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Projection onto the indicatrix bundle of a Finsler manifold

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Abstract. We present a simple geometric description of the process "projection onto the indicatrix bundle" of a Finsler manifold originally introduced by H. IZUMI, in the framework of the theory whose foundations were laid by J. GRIFONE [7], [8]. Next we derive an intrinsic expression for the projection of the Douglas tensor and give a new proof of SAKAGUCHI's important observation that the projected Douglas tensor and the Douglas tensor vanish together. Finally we deduce a concise coordinate-free form for the mixed curvature of the "Landsberg–Douglas manifolds", and conlude that in $n > 2$ dimensions these manifolds are just the Berwald manifolds.

Introduction

In the third part [3] of his famous series on Finsler and Cartan geometries L. BERWALD showed that (in our terminology) *any two-dimensional Landsberg manifold with vanishing Douglas tensor is a Berwald manifold*. More than 40 years later it was announced by H. IZUMI [10] that this is also true in higher dimensions. IZUMI's statement was proved by the group S. BÁCSÓ, F. ILOSVAY and B. KIS in [1], using the machinery of classical tensor calculus and utilizing an important observation of T. SAKAGUCHI [13] on the projected Douglas tensor. Finsler manifolds with vanishing Douglas tensor were baptized *Douglas manifolds* by M. MATSUMOTO and S. BÁCSÓ [2]. Thus, briefly speaking, *any Landsberg–Douglas manifold is a Berwald manifold.*

In a recent paper [15] J. SZILASI and the present author elaborated an intrinsic approach to the projective geometry of sprays and obtained

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compact, coordinate – free representations for the projectively invariant Douglas tensor. These results having been obtained, there arises an exciting question: possible to derive the above theorem *intrinsically*, using the technique developed in [15] and if so, how? The present paper is the outgrowth of attempts to solve this problem.

It was not too difficult to make a promising plan of attack. The idea of *projection onto the indicatrix bundle* initiated by H. Izumi and systematized by M. Matsumoto (see e.g. [11]) seemed to be a suitable starting point. It can be formulated intrinsically straightaway. One of the canonical geometrical objects on the tangent bundle, the Liouville vector field C, is a normal vector field to the "unit sphere bundle" or "indicatrix bundle" $U(M)$ of any Finsler manifold (M, E). It follows that orthogonal projection into $U(M)$ is given by the operator

$$\tau := I - \frac{1}{2E} dE \otimes C.$$

(This operator was first introduced by J. Grifone [9].) The operator

$$\kappa := J^* \circ \tau = J - \frac{1}{2E} d_J E \otimes C,$$

where J is the vertical endomorphism, and its lowered tensor $\kappa_\flat$, will also play an important role in our approach. If $\mathcal{T}M$ is the deleted tangent manifold of M, then, suggested by J. Szilasi, we define the *projected tensor* A^* of a tensor $A \in \mathcal{T}_r^1(\mathcal{T}M)$ as follows:

$$A^* := \tau \circ \tau^* A.$$

The behavior of the Douglas tensor becomes much more transparent after it has been projected onto the indicatrix bundle; this is the main point of the paper. Notice that this process does not give anything in the two-dimensional case, which is – for this reason – a completely different story.

Despite the simplicity of the underlying ideas, we had to face a lot of nontrivial technical difficulties when we tried to carry out our programme. In the course of overcoming them we have deduced a number of useful tensorial relations for the special Finsler manifolds studied; these relations are of interest in their own right.

The paper is organized as follows. In Section 1 we have collected background material concerning the applied technical tools, Finsler manifolds, and distinguished Finsler connections (including basic curvature identities), in a form which best fits our subsequent requirements. Section 2 provides different characterizations of Landsberg and Berwald manifolds in the spirit of the approach presented in [16]. Section 3 is devoted to an intrinsic treatment of the projection onto the indicatrix bundle, especially to the projection of the Douglas tensor $\mathbb{D}$. It is shown that its projected tensor $\mathbb{D}^*$ can be represented in the form

$$\mathbb{D}^* = \overset{\circ}{\mathbb{P}} - \frac{1}{n+1}\overset{\tilde{\circ}}{\mathbb{P}} \odot \kappa + \frac{1}{E}\mathcal{C}'_\flat \otimes C,$$

where $\overset{\circ}{\mathbb{P}}$ is the mixed curvature of the Berwald connection, $\mathcal{C}'_\flat$ is the type $(0,3)$ second Cartan tensor, and the sign $\sim$ means semibasic trace. Then we give a new, intrinsic proof of SAKAGUCHI's important observation that $\mathbb{D}^* = 0$ if and only if $\mathbb{D} = 0$. We note that in local coordinates our formulas, including the coordinate expression of $\mathbb{D}^*$ do not coincide with those of SAKAGUCHI. The reason for this is the difference in the underlying bundle structures, which implies that the notions of "Finsler tensor fields" are also different. Most of the "essential" tensors are semibasic in our framework, while the vector field variables of MATSUMOTO's theory [12] used by SAKAGUCHI correspond to the vertical vector fields in our approach.

In the concluding section the reader will find the deduction of the main result. We show that in any "Landsberg–Douglas manifold" the mixed curvature of the Berwald connection has the form

$$\overset{\circ}{\mathbb{P}} = \frac{1}{n^2-1}\widetilde{\overset{\tilde{\circ}}{\mathbb{P}}^{\#}} \kappa_\flat \odot \kappa,$$

where $\#$ and $\flat$ are the musical isomorphisms with respect to the metric tensor arising from the Finsler structure; and the relation

$$(n-2)\widetilde{\overset{\tilde{\circ}}{\mathbb{P}}^{\#}}\widetilde{\mathcal{C}} = 0$$

holds. The theorem follows immediately from these results.

Unfortunately, this elegant method fails in two dimensions. Then the Douglas tensor has components only along the Liouville vector field. Since the Liouville vector field is orthogonal to the unit sphere bundle, it follows immediately that *the projected Douglas tensor of a two-dimensional*

Finsler manifold vanishes identically. However, after this work was finsished, we felt that it would be worthwile to complete the story discussing the previously cited paper of Berwald on the two-dimensional Finsler manifolds from our present day standpoint. We have done this jointly with Cs. Vincze, and a forthcoming paper of us will be devoted to a modern exposition of the two-dimensional case.

1. Preliminaries

1.1. Throughout the paper we shall freely use the terminology and results of [14] and [15]. However, we collect some of the notations and definitions at this point for easy reference.

1.1.1. The letter M shall stand for an $n(\geqq 2)$-dimensional, connected, paracompact, smooth manifold; $C^\infty(M)$ denotes the ring of the real-valued C^∞ functions on M.

