{k_n(t)} \mu_{n,\ell} \to \mu(s,t) \qquad \text{for all } 0 \leqslant s \leqslant t,$$

*and* $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ *is a convolution hemigroup of continuous* $\mathfrak{F}$*-finite variation with respect to* $\operatorname{Irr}(G)$.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. If $\{X(t) : t \geqslant 0\}$ is a $G$-valued càdlàg process with independent left increments then the distributions

$$\mu(s,t) := \mathbb{P}_{X(s)^{-1}X(t)} \qquad \text{for } 0 \leqslant s \leqslant t$$

of the left increments $X(s)^{-1}X(t)$ form a convolution hemigroup $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ in $\mathfrak{M}^1(G)$. The process $\{X(t) : t \geqslant 0\}$ is stochastically continuous if and only if the convolution hemigroup $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ is continuous. The process $\{X(t) : t \in \mathbb{R}_+\}$ is said to be of (continuous) $\mathfrak{F}$-finite variation with respect to $\Gamma \subset \operatorname{Irr}(G)$ if the convolution hemigroup $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ enjoys the corresponding property.

**Theorem 6.7.** *Let* $G$ *be a second countable locally compact group. Let* $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ *be a convolution hemigroup of* $\mathfrak{F}$*-finite variation in* $\mathfrak{M}^1(G)$ *with respect to* $\operatorname{Irr}(G)$. *Let* $\{\varphi^U : U \in \operatorname{Irr}(G)\}$ *be some related integrating family. Let* $\{X_n(t) : t \in \mathbb{R}_+\}$, $n \geqslant 1$, *be* $G$*-valued càdlàg processes with independent left increments having* $\mathfrak{F}$*-finite variation with respect to* $\operatorname{Irr}(G)$. *Let* $\{\varphi_n^U : U \in \operatorname{Irr}(G)\}$, $n \geqslant 1$, *be some related integrating families. Suppose that for all* $U \in \operatorname{Irr}(G)$,

(a) *there is a dense subset $D$ of $\mathbb{R}_+$ such that* $\lim_{n\to\infty} \varphi_n^U(t) = \varphi^U(t)$ *for all* $t \in D$,

(b) $\lim_{\delta\to 0} \limsup_{n\to\infty} \omega_T(V_{\varphi_n^U}; \delta) = 0$ *for all* $T > 0$.

*Then there exists a stochastically continuous càdlàg process* $\{X(t) : t \in \mathbb{R}_+\}$ *with independent left increments having continuous* $\mathfrak{F}$*-finite variation with respect to* $\operatorname{Irr}(G)$ *such that*

$$X_n \xrightarrow{\mathcal{L}} X$$

*(in the sense of weak convergence of the induced measures on the space* $D(\mathbb{R}_+, G)$*), and* $\mathbb{P}_{X(s)^{-1}X(t)} = \mu(s,t)$ *for all* $0 \leqslant s \leqslant t$.

Proof. It can be carried out exactly as the proof of Theorem 5.10 in Heyer and Pap [12]. □

For a $G$-valued random variable $X$ and for $U \in \operatorname{Rep}(G)$, we can define the expectation $\mathbb{E}(U \circ X) \in \mathcal{L}(\mathcal{H}_U)$ by

$$\langle \mathbb{E}(U \circ X)u, v\rangle = \mathbb{E}\big(\langle (U \circ X)u, v\rangle\big)$$

whenever $u, v \in \mathcal{H}_U$, hence in fact, $\mathbb{E}(U \circ X) = \widehat{\mathbb{P}}_X(U)$. Eventually, we have for all $u \in \mathcal{H}_U$ that

$$\mathbb{E}(U \circ X)u = \int_\Omega (U \circ X)(\omega)u \, \mathbb{P}(d\omega),$$

where the integral on the right hand side is a Pettis integral. However, if $G$ is a second countable locally compact group then

$$\mathbb{E}(U \circ X) = \int_\Omega (U \circ X)(\omega) \, \mathbb{P}(d\omega),$$

where the integral on the right hand side is a Bochner integral. Indeed, the Bochner integral $\int_\Omega Z(\omega)\,\mathbb{P}(d\omega)$ of a random variable $Z : \Omega \to B$ with values in a Banach space $B$ exists if $\mathbb{E}\|Z\| < \infty$ and $Z$ is almost separable-valued, i.e., there exists $\Omega_0 \in \mathcal{A}$ such that $\mathbb{P}(\Omega_0) = 1$ and $\{Z(\omega) : \omega \in \Omega_0\}$ is a separable subset of $B$. A second countable locally compact group $G$ is necessarily separable (since there exists a metric which induces its topology and then the second countability implies separability), consequently, for all $U \in \operatorname{Rep}(G)$, the set $\{U(x) : x \in G\}$ is separable in $\mathcal{L}(\mathcal{H}_U)$. Since $\|U(x)\| = 1$ for all $x \in G$, the Bochner integral $\int_\Omega (U \circ X)(\omega)\,\mathbb{P}(d\omega)$ exists for all $G$-valued random variables $X$ and for all $U \in \operatorname{Rep}(G)$.