1.1.2. $\pi : TM \to M$ is the tangent bundle of M, $\pi_0 : \mathcal{T}M \to M$ is the bundle of the nonzero tangent vectors. $\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M, $\mathcal{T}^r_s(M)$ stands for the $C^\infty(M)$-module of tensor fields of type (r,s), $\Omega^k(M)$ is the $C^\infty(M)$-module of k-forms on M.

1.1.3. i_X, $\mathcal{L}_X$ $(X \in \mathfrak{X}(M))$ and d are the insertion operator, the Lie derivative (with respect to X) and the exterior derivative, respectively.

1.1.4. $\mathfrak{X}^v(TM)$ denotes the $C^\infty(TM)$-module of vertical vector fields on TM. $C \in \mathfrak{X}^v(TM)$ is the *Liouville vector field*; X^v and X^c stand for the vertical and the complete lift of a vector field $X \in \mathfrak{X}(M)$.

1.1.5. $J \in \mathcal{T}^1_1(TM) \cong \operatorname{End}\mathfrak{X}(TM)$ is the *vertical endomorphism* (or the *canonical almost tangent structure*) on TM. We recall, that

$$\operatorname{Im} J = \operatorname{Ker} J = \mathfrak{X}^v(TM), \qquad J^2 = 0. \tag{1}$$

The *adjoint operator* J^* of J is introduced as follows:

$$\forall f \in C^\infty(TM) : J^* f := f; \tag{2a}$$

$$\forall \omega \in \Omega^k(TM) : J^*\omega(X_1, \dots, X_k) := \omega(JX_1, \dots, JX_k) \tag{2b}$$

$$(X_1, \dots, X_k \in \mathfrak{X}(TM);\ k \in \mathbb{N}^+).$$

1.1.6. The *vertical derivation* i_J is defined by the formulas

$$\forall f \in C^\infty(TM) : i_J f := 0; \tag{3a}$$

$$\forall \omega \in \Omega^k(TM) : i_J\omega(X_1, \dots, X_k) := \sum_{i=1}^{k} \omega(X_1, \dots, JX_i, \dots, X_k) \tag{3b}$$

$$(X_i \in \mathfrak{X}(TM);\ 1 \leqq i \leqq k).$$

The operator

$$d_J := i_J \circ d - d \circ i_J \tag{4}$$

is called the *vertical differentiation*. We have

$$\forall f \in C^\infty(TM) : d_J f = i_J df = J^*(df). \tag{5}$$

(For details, see [5], 4.1.)

1.1.7. $\Psi^k(TM)$ is the module of *vector k-forms* on TM, constituted by the skew-symmetric $C^\infty(TM)$-multilinear maps

$$\mathfrak{X}(T\underbrace{M) \times \cdots \times \mathfrak{X}}_{k \text{ times}}(TM) \to \mathfrak{X}(TM)$$

if $k \geqq 1$, and $\Psi^0(TM) := \mathfrak{X}(TM)$. *A priori*, the differentiability of vector k-forms will be required only over $\mathcal{T}M$.

1.1.8. We shall employ the *Frölicher–Nijenhuis bracket* [,] of vector forms in the following simplest, but non-trivial cases:

(i) $K \in \Psi^1(TM)$, $Y \in \Psi^0(TM)$. Then $[K, Y] \in \Psi^1(TM)$, and for any vector field $X \in \mathfrak{X}(TM)$

$$[K, Y](X) = [K(X), Y] - K[X, Y]. \tag{6}$$

(ii) $K \in \Psi^1(TM)$, $L \in \Psi^1(TM)$. Then $[K, L] \in \Psi^2(TM)$ is given by the famous 8-terms formula

$$\begin{aligned}[K, L](X, Y) = {} & [K(X), L(Y)] + [L(X), K(Y)] + K \circ L[X, Y] \\ & + L \circ K[X, Y] - K[L(X), Y] - K[X, L(Y)] \\ & - L[X, K(Y)] - L[K(X), Y] \qquad (X, Y \in \mathfrak{X}(TM)).\end{aligned} \tag{7}$$

1.2 Horizontal endomorphisms. A $(1,1)$ tensor $h \in \mathcal{T}_1^1(TM) \cong \Psi^1(TM)$, smooth only over $\mathcal{T}M$ (**1.1.7**), is said to be a *horizontal endomorphism* on M if it is a projector (i.e., $h^2 = h$) and $\operatorname{Ker} h = \mathfrak{X}^v(TM)$. The mapping

$$X \in \mathfrak{X}(M) \mapsto X^h := hX^c \in \mathfrak{X}(\mathcal{T}M) \tag{8}$$

is called *horizontal lifting* by h.

Any horizontal endomorphism h determines a unique $(1,1)$ tensor $F \in \mathcal{T}_1^1(TM)$ such that

$$F \circ h = -J, \qquad F \circ J = h. \tag{9}$$

Then F is an almost complex structure on TM, called the *almost complex structure associated with h*. Note that

$$\forall X \in \mathfrak{X}(M) : FX^v = X^h. \tag{10}$$

1.3 Local basis properties. Most of our local calculations will be based on the following simple observation: If $(X_1, \dots, X_n)$ is a local basis of the module $\mathfrak{X}(M)$, then

$$(X_1^v, \dots, X_n^v, X_1^c, \dots, X_n^c) \quad \text{or} \quad (X_1^v, \dots, X_n^v, X_1^h, \dots, X_n^h)$$

(where X_i^h, $1 \leqq i \leq n$, are horizontal lifts by a horizontal endomorphism) are local bases for the module $\mathfrak{X}(TM)$.

1.4 Semibasic tensors, semibasic trace. A symmetric or skew-symmetric tensor $A \in \mathcal{T}_s^0(TM)$ $(s \in \mathbb{N}^+)$ is said to be *semibasic* if for any vector fields $X \in \mathfrak{X}(TM)$ we have $i_{JX}A = 0$. Analogously, a symmetric or skew-symmetric tensor $L \in \mathcal{T}_s^1(TM)$ is called semibasic if

$$J \circ L = 0 \quad \text{and} \quad \forall X \in \mathfrak{X}(TM) : i_{JX}L = 0.$$

The *semibasic trace* $\widetilde{L} \in \mathcal{T}_{s-1}^0(TM)$ of a semibasic tensor $L \in \mathcal{T}_s^1(TM)$ is defined by recurrence as follows:

(i) if $s = 1$, then $\widetilde{L} := tr\ (F \circ L)$, where F is the associated almost complex structure of an arbitrarily chosen horizontal endomorphism;

(ii) if $s > 1$, then for any vector field $X \in \mathfrak{X}(TM)$ $i_X\widetilde{L} := \widetilde{i_X L}$.

1.4.1 Lemma. *Let $A \in \mathcal{T}_s^0(TM)$, $L \in \mathcal{T}_1^1(TM)$, $K \in \mathcal{T}_2^1(TM)$ be semibasic tensors, and suppose that A is symmetric. Then*

$$\widetilde{A \otimes L} = \widetilde{L} A, \tag{11}$$

$$\forall X \in \mathfrak{X}(TM) : i_X \widetilde{L \otimes A} = i_{FL(X)} A, \tag{12}$$

$$\widetilde{A \otimes K} = A \otimes \widetilde{K}, \tag{13}$$

$$\forall X, Y \in \mathfrak{X}(TM) : i_Y i_X \widetilde{K \otimes A} = i_{FK(X,Y)} A. \tag{14}$$

The *proof* is straightforward and we omit it.

1.5 Finsler manifolds. Let a function $E : TM \to \mathbb{R}$ be given. The pair (M, E) is said to be a *Finsler manifold* if the following conditions are satisfied:

(i) $\forall v \in \mathcal{T}M : E(v) > 0$, $E(0) = 0$;

(ii) E is of class C^1 on TM and smooth on $\mathcal{T}M$;

(iii) $CE = 2E$, i.e., E is homogeneous of degree 2;

(iv) the fundamental 2-form $\omega := dd_J E$ is symplectic.

Then the function E is called the *energy function* of the Finsler manifold. It follows at once that

$$i_C \omega = d_J E. \tag{15}$$

1.5.1 The canonical spray. On any Finsler manifold (M, E) there is a spray $S : TM \to TTM$ uniquely determined on $\mathcal{T}M$ by the relation

$$i_S \omega = -dE \tag{16}$$

and prolonged to a C^1 mapping of TM such that $S \upharpoonright TM \setminus \mathcal{T}M = 0$. The spray S in called the *canonical spray* of the Finsler manifold.

1.5.2 The Barthel endomorphism. Let (M, E) be a Finsler manifold with canonical spray S. The $(1,1)$ tensor

$$h := \frac{1}{2}\left(1_{\mathfrak{X}(TM)} + [J, S]\right)$$

is a horizontal endomorphism on M which we call the *Barthel endomorphism*. It has the following characteristic properties:

$$d_h E = 0 \quad (\text{“}h \text{ is conservative”}), \tag{17}$$

$$[h, C] = 0 \quad (\text{“}h \text{ is homogeneous”}), \tag{18}$$

$$[J, h] = 0 \quad (\text{“the weak torsion of } h \text{ vanishes”}). \tag{19}$$

This fundamental result is due to J. Grifone [7].

1.5.3 Metric. Cartan tensors. Let F be the almost complex structure associated to the Barthel endomorphism. Then the mapping

$$\begin{gathered} g : \mathfrak{X}(TM) \times \mathfrak{X}(TM) \to C^\infty(TM), \\ (X, Y) \mapsto g(X, Y) := \omega(X, FY) \end{gathered} \tag{20}$$

is a well-defined pseudo-Riemannian metric on $\mathcal{T}M$. We introduce the *first Cartan tensors* $\mathcal{C}$ and $\mathcal{C}_\flat$ by the relations

$$g(\mathcal{C}(X, Y), JZ) := \frac{1}{2}\mathcal{L}_{JX}(J^* g)(Y, Z) \quad \text{and} \quad J \circ \mathcal{C} = 0, \tag{21}$$

$$\mathcal{C}_\flat(X, Y, Z) := g(\mathcal{C}(X, Y), JZ); \qquad X, Y, Z \in \mathfrak{X}(\mathcal{T}M). \tag{22}$$

The *second Cartan tensors* $\mathcal{C}'$, $\mathcal{C}'_\flat$ are defined as follows: for any vector fields $X, Y, Z \in \mathfrak{X}(\mathcal{T}M)$

$$g(\mathcal{C}'(X, Y), JZ) := \frac{1}{2}\left(\mathcal{L}_{hX} g\right)(JY, JZ) \quad \text{and} \quad J \circ \mathcal{C}' = 0, \tag{23}$$

$$\mathcal{C}'_\flat(X, Y, Z) := g(\mathcal{C}'(X, Y), JZ). \tag{24}$$

We now summarize the basic properties of the Cartan tensors.

(25) *$\mathcal{C}$ and $\mathcal{C}'$ are semibasic.*

(26) *$\mathcal{C}_\flat$ and $\mathcal{C}'_\flat$ are totally symmetric.*

For any semispray S_0,

$$i_{S_0}\mathcal{C} = i_{S_0}\mathcal{C}_\flat = 0, \qquad i_{S_0}\mathcal{C}' = i_{S_0}\mathcal{C}'_\flat = 0. \tag{27}$$

We have the following computational formulas:

$$\forall X, Y, Z \in \mathfrak{X}(M) : 2\mathcal{C}_\flat(X^c, Y^c, Z^c) = X^v[Y^v(Z^v E)]; \tag{28}$$

$$\forall X, Y, Z \in \mathfrak{X}(M) : 2\mathcal{C}'_\flat(X^c, Y^c, Z^c) = [Y^v, [X^h, Z^v]]E. \tag{29}$$

For details and a somewhat more general approach we refer to [16].

1.6 The Berwald and the Cartan connection. As it is well-known, there exist a number of notable connections (more precisely, *Finsler connections*) on a Finsler manifold [14]. In our subsequent considerations we shall employ only the Berwald connection $\overset{\circ}{D}$ and the Cartan connection D. The rules for calculation with respect to these connections can be summarized in the following table:

Berwald connection	**Cartan connection**
$\overset{\circ}{D}_{JX} JY = J[JX, Y]$	$D_{JX} JY = J[JX, Y] + \mathcal{C}(X, Y)$
$\overset{\circ}{D}_{hX} JY = v[hX, JY]$	$D_{hX} JY = v[hX, JY] + \mathcal{C}'(X, Y)$
$\overset{\circ}{D}_{JX} hY = h[JX, Y]$	$D_{JX} hY = h[JX, Y] + F\mathcal{C}(X, Y)$
$\overset{\circ}{D}_{hX} hY = hF[hX, JY]$	$D_{hX} hY = hF[hX, JY] + F\mathcal{C}'(X, Y)$

$(X, Y \in \mathfrak{X}(\mathcal{T}M))$.