**Corollary 6.8.** *Let* $\{X_{n,\ell} : n, \ell \in \mathbb{N}\}$ *be an array of rowwise independent* $G$*-valued random variables. For all* $n \in \mathbb{N}$*, let* $k_n : \mathbb{R}_+ \to \mathbb{Z}_+$ *be an increasing, right continuous function with* $k_n(0) = 0$ *and* $k_n(\mathbb{R}_+) = \mathbb{Z}_+$*.*

*Let* $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ *be a convolution hemigroup of* $\mathfrak{F}$*-finite variation in* $\mathfrak{M}^1(G)$ *with respect to* $\mathrm{Irr}(G)$*. Let* $\{\varphi^U : U \in \mathrm{Irr}(G)\}$ *be some related integrating family. Suppose that for all* $U \in \mathrm{Irr}(G)$*,*

(a) *there is a dense subset* $D$ *of* $\mathbb{R}_+$ *such that* $\lim\limits_{n\to\infty} \sum\limits_{\ell=1}^{k_n(t)} (\mathbb{E}(U \circ X_{n,\ell}) - I) = \varphi^U(t)$ *for all* $t \in D$*,*

(b) $\lim\limits_{\delta\to 0} \limsup\limits_{n\to\infty} \sup\limits_{\substack{0\leqslant s\leqslant t\leqslant T \\ t-s\leqslant\delta}} \sum\limits_{\ell=k_n(s)+1}^{k_n(t)} \|\mathbb{E}(U \circ X_{n,\ell}) - I\| = 0$ *for all* $T > 0$*.*

*Then there exists a stochastically continuous càdlàg process* $\{X(t) : t \in \mathbb{R}_+\}$ *with independent left increments having continuous* $\mathfrak{F}$*-finite variation with respect to* $\mathrm{Irr}(G)$ *such that*

$$\prod_{\ell=1}^{k_n(\cdot)} X_{n,\ell} \xrightarrow{\mathcal{L}} X(\cdot),$$

*and* $\mathbb{P}_{X(s)^{-1}X(t)} = \mu(s,t)$ *for all* $0 \leqslant s \leqslant t$*.*

If one wishes to consider the convergence of a sequence of hemigroups in $\mathfrak{M}^1(G)$ without specifying the limiting hemigroup a more sophisticated approach is necessary. For a cardinal $\alpha$, let $\mathcal{H}(\alpha)$ be an $\alpha$-dimensional complex Hilbert space and let $\mathrm{Rep}_\alpha(G)$ denote the set of representations $U \in \mathrm{Rep}(G)$ with representing Hilbert space $\mathcal{H}(\alpha)$.

*Definition 6.9.* A locally compact group $G$ is said to have the *Lévy continuity property* with respect to a subset $\Gamma \subset \mathrm{Rep}(G)$ if there is a topology in $\Gamma$ such that if $(\mu_n)_{n\geqslant 1}$ is a sequence in $\mathfrak{M}^1(G)$, the mapping $h : \Gamma \to \bigcup\limits_{U\in\Gamma} \mathcal{L}(\mathcal{H}_U)$ is continuous on $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ for each cardinal $\alpha$ and

$$\widehat{\mu}_n(U) \to h(U) \qquad \text{for all } U \in \Gamma$$

then there exists a measure $\mu \in \mathfrak{M}^1(G)$ satisfying $\mu_n \to \mu$ and $\widehat{\mu}(U) = h(U)$ for all $U \in \Gamma$.

*Remark 6.10.* The Fell topology in $\Gamma$ (see, e.g., DIXMIER [7]) could be a natural candidate. We note that the Lévy continuity property used in HEYER [10] and BOUGEROL [6] is slightly different since it contains convergence $\widehat{\mu}_n(U) \to h(U)$ only in the strong operator topology on $\mathcal{L}(\mathcal{H}_U)$ (and not in the norm topology as in the Definition 6.9), but only the subset $\Gamma = \mathrm{Rep}^{(f)}(G)$ of all finite-dimensional representations of $G$ is considered and continuity of the mapping $h : \mathrm{Rep}^{(f)}(G) \to \bigcup_{U\in\Gamma} \mathcal{L}(\mathcal{H}_U)$ on the whole set $\mathrm{Rep}^{(f)}(G)$ is supposed. A locally compact group $G$ is called a *Moore group* once $\mathrm{Irr}(G) \subset \mathrm{Rep}^{(f)}(G)$. A Moore group has the Lévy continuity property with respect to the set $\mathrm{Rep}^{(f)}(G)$ (see HEYER [10]). Further classes of locally compact groups having the Lévy continuity property have been discussed in the work [6] of BOUGEROL.