Using vertically and horizontally lifted vector fields, these formulas take the following more convenient form:

Berwald connection	**Cartan connection**
$\overset{\circ}{D}_{X^v} Y^v = 0$	$D_{X^v} Y^v = \mathcal{C}(X^h, Y^h)$
$\overset{\circ}{D}_{X^h} Y^v = [X^h, Y^v]$	$D_{X^h} Y^v = [X^h, Y^v] + \mathcal{C}'(X^h, Y^h)$
$\overset{\circ}{D}_{X^v} Y^h = 0$	$D_{X^v} Y^h = F\mathcal{C}(X^h, Y^h)$
$\overset{\circ}{D}_{X^h} Y^h = F[X^h, Y^v]$	$D_{X^h} Y^h = F[X^h, Y^v] + F\mathcal{C}'(X^h, Y^h)$

$(X, Y \in \mathfrak{X}(M)$ are arbitrary vector fields).

Let $\widetilde{D} \in \{\overset{\circ}{D}, D\}$. It will be convenient to introduce the operators

$$\widetilde{D}_J, \widetilde{D}_h : \mathcal{T}^r_s(TM) \to \mathcal{T}^r_{s+1}(TM)$$

as follows: for any tensor $A \in \mathcal{T}^r_s(TM)$ and any vector field $X \in \mathfrak{X}(TM)$

$$i_X \widetilde{D}_J A := \widetilde{D}_{JX} A, \quad i_X \widetilde{D}_h A := \widetilde{D}_{hX} A. \tag{30}$$

1.7 Torsion and curvature. The classical torsion and curvature tensor of a Finsler connection can be decomposed into five "partial" torsions and three partial curvatures. As for the Berwald and the Cartan connection, we have the following data ([8], [14]):

	Berwald connection	Cartan connection
h-horizontal torsion	$\overset{\circ}{\mathbb{A}} = 0$	$\mathbb{A} = 0$
h-mixed torsion	$\overset{\circ}{\mathbb{B}} = 0$	$\mathbb{B} = -F \circ \mathcal{C}$
v-horizontal torsion	$\overset{\circ}{\mathbb{R}}{}^1 = -\frac{1}{2}[h, h]$	$\mathbb{R}^1 = -\frac{1}{2}[h, h]$
v-mixed torsion	$\overset{\circ}{\mathbb{P}}{}^1 = 0$	$\mathbb{P}^1 = \mathcal{C}'$
v-vertical torsion	$\overset{\circ}{\mathbb{S}}{}^1 = 0$	$\mathbb{S}^1 = 0$
horizontal curvature	$\overset{\circ}{\mathbb{R}} = \overset{\circ}{D}_J \overset{\circ}{\mathbb{R}}{}^1$	$\mathbb{R}$
mixed curvature	$\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h = \left[[X^h, Y^v], Z^v\right]$	$\mathbb{P}$
vertical curvature	$\overset{\circ}{\mathbb{Q}} = 0$	$\mathbb{Q}(X^h, Y^h)Z^h$ $= \mathcal{C}(Y^h, F\mathcal{C}(X^h, Z^h))$ $-\mathcal{C}(X^h, F\mathcal{C}(Y^h, Z^h))$

$(X, Y, Z \in \mathfrak{X}(M))$. $\mathbb{R}$ and $\mathbb{P}$ can be expressed explicitly with the help of $\overset{\circ}{\mathbb{R}}$ and $\overset{\circ}{\mathbb{P}}$ respectively, but we shall not need these quite complicated expressions.

1.7.1 Lemma. *The mixed curvature $\overset{\circ}{\mathbb{P}}$ of the Berwald connection has the following properties:*

(31) *it is totally symmetric*;

(32) *it is homogeneous of degree -1, i.e., $\mathcal{L}_C \overset{\circ}{\mathbb{P}} = -2\overset{\circ}{\mathbb{P}}$;*

(33) *for any semispray S_0, $i_{S_0} \overset{\circ}{\mathbb{P}} = 0$;*

(34) *$\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}$ is totally symmetric.*

Proof. See [15], **4.3**, **4.4**. □

1.7.2 Lemma (the fifth Bianchi identity)**.** *For any vector fields $X, Y, Z \in \mathfrak{X}(TM)$,*

$$\begin{aligned}
&(D_{hX}\mathbb{Q})\,(Y, Z) - (D_{JY}\mathbb{P})\,(X, Z) + (D_{JZ}\mathbb{P})\,(X, Y) \\
&\qquad = \mathbb{P}(F\mathcal{C}(X, Y), Z) - \mathbb{P}(F\mathcal{C}(Z, X), Y) \\
&\qquad - \mathbb{Q}(F\mathcal{C}'(X, Y), Z) + \mathbb{Q}(F\mathcal{C}'(Z, X), Y).
\end{aligned}$$

PRROF. See [8], A.26. □

1.7.3 Lemma. *For any vector fields* $X, Y, Z, U \in \mathfrak{X}(M)$,

$$\begin{aligned}(D_h\mathbb{Q})(X^h, Y^h, Z^h, U^h) &= (D_{X^h}\mathcal{C})\,(Z^h, F\mathcal{C}(Y^h, U^h)) \\ &\quad + (D_{X^h}\mathcal{C})\,(Y^h, F\mathcal{C}(Z^h, U^h)) + \mathcal{C}\left(Z^h, F\,(D_{X^h}\mathcal{C})\,(Y^h, U^h)\right) \\ &\quad - \mathcal{C}\left(Y^h, F\,(D_{X^h}\mathcal{C})\,(Z^h, U^h)\right).\end{aligned}$$

The proof is a routine but little lengthy calculation so we omit it.

2. General remarks on Landsberg and Berwald manifolds

2.1 Proposition and definition (J.-G. DIAZ). *Keeping the notation introduced in Section 1, let* (M, E) *be a Finsler manifold. The following assertions are equivalent:*

(i) *The mixed curvature* $\mathbb{P}$ *of the Cartan connection vanishes.*

(ii) *The second Cartan tensor* $\mathcal{C}'$ *vanishes.*

(iii) *The tensor* $D_h\mathcal{C}$ *is totally symmetric.*

(iv) $\overset{\circ}{\mathbb{P}} = -D_h\mathcal{C}$.

If one, and therefore all, of the conditions (i)–(iv) *are satisfied, then* (M, E) *is called a* Landsberg manifold.

An intrinsic *proof* of this important result (presented in the language of *Dazord*'s formalism) can be found in DIAZ's thesis [6]. As for the traditional treatment, we refer to [12].

2.2 Further characterizations of a Landsberg manifold.

2.2.1. (M, E) is a Landsberg manifold if and only if $\overset{\circ}{D}_h = D_h$.

PROOF. It can be seen immediately from the table in **1.6** that the properties $\mathcal{C}' = 0$ and $\overset{\circ}{D}_h = D_h$ are equivalent. □

2.2.2. Since $\mathbb{P}^1 = \mathcal{C}'$ (**1.7**), *the vanishing of the v-mixed torsion of the Cartan connection characterizes the Landsberg manifolds.*

2.2.3. A Finsler manifold is a Landsberg manifold if and only if the Berwald connection is h-metrical, i.e., $\overset{\circ}{D}_h g = 0$.

Proof. For any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$\begin{aligned}
&\mathcal{C}'_\flat(X^h, Y^h, Z^h) \overset{(24)}{:=} g(\mathcal{C}'(X^h, Y^h), Z^v)\\
&\overset{(23)}{:=} \frac{1}{2}\left(\mathcal{L}_{X^h} g\right)(JY^h, JZ^h) = \frac{1}{2}\left(\mathcal{L}_{X^h} g\right)(Y^v, Z^v)\\
&= \frac{1}{2}\left[X^h g(Y^v, Z^v) - g([X^h, Y^v], Z^v) - g(Y^v, [X^h, Z^v])\right]\\
&\overset{\mathbf{1.6}}{=} \frac{1}{2}\left[X^h g(Y^v, Z^v) - g\left(\overset{\circ}{D}_{X^h} Y^v, Z^v\right) - g\left(Y^v, \overset{\circ}{D}_{X^h} Z^v\right)\right]\\
&= \frac{1}{2}\left(\overset{\circ}{D}_{X^h} g\right)(Y^v, Z^v).
\end{aligned}$$

Hence

$$2\mathcal{C}'_\flat = \overset{\circ}{D}_h g, \tag{35}$$

which proves our assertion. □

Remark that in Matsumoto's monograph [12] Landsberg manifolds are defined by the property $\overset{\circ}{D}_h g = 0$.

2.3 Corollary. *In any Landsberg manifold*

$$\overset{\circ}{D}_h \mathbb{Q} = D_h \mathbb{Q} = 0. \tag{36}$$

Proof. This follows from the fifth Bianchi identity (**1.7.2**), since – as we have just seen – in Landsberg manifolds $\mathbb{P} = 0$, $\mathcal{C}' = 0$ and $\overset{\circ}{D}_h = D_h$. □

2.4 Corollary. *The mixed curvature of the Berwald connection in a Landsberg manifold satisfies the identity*

$$\begin{aligned}
&\overset{\circ}{\mathbb{P}}(X^h, Y^h) F\mathcal{C}(Z^h, U^h) + \mathcal{C}\left(Y^h, F\left(\overset{\circ}{\mathbb{P}}(X^h, Z^h) U^h\right)\right)\\
&= \overset{\circ}{\mathbb{P}}(Z^h, X^h) F\mathcal{C}(Y^h, U^h) + \mathcal{C}\left(Z^h, F\left(\overset{\circ}{\mathbb{P}}(X^h, Y^h) U^h\right)\right)\\
&\qquad (X, Y, Z, U \in \mathfrak{X}(M)).
\end{aligned} \tag{37}$$

Proof. This is an immediate consequence of **1.7.3**, taking into account the fact that in Landsberg manifolds $D_h\mathbb{Q} = 0$ and $D_h\mathcal{C} = -\overset{\circ}{\mathbb{P}}$. □

2.5 Proposition. *The lowered mixed curvature tensor $\overset{\circ}{\mathbb{P}}_\flat$ of the Berwald connection defined by*

$$\overset{\circ}{\mathbb{P}}_\flat(X^h, Y^h, Z^h, U^h) := g\Big(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, JU^h\Big) \qquad (38)$$
$$(X, Y, Z, U \in \mathfrak{X}(M))$$

is totally symmetric in any Landsberg manifold.

Proof. Starting with the definition, we obtain

$$\begin{aligned}\overset{\circ}{\mathbb{P}}_\flat(X^h, Y^h, Z^h, U^h) &= g\Big(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, JU^h\Big) \overset{(20),(9)}{=} \omega\Big(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, U^h\Big)\\ &= d(d_JE)\Big(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, U^h\Big) = \overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h(d_JE)(U^h)\\ &- U^h(d_JE)\Big(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h\Big) - d_JE\Big(\Big[\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, U^h\Big]\Big).\end{aligned}$$

Since $\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h \in \mathfrak{X}^v(TM)$, the second and the third term of the last expression vanish by (1) and (5). The first term can be formed as follows:

$$\begin{aligned}\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h(d_JE)(U^h) &= \overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h(U^vE)\\ &= \Big[\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h, U^v\Big]E - U^v\Big(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h(E)\Big).\end{aligned}$$

Applying Lemma 3.7 of [15], it follows that here the first term is $[\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h, U^h)]E$, while

$$\begin{aligned}U^v\Big(\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h(E)\Big) &\overset{\mathbf{1.7}}{=} U^v([[X^h, Y^v], Z^v]E)\\ &\overset{(29)}{=} -2U^v(\mathcal{C}'_\flat(X^h, Y^h, Z^h)) = 0,\end{aligned}$$

since the underlying manifold is Landsberg. Thus we obtain the relation

$$\overset{\circ}{\mathbb{P}}_\flat(X^h, Y^h, Z^h, U^h) = \Big[\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h, U^h)\Big]E.$$

In view of (34) this shows that $\overset{\circ}{\mathbb{P}}_\flat$ is totally symmetric. □

2.6 Definition [16]. *A Finsler manifold* (M,E) *is said to be a* Berwald manifold *if there is a linear connection* ∇ *on* M *such that for any vector fields* $X,Y\in\mathfrak{X}(M)$, $(\nabla_X Y)^v=[X^h,Y^v]$.

2.7 Remark. The linear connection ∇ in the previous definition is clearly unique, so it can be mentioned as *the* linear connection of the Berwald manifold. It is readily seen that the horizontal endomorphism induced by ∇ is just the Barthel endomorphism, therefore *the canonical spray of a Berwald manifold is smooth on the whole tangent manifold* TM. The converse of the last statement is also true. Let us note finally that the Berwald connection of a Berwald manifold is just the horizontal lift ([5], p. 134) of the linear connection ∇.

2.8 Lemma. *A Finsler manifold* (M,E) *is a Berwald manifold if and only if the condition*

$$\forall X,Y\in\mathfrak{X}(M):[X^h,Y^v] \text{ is a vertical lift} \tag{39}$$

is fulfilled.