**Theorem 6.11.** *Let $G$ be a locally compact group with Lévy continuity property with respect to a subset $\Gamma \subset \mathrm{Rep}(G)$. Let $\{\mu_n(s,t) : 0 \leqslant s \leqslant t\}$, $n \geqslant 1$, be convolution hemigroups of $\mathfrak{F}$-finite variation in $\mathfrak{M}^1(G)$ with respect to $\Gamma$. Let $\{\varphi_n^U : U \in \Gamma\}$, $n \geqslant 1$, be some related integrating families. Suppose that for all $U \in \Gamma$,*

(a) *there is a dense subset $D$ of $\mathbb{R}_+$ such that the sequence $(\varphi_n^U(t))_{n\geqslant 1}$ in $\mathcal{L}(\mathcal{H}_U)$ is convergent for all $t \in D$,*

(b) $\lim_{\delta\to 0} \limsup_{n\to\infty} \omega_T(V_{\varphi_n^U}; \delta) = 0$ *for all $T > 0$.*

*Then there is a family of mappings $\varphi^U \in FV(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U)) \bigcap C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$, $U \in \Gamma$, such that $\varphi_n^U \to \varphi^U$ locally uniformly for all $U \in \Gamma$.*

*If in addition, for each cardinal $\alpha$,*

(c) *the mapping $U \mapsto \varphi^U$ from $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ into $C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}(\alpha)))$ is continuous (i.e., for all $T > 0$, $U \in \Gamma \bigcap \mathrm{Rep}_\alpha(G)$ and $\delta > 0$, there exists a neighborhood $W \subset \Gamma \bigcap \mathrm{Rep}_\alpha(G)$ of $U$ such that $\|\varphi^U - \varphi^{U'}\|_T < \delta$ for all $U' \in W$),*

(d) *the mapping $U \mapsto V_{\varphi^U}$ from $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ into $C(\mathbb{R}_+, \mathbb{R}_+)$ is locally bounded (i.e., for all $T > 0$ and $U \in \Gamma \bigcap \mathrm{Rep}_\alpha(G)$, there exists a neighborhood $W \subset \Gamma \bigcap \mathrm{Rep}_\alpha(G)$ of $U$ such that $\sup_{U'\in W} V_{\varphi^{U'}}(T) < \infty$),*

*then there is a convolution hemigroup $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ of continuous $\mathfrak{F}$-finite variation with respect to $\Gamma$ such that*

$$\mu_n(s,t) \to \mu(s,t) \qquad \text{for all } 0 \leqslant s \leqslant t,$$

*and* $\{\varphi^U : U \in \Gamma\}$ *is an integrating family related to* $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$.

Proof. Let $U \in \Gamma$. Assumption (b) implies that for all $T > 0$, we have

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \omega_T(\varphi_n^U; \delta) = 0.$$

Assumption (a) implies that for all $t \in D$, the set $\{\varphi_n^U(t) : n \geqslant 1\}^-$ is compact in $\mathcal{L}(\mathcal{H}_U)$. Applying Theorem 5.3 we conclude that for each subsequence $(n')$ of $(n)$ there are a subsequence $(n'')$ of $(n')$ and a function $\varphi^U \in C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$ such that $\varphi_{n''}^U \to \varphi^U$ locally uniformly. Assumption (a) implies that the function $\varphi^U$ is uniquely determined, hence $\varphi_n^U \to \varphi^U$ locally uniformly.

Applying Theorem 5.4 (iii) for the sequence $(\varphi_n^U)_{n\geqslant 1}$ in $FV(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$ we obtain

$$\|\widehat{\mu}_n(s,t)(U) - h_U(s,t)\| \to 0 \qquad \text{for all } 0 \leqslant s \leqslant t,$$

where $\{h_U(s,t) : 0 \leqslant s \leqslant t\}$ is an evolution family of continuous finite variation in $\mathcal{L}(\mathcal{H}_U)$ with integrating function $\varphi^U$:

$$h_U(s,t) = I + \int_{]s,t]} h_U(s,\tau-)\, d\varphi^U(\tau) \qquad \text{for all } 0 \leqslant s \leqslant t.$$

Using conditions (i) and (ii), and applying Theorem 5.5 to the family $\{\varphi^U : U \in \Gamma \bigcap \mathrm{Rep}_\alpha(G)\}$ with a cardinal $\alpha$ we obtain that for all $0 \leqslant s \leqslant t$, the function $U \mapsto h_U(s,t)$ from $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ into $\mathcal{L}(\mathcal{H}(\alpha))$ is continuous. Since the group $G$ has the Lévy continuity property, we arrive at the assertion. □

**Corollary 6.12.** *Let $G$ be a locally compact group with Lévy continuity property with respect to a subset $\Gamma \subset \mathrm{Rep}(G)$. Let $\{\mu_{n,\ell} : (n,\ell) \in \mathbb{N}^2\}$ be an array in $\mathfrak{M}^1(G)$. For all $n \in \mathbb{N}$ let $k_n : \mathbb{R}_+ \to \mathbb{Z}_+$ be an increasing, right continuous function with $k_n(0) = 0$ and $k_n(\mathbb{R}_+) = \mathbb{Z}_+$.*

*Suppose that for all $U \in \Gamma$,*

(a) *there is a dense subset $D$ of $\mathbb{R}_+$ such that the sequence* $\left(\sum_{\ell=1}^{k_n(t)} (\widehat{\mu}_{n,\ell}(U) - I)\right)_{n\geqslant 1}$ *in $\mathcal{L}(\mathcal{H}_U)$ is convergent for all $t \in D$,*

(b) $\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{\substack{0\leqslant s\leqslant t\leqslant T\\ t-s\leqslant\delta}} \sum_{\ell=k_n(s)+1}^{k_n(t)} \|\widehat{\mu}_{n,\ell}(U) - I\| = 0$ *for all $T > 0$.*

*Then there is a family of mappings* $\varphi^U \in FV(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U)) \bigcap C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$, $U \in \Gamma$, *such that*

$$\sup_{t\in[0,T]} \left\| \sum_{\ell=1}^{k_n(t)} (\widehat{\mu}_{n,\ell}(U) - I) - \varphi^U(t) \right\| \to 0 \qquad \text{for all } T > 0,\ U \in \Gamma.$$

*If in addition, for each cardinal* $\alpha$,

(c) *the mapping* $U \mapsto \varphi^U$ *from* $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ *into* $C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}(\alpha)))$ *is continuous,*

(d) *the mapping* $U \mapsto V_{\varphi^U}$ *from* $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ *into* $C(\mathbb{R}_+, \mathbb{R}_+)$ *is locally bounded,*

*then*

$$\mathop{\Huge *}_{\ell=k_n(s)+1}^{k_n(t)} \mu_{n,\ell} \to \mu(s,t) \qquad \text{for all } 0 \leqslant s \leqslant t,$$

*where* $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$ *is a convolution hemigroup of continuous* $\mathfrak{F}$*-finite variation with respect to* $\Gamma$, *and* $\{\varphi^U : U \in \Gamma\}$ *is an integrating family related to* $\{\mu(s,t) : 0 \leqslant s \leqslant t\}$.

**Theorem 6.13.** *Let* $G$ *be a second countable locally compact group with Lévy continuity property with respect to a subset* $\Gamma \subset \mathrm{Rep}(G)$. *Let* $\{X_n(t) : t \in \mathbb{R}_+\}$, $n \geqslant 1$, *be* $G$*-valued càdlàg processes with independent left increments having* $\mathfrak{F}$*-finite variation with respect to* $\Gamma$. *Let* $\{\varphi_n^U : U \in \Gamma\}$, $n \geqslant 1$, *be some related integrating families. Suppose that for all* $U \in \Gamma$,

(a) *there is a dense subset* $D$ *of* $\mathbb{R}_+$ *such that the sequence* $(\varphi_n^U(t))_{n\geqslant1}$ *in* $\mathcal{L}(\mathcal{H}_U)$ *is convergent for all* $t \in D$,

(b) $\lim_{\delta\to0} \limsup_{n\to\infty} \omega_T(V_{\varphi_n^U}; \delta) = 0$ *for all* $T > 0$.

*Then there is a family of mappings* $\varphi^U \in FV(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U)) \bigcap C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$, $U \in \Gamma$, *such that* $\varphi_n^U \to \varphi^U$ *locally uniformly for all* $U \in \Gamma$.

*If in addition, for each cardinal* $\alpha$,

(c) *the mapping* $U \mapsto \varphi^U$ *from* $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ *into* $C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}(\alpha)))$ *is continuous,*

(d) *the mapping* $U \mapsto V_{\varphi^U}$ *from* $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ *into* $C(\mathbb{R}_+, \mathbb{R}_+)$ *is locally bounded,* *then there exists a stochastically continuous càdlàg process* $\{X(t) : t \in \mathbb{R}_+\}$ *with independent left increments having continuous* $\mathfrak{F}$*-finite variation with respect to* $\Gamma$ *such that*

$$X_n \xrightarrow{\mathcal{L}} X,$$

*and* $\{\varphi^U : U \in \Gamma\}$ *is an integrating family related to* $\{\mathbb{P}_{X(s)^{-1}X(t)} : 0 \leqslant s \leqslant t\}$.

**Corollary 6.14.** *Let* $G$ *be a second countable locally compact group with Lévy continuity property with respect to a subset* $\Gamma \subset \operatorname{Rep}(G)$. *Let* $\{X_{n,\ell} : (n,\ell) \in \mathbb{N}^2\}$ *be an array of rowwise independent* $G$*-valued random variables. For all* $n \in \mathbb{N}$ *let* $k_n : \mathbb{R}_+ \to \mathbb{Z}_+$ *be an increasing, right continuous function with* $k_n(0) = 0$ *and* $k_n(\mathbb{R}_+) = \mathbb{Z}_+$.

*Suppose that for all* $U \in \Gamma$,

(a) *there is a dense subset* $D$ *of* $\mathbb{R}_+$ *such that the sequence* $\left(\sum_{\ell=1}^{k_n(t)} (\mathbb{E}(U \circ X_{n,\ell}) - I)\right)_{n \geqslant 1}$ *in* $\mathcal{L}(\mathcal{H}_U)$ *is convergent for all* $t \in D$,

(b) $\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{\substack{0 \leqslant s \leqslant t \leqslant T \\ t-s \leqslant \delta}} \sum_{\ell=k_n(s)+1}^{k_n(t)} \|\mathbb{E}(U \circ X_{n,\ell}) - I\| = 0$ *for all* $T > 0$.