Proof. The necessity of (39) is evident. To see the sufficiency, let us consider the mapping

$$\nabla:(X,Y)\in\mathfrak{X}(M)\times\mathfrak{X}(M)\mapsto\nabla_X Y\in\mathfrak{X}(M),\quad (\nabla_X Y)^v=[X^h,Y^v],$$

where X^h is the horizontal lift of X by the Barthel endomorphism. Then an easy calculation shows that ∇ is a well-defined linear connection on M. $\square$

2.9 Corollary. *A Finsler manifold is a Berwald manifold if and only if the mixed curvature of the Berwald connection vanishes.*

Proof. If (M,E) is a Berwald manifold, then for any vector fields $X,Y,Z\in\mathfrak{X}(M)$,

$$\overset{\circ}{\mathbb{P}}(X^h,Y^h)Z^h\overset{\mathbf{1.7}}{=}[[X^h,Y^v],Z^v]=[(\nabla_X Y)^v,Z^v]=0.$$

Conversely, if $\overset{\circ}{\mathbb{P}}$ vanishes, then
$\forall X,Y,Z\in\mathfrak{X}(M):0=[[X^h,Y^v],Z^v]\Rightarrow\forall X,Y\in\mathfrak{X}(M):[X^h,Y^v]$ *is a vertical lift*, and it follows (**2.8**) that (M,E) is a Berwald manifold. $\square$

2.10 Remark. This is a well-known observation, of course. As for the usual "coordinate-proof", we also refer to [12].

3. Projection onto the indicatrix bundle

3.1. Suppose that (M, E) is a Finsler manifold. Let $\mathcal{L} := \sqrt{2E}$ and

$$U(M) := \{v \in TM \mid \mathcal{L}(v) = 1\}.$$

By abuse of language, we shall call the $(2n-1)$-dimensional manifold $U(M)$ the *indicatrix bundle* of (M, E).

3.2 Lemma [9]. *Let us denote by $C_0^\infty(\mathcal{T}M)$ the ring of those smooth functions on $\mathcal{T}M$ which are homogeneous of degree 0. The module $\mathfrak{X}[U(M)]$ is canonically isomorphic to the $C_0^\infty(\mathcal{T}M)$-module*

$$\mathfrak{X}_0(\mathcal{T}M) := \{X \in \mathfrak{X}(\mathcal{T}M) \mid [C, X] = -X \text{ and } XE = 0\}. \qquad (40) \qquad \square$$

3.3 Lemma. *With respect to the metric g given by (20), the Liouville vector field is everywhere orthogonal to the indicatrix bundle, i.e., $g(C, X) = 0$ for any vector fields $X \in \mathfrak{X}[U(M)]$. The vector field $\frac{1}{\sqrt{2E}} C$ is a unit normal vector field of the indicatrix bundle.*

Proof. In view of Lemma 3.2 it is enough to show that for any vector field $X \in \mathfrak{X}_0(\mathcal{T}M)$, we have $g(C, X) = 0$. This can be seen by an easy calculation:

$$\begin{aligned} g(C, X) &\overset{(20)}{:=} \omega(C, FX) = i_C\omega(FX) \overset{(15)}{=} d_J E(FX) \\ &= dE(J \circ F(X)) = dE(X - hX) \overset{(17)}{=} dE(X) = XE \overset{(40)}{=} 0. \qquad \square \end{aligned}$$

3.4 Lemma. *Suppose that (M, E) is a Finsler manifold. The mapping*

$$\tau := I - \frac{1}{2E}\, dE \otimes C$$

is a projector. For any vector field $X \in \mathfrak{X}(\mathcal{T}M)$, $\tau(X) \upharpoonright U(M)$ is a tangent vector field to the indicatrix bundle.

Proof.

$$\forall X \in \mathfrak{X}(\mathcal{T}M) : \tau(X) = X - \frac{1}{2E} dE(X)C \overset{(16)}{=} X + \frac{1}{2E} i_S\omega(X)C$$
$$= X - \frac{1}{2E}\omega(X,S)C = X - \frac{1}{2E}\omega(X,hS)C \overset{(9)}{=} X - \frac{1}{2E}\omega(X,FC)C$$
$$\overset{(20)}{=} X - \frac{1}{2E} g(X,C)C = X - \frac{g(X,C)}{g(C,C)}C.$$

Thus we have

$$\forall X \in \mathfrak{X}(\mathcal{T}M) : \tau(X) = X - \frac{g(X,C)}{g(C,C)}C. \tag{41}$$

In particular, for any vector field $X \in \mathfrak{X}(\mathcal{T}M)$,

$$\tau(\tau(X)) = \tau(X) - \frac{g(\tau(X),C)}{g(C,C)}C.$$

Since $g(\tau(X),C) = g(X,C) - \frac{g(X,C)}{g(C,C)} g(C,C) = 0$, $\tau(X) \restriction U(M)$ is tangent to $U(M)$ and $\tau(\tau(X)) = \tau(X)$. □

3.5 Proposition. *Using the above notation, let $\kappa := J^* \circ \tau$. Then*

$$\kappa = J - \frac{1}{2E} d_J E \otimes C. \tag{42}$$

The semibasic trace of κ is

$$\widetilde{\kappa} = n - 1. \tag{43}$$

The lowered tensor $\kappa_\flat$ of κ defined by

$$\kappa_\flat(X,Y) := g(\kappa X, JY) \quad (X,Y \in \mathfrak{X}(\mathcal{T}M)) \tag{44}$$

is symmetric.

Proof. The relation (42) is trivial from the definition. It is also clear that κ is semibasic. Since

$$F \circ \kappa \overset{(9)}{=} h - \frac{1}{2E} F \circ (d_J E \otimes C),$$

in view of definition **1.4** (i), we have

$$\widetilde{\kappa} := tr(F \circ \kappa) = tr(h) - \frac{1}{2E} tr\left[F \circ (d_J E \otimes C)\right] = tr(h) - \frac{1}{2E} d_J \widetilde{E \otimes C}.$$

An easy calculation shows that $tr(h) = n$, while, using formula (26) of [17],

$$\widetilde{d_J E \otimes C} = i_S d_J E = dE(C) = CE = 2E,$$

thus (43) is proved. Finally, for any vector fields $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
\kappa_\flat(X^h, Y^h) := g(\kappa X^h, JY^h) &\overset{(42)}{=} g(X^v - \frac{1}{2E}(X^v E)C, Y^v) \\
&= g(X^v, Y^v) - \frac{1}{2E}(X^v E)\omega(Y^v, S) \\
&= g(X^v, Y^v) + \frac{1}{2E}(X^v E)i_S\omega(Y^v) \\
&\overset{(16)}{=} g(X^v, Y^v) - \frac{1}{2E}(X^v E)dE(Y^v) \\
&= g(X^v, Y^v) - \frac{1}{2E}(X^v E)(Y^v E),
\end{aligned}$$

showing the symmetry of $\kappa_\flat$. □

3.6 Lemma. *Consider the operator κ given by (42). For any vector fields*
$X, Y \in \mathfrak{X}(M)$, we have

$$\kappa F\mathcal{C}(X^h, Y^h) = \mathcal{C}(X^h, Y^h) = \mathcal{C}(X^h, F\kappa Y^h). \tag{45}$$

Proof.