*Then there is a family of mappings* $\varphi^U \in FV(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U)) \bigcap C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$, $U \in \Gamma$, *such that* $\varphi_n^U \to \varphi^U$ *locally uniformly for all* $U \in \Gamma$.

*If in addition, for each cardinal* $\alpha$,

(c) *the mapping* $U \mapsto \varphi^U$ *from* $\Gamma \bigcap \operatorname{Rep}_\alpha(G)$ *into* $C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}(\alpha)))$ *is continuous,*

(d) *the mapping* $U \mapsto V_{\varphi^U}$ *from* $\Gamma \bigcap \operatorname{Rep}_\alpha(G)$ *into* $C(\mathbb{R}_+, \mathbb{R}_+)$ *is locally bounded,*

*then there exists a stochastically continuous càdlàg process* $\{X(t) : t \in \mathbb{R}_+\}$ *with independent left increments having continuous* $\mathfrak{F}$*-finite variation with respect to* $\Gamma$ *such that*

$$\prod_{\ell=1}^{k_n(\cdot)} X_{n,\ell} \xrightarrow{\mathcal{L}} X(\cdot),$$

*and* $\{\varphi^U : U \in \Gamma\}$ *is an integrating family related to* $\{\mathbb{P}_{X(s)^{-1}X(t)} : 0 \leqslant s \leqslant t\}$.

Next we present a convergence theorem for randomly scaled sums of an array of not rowwise independent $G$-valued random variables but having a property similar to martingale differences. We will use conditional expectation $\mathbb{E}(Z \mid \mathcal{F})$ of a random variable $Z : \Omega \to B$ with values in a Banach space $B$, where $\mathcal{F} \subset \mathcal{A}$ is a $\sigma$-algebra. It is defined for almost separable-valued random variables $Z$ with $\mathbb{E}\|Z\| < \infty$ (see [18]). Hence if $G$ is a second countable locally compact group then the conditional expectation $\mathbb{E}(U \circ X \mid \mathcal{F})$ is defined for all $G$-valued random variables $X$ and for all $U \in \mathrm{Rep}(G)$.

**Theorem 6.15.** *Let $G$ be a second countable locally compact group with Lévy continuity property with respect to a subset $\Gamma \subset \mathrm{Rep}(G)$. For all $n \in \mathbb{N}$, let $\mathcal{F}_{n,1} \subset \mathcal{F}_{n,2} \subset \dots$ be a filtration of a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ endowed with an adapted sequence $(X_{n,\ell})_{\ell\geqslant 1}$ of $G$-valued random variables, and let $\mathcal{F}_{n,0} := \{\emptyset, \Omega\}$. For all $n \in \mathbb{N}$, let $\{\sigma_n(t) : t \in \mathbb{R}_+\}$ be a family of stopping times with respect to the filtration $\mathcal{F}_{n,1} \subset \mathcal{F}_{n,2} \subset \dots$ such that $\sigma_n(0) = 0$ and $t \mapsto \sigma_n(t)$ is increasing and right continuous.*

*Suppose that for all $U \in \Gamma$,*

(a) *there is a dense subset $D$ of $\mathbb{R}_+$ such that the sequence*

$$\left( \sum_{\ell=1}^{\sigma_n(t)} (\mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I) \right)_{n\geqslant 1}$$

*of $\mathcal{L}(\mathcal{H}_U)$-valued random variables converges in probability to a non-random limit for all $t \in D$,*

(b) $\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{\substack{0\leqslant s\leqslant t\leqslant T \\ t-s\leqslant\delta}} \sum_{\ell=\sigma_n(s)+1}^{\sigma_n(t)} \|\mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I\| = 0$ *$\mathbb{P}$-a.s. for all $T > 0$.*

*Then there is a family of mappings $g^U \in FV(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U)) \bigcap C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$, $U \in \Gamma$, such that for all $T > 0$, $U \in \Gamma$ we have*

$$\sup_{t\in[0,T]} \left\| \sum_{\ell=1}^{\sigma_n(t)} (\mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I) - g^U(t) \right\| \to 0 \qquad \text{in probability.}$$

*If in addition, for each cardinal $\alpha$,*

(c) *the mapping $U \mapsto g^U$ from $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ into $C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}(\alpha)))$ is continuous,*

(d) *the mapping* $U \mapsto V_{g^U}$ *from* $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ *into* $C(\mathbb{R}_+, \mathbb{R}_+)$ *is locally bounded,*

*then there exists a stochastically continuous càdlàg process* $\{X(t) : t \in \mathbb{R}_+\}$ *with independent left increments having continuous* $\mathfrak{F}$*-finite variation with respect to* $\Gamma$ *such that*

$$\prod_{\ell=1}^{\sigma_n(\cdot)} X_{n,\ell} \xrightarrow{\mathcal{L}} X(\cdot),$$

*and* $\{g^U : U \in \Gamma\}$ *is an integrating family related to* $\{\mathbb{P}_{X(s)^{-1}X(t)} : 0 \leqslant s \leqslant t\}$.

We need some preparations for the proof of Theorem 6.15.

**Lemma 6.16.** *For all* $n \in \mathbb{N}$, *let* $\mathcal{F}_{n,1} \subset \mathcal{F}_{n,2} \subset \dots$ *be a filtration of a probability space* $(\Omega, \mathcal{A}, \mathbb{P})$ *endowed with an adapted sequence* $(Z_{n,\ell})_{\ell \geqslant 1}$ *of almost separable-valued random variables in* $\mathcal{L}(B)$, *and let* $\mathcal{F}_{n,0} := \{\emptyset, \Omega\}$. *For all* $n \in \mathbb{N}$, *let* $\tau_n$ *be a stopping time with respect to the filtration* $\mathcal{F}_{n,1} \subset \mathcal{F}_{n,2} \subset \dots$.

*Suppose that*

(a) *there exists* $C_1 \in \mathbb{R}_+$ *such that for all* $n \in \mathbb{N}$ *and* $1 \leqslant i_1 < i_2 < \dots < i_m \leqslant \tau_n(\omega)$:

$$\left\| \prod_{j=1}^{m} Z_{n,i_j}(\omega) \right\| \leqslant C_1 \qquad \mathbb{P}\text{-a.s.},$$

(b) $\max\limits_{1 \leqslant \ell \leqslant \tau_n} \|\mathbb{E}(Z_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I\| \to 0$ *in probability,*

(c) *there exists* $C_2 \in \mathbb{R}_+$ *such that* $\sup\limits_{n \geqslant 1} \sum\limits_{\ell=1}^{\tau_n(\omega)} \|\mathbb{E}(Z_{n,\ell} \mid \mathcal{F}_{n,\ell-1})(\omega) - I\| \leqslant C_2$ $\mathbb{P}$-a.s.,

(d) *there exists* $z_0 \in \mathcal{L}(B)$ *such that* $\prod\limits_{\ell=1}^{\tau_n} \mathbb{E}(Z_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) \to z_0$ *in probability.*

*Then*

$$\mathbb{E}\left( \prod_{\ell=1}^{\tau_n} Z_{n,\ell} \right) \to z_0.$$

Proof. For a sequence $(V_n)_{n\geqslant 0}$ of $\mathcal{L}(B)$-valued random variables, $V_n \to V_0$ in probability if and only if any subsequence $(n') \subset (n)$ contains a subsequence $(n'') \subset (n')$ such that $V_{n''} \to V_0$ $\mathbb{P}$-a.s., hence the lemma will be proved if we show the statement assuming $\mathbb{P}$-a.s. convergence in (b) and in (d).

Let

$$Y_{n,\ell} := Z_{n,\ell} \cdot \mathbb{1}_{A_{n,\ell}} + I \cdot \mathbb{1}_{\complement A_{n,\ell}}$$

where

$$A_{n,\ell} := \left\{ \omega \in \Omega : \|\mathbb{E}(Z_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I\| \leqslant \frac{1}{2} \right\} \subset \mathcal{F}_{n,\ell-1}.$$

Clearly $Y_{n,\ell}$ is $\mathcal{F}_{n,\ell}$-measurable,

$$\mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) = \mathbb{E}(Z_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) \cdot \mathbb{1}_{A_{n,\ell}} + I \cdot \mathbb{1}_{\complement A_{n,\ell}},$$

and the inverse $\mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1}$ exists. For all $n$, the sequence

$$\left( \prod_{\ell=1}^{m} Y_{n,\ell} \prod_{\ell=m}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} \right)_{m\geqslant 1}$$

is a martingale with respect to the filtration $\mathcal{F}_{n,1} \subset \mathcal{F}_{n,2} \subset \dots$, since

$$\begin{aligned}
\mathbb{E} &\left( \prod_{\ell=1}^{m} Y_{n,\ell} \prod_{\ell=m}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} \mid \mathcal{F}_{n,m-1} \right) \\
&\quad = \prod_{\ell=1}^{m-1} Y_{n,\ell} \mathbb{E}(Y_{n,m} \mid \mathcal{F}_{n,m-1}) \prod_{\ell=m}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} \\
&\quad = \prod_{\ell=1}^{m-1} Y_{n,\ell} \prod_{\ell=m-1}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1}.
\end{aligned}$$

We have also

$$\mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} = I + (\mathbb{E}(Z_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} - I) \cdot \mathbb{1}_{A_{n,\ell}},$$

hence

$$\prod_{\ell=\tau_n}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} - I$$
$$= \sum_{m=1}^{\tau_n} \sum_{1 \leqslant i_1,\dots,i_m \leqslant \tau_n} \prod_{\ell=m}^{1} \left[ (\mathbb{E}(Z_{n,i_\ell} \mid \mathcal{F}_{n,i_\ell-1})^{-1} - I) \cdot \mathbb{1}_{A_{n,i_\ell}} \right],$$

and using condition (a) we obtain

$$\left\| \prod_{\ell=\tau_n}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} - I \right\|$$
$$\leqslant \sum_{m=1}^{\infty} \frac{1}{m!} \left( \sum_{j=1}^{\tau_n} \|\mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})^{-1} - I\| \cdot \mathbb{1}_{A_{n,j}} \right)^m$$
$$= \exp\left\{ \sum_{j=1}^{\tau_n} \|\mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})^{-1} - I\| \cdot \mathbb{1}_{A_{n,j}} \right\}.$$