$$\begin{aligned}
\kappa F\mathcal{C}(X^h, Y^h) &= \mathcal{C}(X^h, Y^h) - \frac{1}{2E}d_J E(F\mathcal{C}(X^h, Y^h))C \overset{(15)}{=} \mathcal{C}(X^h, Y^h) \\
&\quad - \frac{1}{2E}i_C\omega(F\mathcal{C}(X^h, Y^h))C = \mathcal{C}(X^h, Y^h) - \frac{1}{2E}g(C, \mathcal{C}(X^h, Y^h))C \\
&\overset{(22)}{=} \mathcal{C}(X^h, Y^h) - \frac{1}{2E}\mathcal{C}_\flat(X^h, Y^h, S)C \overset{(27)}{=} \mathcal{C}(X^h, Y^h);
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}(X^h, F\kappa Y^h) &= \mathcal{C}\left(X^h, F\left(Y^v - \frac{1}{2E}(Y^v E)C\right)\right) \\
&= \mathcal{C}(X^h, Y^h) - \frac{1}{2E}(Y^v E)\mathcal{C}(X^h, S) = \mathcal{C}(X^h, Y^h).
\end{aligned}$$

□

3.7 Corollary. *For any vector fields $X, Y, Z \in \mathfrak{X}(M)$, we have*

$$\kappa_\flat(X^h, F\mathcal{C}(Y^h, Z^h)) = \mathcal{C}_\flat(X^h, Y^h, Z^h). \tag{46}$$

Proof. $\kappa_\flat(X^h, F\mathcal{C}(Y^h, Z^h)) \overset{\mathbf{3.5}}{=} \kappa_\flat(F\mathcal{C}(Y^h, Z^h), X^h) \overset{(44)}{:=} g(\kappa F\mathcal{C}(Y^h, Z^h), JX^h) \overset{(45)}{=} g(\mathcal{C}(Y^h, Z^h), JX^h) \overset{(22)}{=} \mathcal{C}_\flat(Y^h, Z^h, X^h) \overset{(26)}{=} \mathcal{C}_\flat(X^h, Y^h, Z^h)$. $\square$

3.8 Definition. *Let (M, E) be a Finsler manifold, $A \in \mathcal{T}_r^0(\mathcal{T}M)$, $K \in \mathcal{T}_r^1(\mathcal{T}M)$ $(r \in \mathbb{N}^+)$; and let us consider the operator $\tau = I - \frac{1}{2E} dE \otimes C$. The tensors A^* and K^*, given by*

$$A^*(X_1, \dots, X_r) := A(\tau(X_1), \dots, \tau(X_r))$$

and

$$K^*(X_1, \dots, X_r) := \tau\left[K(\tau(X_1), \dots, \tau(X_r))\right]$$

$(X_i \in \mathfrak{X}(\mathcal{T}M),\ 1 \leqq i \leqq r)$ are called the projected tensors *of A and K, respectively.*

3.9 Remark. Since for any vector field $X \in \mathfrak{X}(M)$

$$\tau(X^h) = X^h - \frac{1}{2E}(X^h E)C \overset{(17)}{=} X^h,$$

it follows that in case of *semibasic* tensors $A \in \mathcal{T}_r^0(\mathcal{T}M)$ and $K \in \mathcal{T}_r^1(\mathcal{T}M)$

$$A^* = A, \qquad K^* = \tau \circ K.$$

3.10 Proposition. *Consider the Douglas tensor*

$$\mathbb{D} = \overset{\circ}{\mathbb{P}} - \frac{1}{n+1}\Big(\overset{\circ}{D}_J \overset{\tilde{\circ}}{\mathbb{P}} \otimes C + \overset{\tilde{\circ}}{\mathbb{P}} \odot J\Big) \in \mathcal{T}_3^1(\mathcal{T}M) \tag{47}$$

*of the Finsler manifold (M, E) ([15]/**6.1**; the symbol $\odot$ means symmetric product). The projected tensor of $\mathbb{D}$ is*

$$\mathbb{D}^* = \overset{\circ}{\mathbb{P}} - \frac{1}{n+1}\overset{\tilde{\circ}}{\mathbb{P}} \odot \kappa + \frac{1}{E}\mathcal{C}_\flat' \otimes C. \tag{48}$$

Proof. Since $\mathbb{D}$ is semibasic, by the preceding remark $\mathbb{D}^* = \tau \circ \mathbb{D}$. So, for any vector fields $X, Y, Z \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
\mathbb{D}^*(X^h, Y^h, Z^h) &= \mathbb{D}(X^h, Y^h, Z^h) - \frac{1}{2E}\mathbb{D}(X^h, Y^h, Z^h)(E)C \\
&\overset{(47)}{=} \overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h) - \frac{1}{n+1}\,\overset{\circ}{D}_J\tilde{\overset{\circ}{\mathbb{P}}}(X^h, Y^h, Z^h)C \\
&\quad - \frac{1}{n+1}\,\tilde{\overset{\circ}{\mathbb{P}}} \odot J(X^h, Y^h, Z^h) - \frac{1}{2E}\,\overset{\circ}{\mathbb{P}}(X^h, Y^h, Z^h)(E)C \\
&\quad + \frac{1}{n+1}\,\frac{1}{2E}\,\overset{\circ}{D}_J\tilde{\overset{\circ}{\mathbb{P}}}(X^h, Y^h, Z^h)(CE)C \\
&\quad + \frac{1}{n+1}\,\frac{1}{2E}\Big[\tilde{\overset{\circ}{\mathbb{P}}} \odot J(X^h, Y^h, Z^h)\Big](E)C.
\end{aligned}$$

The second and the fifth term on the left hand side cancel. Applying (29) and **1.7** we obtain that

$$\begin{aligned}
\Big[\overset{\circ}{\mathbb{P}}(X^h, Y^h)Z^h\Big]E &= \big[[X^h, Y^v], Z^v\big]\,E = -\,\big[Z^v, [X^h, Y^v]\big]\,E \\
&= -2\mathcal{C}'_\flat(X^h, Z^h, Y^h) = -2\mathcal{C}'_\flat(X^h, Y^h, Z^h),
\end{aligned}$$

hence the fourth term is

$$\frac{1}{E}\mathcal{C}'_\flat \otimes C(X^h, Y^h, Z^h).$$

It is also easy to check that the sixth term can be written in the form

$$\frac{1}{n+1}\,\frac{1}{2E}\Big(\tilde{\overset{\circ}{\mathbb{P}}} \odot d_J E\Big) \otimes C(X^h, Y^h, Z^h),$$

which, together with the third term, yields the second term of (48). $\square$

3.11 Corollary. *The Douglas tensor and its projected tensor are related as follows:*

(49) $$\mathbb{D}^* = \mathbb{D} + \left[\frac{1}{E}\mathcal{C}'_\flat + \frac{1}{n+1}\left(\overset{\circ}{D}_J\tilde{\overset{\circ}{\mathbb{P}}} + \frac{1}{2E}\,\tilde{\overset{\circ}{\mathbb{P}}} \odot d_J E\right)\right] \otimes C. \qquad \square$$

3.12 Theorem. *If (M, E) is a Finsler manifold of dimension $n > 2$, then the vanishing of the projected Douglas tensor is equivalent with the vanishing of the Douglas tensor.*

Proof. Let us consider the Douglas tensor $\mathbb{D}$ and the projected Douglas tensor $\mathbb{D}^*$ of (M,E). Since $\mathbb{D}^* = \tau \circ \mathbb{D}$, if $\mathbb{D} = 0$, then $\mathbb{D}^*$ also vanishes.