Moreover,

$$\|\mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})^{-1} - I\| \cdot \mathbb{1}_{A_{n,j}} \leqslant 2\|\mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1}) - I\|,$$

since for $\omega \in A_{n,j}$ we have

$$\mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})^{-1}(\omega) - I = \mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})^{-1}(\omega)$$
$$\times \left[ I - \mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})(\omega) \right],$$

and

$$\mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})^{-1}(\omega) = \left[ I - (I - \mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})^{-1}(\omega)) \right]^{-1}$$
$$= \sum_{k=0}^{\infty} (I - \mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})^{-1}(\omega))^k$$

implies

$$\|\mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})^{-1}(\omega)\| \leqslant \sum_{k=0}^{\infty} \|I - \mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1})^{-1}(\omega)\|^k \leqslant \sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

Consequently by condition (c),

$$\left\| \prod_{\ell=\tau_n}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} - I \right\| \leqslant \exp\left\{ 2 \sum_{j=1}^{\tau_n} \|\mathbb{E}(Z_{n,j} \mid \mathcal{F}_{n,j-1}) - I\| \right\}$$
$$\leqslant e^{2C_2} \qquad \mathbb{P}\text{-a.s.},$$

hence

$$\left\| \prod_{\ell=\tau_n}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} \right\| \leqslant 1 + e^{2C_2} \qquad \mathbb{P}\text{-a.s.}$$

Condition (a) implies also

$$\left\| \prod_{\ell=1}^{\tau_n} Y_{n,\ell} \right\| \leqslant C_1 \qquad \mathbb{P}\text{-a.s.},$$

hence using the above martingale, we conclude

$$\mathbb{E}\left( \prod_{\ell=1}^{\tau_n} Y_{n,\ell} \prod_{\ell=\tau_n}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} \right) = I.$$

Thus

$$\mathbb{E}\left( \prod_{\ell=1}^{\tau_n} Y_{n,\ell} \right) - z_0 = \mathbb{E}\left( \prod_{\ell=1}^{\tau_n} Y_{n,\ell} \right) - \mathbb{E}\left( \prod_{\ell=1}^{\tau_n} Y_{n,\ell} \prod_{\ell=\tau_n}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} z_0 \right)$$
$$= \mathbb{E}\left( \prod_{\ell=1}^{\tau_n} Y_{n,\ell} \prod_{\ell=\tau_n}^{1} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1})^{-1} \left( \prod_{\ell=1}^{\tau_n} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - z_0 \right) \right),$$

hence

$$\left\| \mathbb{E}\left( \prod_{\ell=1}^{\tau_n} Y_{n,\ell} \right) - z_0 \right\|$$
$$\leqslant C_1(1 + e^{2C_2}) \mathbb{E} \left\| \prod_{\ell=1}^{\tau_n} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - z_0 \right\| \to 0 \qquad \mathbb{P}\text{-a.s.},$$

since conditions (b) and (d) imply

$$\prod_{\ell=1}^{\tau_n} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) \to z_0 \qquad \mathbb{P}\text{-a.s.},$$

and the above arguments leads also to

$$\left\| \prod_{\ell=1}^{\tau_n} \mathbb{E}(Y_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - z_0 \right\| \leqslant 1 + e^{2C_2} + \|z_0\| \qquad \mathbb{P}\text{-a.s.}$$

Finally, condition (b) implies

$$\prod_{\ell=1}^{\tau_n} Y_{n,\ell} - \prod_{\ell=1}^{\tau_n} Z_{n,\ell} \to 0 \qquad \mathbb{P}\text{-a.s.},$$

and we have

$$\left\| \prod_{\ell=1}^{\tau_n} Y_{n,\ell} - \prod_{\ell=1}^{\tau_n} Z_{n,\ell} \right\| \leqslant \left\| \prod_{\ell=1}^{\tau_n} Y_{n,\ell} \right\| + \left\| \prod_{\ell=1}^{\tau_n} Z_{n,\ell} \right\| \leqslant 2C_1 \qquad \mathbb{P}\text{-a.s.},$$

thus

$$\mathbb{E}\left( \prod_{\ell=1}^{\tau_n} Y_{n,\ell} - \prod_{\ell=1}^{\tau_n} Z_{n,\ell} \right) \to 0,$$

which implies the statement. □

PROOF of Theorem 6.15. Clearly $\mathbb{P}$-a.s.,

$$g_n^U(t) := \sum_{\ell=1}^{\sigma_n(t)} \left(\mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I\right) \qquad \text{for } t \in \mathbb{R}_+,\ U \in \Gamma$$

defines a (random) integrating family related to the (random) evolution family $h_n^U(s,t) : 0 \leqslant s \leqslant t\}$ of finite variation in $\mathcal{L}(\mathcal{H}_U)$, defined by

$$h_n^U(s,t) := \prod_{\ell=\sigma_n(s)+1}^{\sigma_n(t)} \mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}).$$

Moreover,

$$V_{g_n^U}(t) = \sum_{\ell=1}^{\sigma_n(t)} \|\mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I\|.$$

Condition (a) implies that for each subsequence $(n')$ of $(n)$ there is a subsequence $(n'')$ of $(n')$ such that $\{g^U_{n''}(t) : n'' \geqslant 1\}^-$ is compact in $\mathcal{L}(\mathcal{H}_U)$ for all $t \in D$ $\mathbb{P}$-a.s. Applying Theorem 5.3 we conclude that for each subsequence $(n''')$ of $(n'')$ there are a subsequence $(n'''')$ of $(n''')$ and a function $g^U \in C(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$ such that $g^U_{n''''} \to g^U$ locally uniformly $\mathbb{P}$-a.s. Assumption (a) implies that the function $g^U$ is uniquely determined, hence $g^U_{n''} \to g^U$ locally uniformly $\mathbb{P}$-a.s.

Applying Theorem 5.4 (iii) for the sequence $(g^U_{n''})_{n''\geqslant 1}$ in $FV(\mathbb{R}_+, \mathcal{L}(\mathcal{H}_U))$ we obtain

$$\|h^U_{n''}(s,t) - h^U(s,t)\| \to 0 \qquad \text{for all } 0 \leqslant s \leqslant t \ \mathbb{P}\text{-a.s,}$$

where $\{h^U(s,t) : 0 \leqslant s \leqslant t\}$ is an evolution family of continuous finite variation in $\mathcal{L}(\mathcal{H}_U)$ with integrating function $g^U$:

$$h^U(s,t) = I + \int_{]s,t]} h^U(s,\tau-)\, dg^U(\tau) \qquad \text{for all } 0 \leqslant s \leqslant t.$$

Thus

$$h^U_n(s,t) = \prod_{\ell=\sigma_n(s)+1}^{\sigma_n(t)} \mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) \to h^U(s,t) \qquad \text{in probability.}$$

We have for arbitrary $1 \leqslant i_1 < \ldots < i_m$,

$$\left\| \prod_{\ell=1}^{m} U \circ X_{n,i_j} \right\| \leqslant \prod_{\ell=1}^{m} \|U \circ X_{n,i_j}\| \leqslant 1,$$

and

$$\max_{1\leqslant \ell \leqslant \sigma_n(T)} \|\mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I\| \leqslant \sup_{\substack{0\leqslant s\leqslant t\leqslant T \\ t-s\leqslant\delta}} \sum_{\ell=\sigma_n(s)+1}^{\sigma_n(t)} \|\mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I\|$$

for all $n \in \mathbb{N}$ and for all $\delta > 0$, hence (b) implies

$$\max_{1\leqslant \ell \leqslant \sigma_n(T)} \|\mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I\| \to 0 \qquad \mathbb{P}\text{-a.s.}$$

Moreover, for all $T > 0$, for $\mathbb{P}$-a.e. $\omega \in \Omega$ and for sufficiently large $r(\omega)$, $n(\omega)$, we have

$$\sum_{\ell=\sigma_n(u)+1}^{\sigma_n(v)} \|\mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I\| \leqslant 1$$

for all $0 \leqslant u < v \leqslant T$ with $v - u < 1/r$, which implies

$$\sum_{\ell=1}^{\sigma_n(T)} \|\mathbb{E}(U \circ X_{n,\ell} \mid \mathcal{F}_{n,\ell-1}) - I\| \leqslant r \qquad \mathbb{P}\text{-a.s.}$$

Consequently, we can apply Lemma 6.16 for $Z_{n\ell} := U \circ X_{n,\ell}$ and for $\tau_n := \sigma_n(T)$, $n \geqslant 1$, and we obtain

$$\mathbb{E}\left(U \circ \left(\prod_{\ell=\sigma_n(s)+1}^{\sigma_n(t)} X_{n,\ell}\right)\right) = \mathbb{E}\left(\prod_{\ell=\sigma_n(s)+1}^{\sigma_n(t)} (U \circ X_{n,\ell})\right) \to h^U(s,t)$$

for all $U \in \Gamma$. Using conditions (i) and (ii), and applying Theorem 5.5 to the family $\{g^U : U \in \Gamma \bigcap \mathrm{Rep}_\alpha(G)\}$ with a cardinal $\alpha$ we obtain that for all $0 \leqslant s \leqslant t$, the function $U \mapsto h^U(s,t)$ from $\Gamma \bigcap \mathrm{Rep}_\alpha(G)$ into $\mathcal{L}(\mathcal{H}(\alpha))$ is continuous. Since the group $G$ has the Lévy continuity property, we arrive at the assertion. $\square$

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