Conversely, assume that $\mathbb{D}^* = 0$. Then, from (48) we obtain

$$\overset{\circ}{\mathbb{P}} = \frac{1}{n+1} \widetilde{\overset{\circ}{\mathbb{P}}} \odot \kappa - \frac{1}{E} \mathcal{C}'_\flat \otimes C,$$

and hence

$$\left(\overset{\circ}{D}_J \overset{\circ}{\mathbb{P}}\right)^* = \frac{1}{n+1} \left[\overset{\circ}{D}_J \left(\widetilde{\overset{\circ}{\mathbb{P}}} \odot \kappa\right)\right]^* - \left[\overset{\circ}{D}_J \left(\frac{1}{E} \mathcal{C}'_\flat \otimes C\right)\right]^*. \tag{50}$$

Step 1. We evaluate the left hand side of (50) term by term. Let $X, Y, Z, U \in \mathfrak{X}(M)$ be arbitrary vector fields.

(a) $\left[\overset{\circ}{D}_J \left(\widetilde{\overset{\circ}{\mathbb{P}}} \odot \kappa\right)\right]^* (X^h, Y^h, Z^h, U^h) = \left[\overset{\circ}{D}_J \left(\widetilde{\overset{\circ}{\mathbb{P}}} \odot \kappa\right)\right] (X^h, Y^h, Z^h, U^h)$

$- \frac{1}{2E} \left[\left(\overset{\circ}{D}_J \left(\widetilde{\overset{\circ}{\mathbb{P}}} \odot \kappa\right)\right)(X^h, Y^h, Z^h, U^h)\right](E) C = \left[\overset{\circ}{D}_{X^v} \left(\widetilde{\overset{\circ}{\mathbb{P}}} \odot \kappa\right)\right]$

$\times (Y^h, Z^h, U^h) - \frac{1}{2E} \left[\left(\overset{\circ}{D}_{X^v} \left(\widetilde{\overset{\circ}{\mathbb{P}}} \odot \kappa\right)\right)(Y^h, Z^h, U^h)\right](E) C.$

Using the product rule and taking into account the facts that

(i) $\overset{\circ}{D}_{X^v} \kappa Y^h \overset{(42)}{=} \overset{\circ}{D}_{X^v} \left(Y^v - \frac{1}{2E} (Y^v E) C\right) \overset{\mathbf{1.6}}{=} -X^v \left(\frac{1}{2E} Y^v E\right) C$ $- \frac{1}{2E} (Y^v E) \overset{\circ}{D}_{X^v} C = -X^v \left(\frac{1}{2E} Y^v E\right) C - \frac{1}{2E} (Y^v E) X^v$ (since $\overset{\circ}{D}_{X^v} C = \overset{\circ}{D}_{JX^h} JS \overset{[14]/(27)}{=} J[JX^h, S] \overset{[7],\ \text{Prop. I.7}}{=} JX^h = X^v$);

(ii) $(\kappa Y^h) E = Y^v E - \frac{1}{2E} (Y^v E)(CE) = Y^v E - Y^v E = 0$;

(iii) $\left(\overset{\circ}{D}_{X^v} \kappa Y^h\right) E \overset{\text{(i)}}{=} -2E \cdot X^v \left(\frac{1}{2E} Y^v E\right) - \frac{1}{2E} (X^v E)(Y^v E)$, after a somewhat lengthy but quite straightforward calculation we obtain

$$\begin{aligned} &\left[\overset{\circ}{D}_J \left(\widetilde{\overset{\circ}{\mathbb{P}}} \odot \kappa\right)\right]^* (X^h, Y^h, Z^h, U^h) \\ &= \left[\left(\overset{\circ}{D}_{X^v} \widetilde{\overset{\circ}{\mathbb{P}}}\right) \odot \kappa - \frac{1}{2E} \left(\widetilde{\overset{\circ}{\mathbb{P}}} \odot d_J E\right) \otimes \kappa X^h\right] (Y^h, Z^h, U^h). \end{aligned} \tag{51}$$

(b) $$\Big[\overset{\circ}{D}_J\Big(\frac{1}{E}\mathcal{C}'_\flat\otimes C\Big)\Big]^*(X^h,Y^h,Z^h,U^h)=\Big[\overset{\circ}{D}_J\Big(\frac{1}{E}\mathcal{C}'_\flat\otimes C\Big)\Big]$$
$$\times(X^h,Y^h,Z^h,U^h)-\frac{1}{2E}\Big[\Big(\overset{\circ}{D}_J\Big(\frac{1}{E}\mathcal{C}'_\flat\otimes C\Big)\Big)$$
$$\times(X^h,Y^h,Z^h,U^h)\Big](E)C=\Big[\overset{\circ}{D}_{X^v}\Big(\frac{1}{E}\mathcal{C}'_\flat\otimes C\Big)\Big](Y^h,Z^h,U^h)$$
$$-\frac{1}{2E}\Big[\Big(\overset{\circ}{D}_{X^v}\Big(\frac{1}{E}\mathcal{C}'_\flat\otimes C\Big)\Big)(Y^h,Z^h,U^h)\Big](E)C$$
$$\overset{\mathbf{1.6}}{=}X^v\Big[\frac{1}{E}\mathcal{C}'_\flat(Y^h,Z^h,U^h)\Big]C+\frac{1}{E}\mathcal{C}'_\flat(Y^h,Z^h,U^h)\overset{\circ}{D}_{X^v}C$$
$$-\frac{1}{2E}X^v\Big[\frac{1}{E}\mathcal{C}'_\flat(Y^h,Z^h,U^h)\Big](CE)C$$
$$-\frac{1}{2E^2}\mathcal{C}'_\flat(Y^h,Z^h,U^h)\Big(\overset{\circ}{D}_{X^v}C\Big)(E)C\overset{\text{(i)}}{=}\frac{1}{E}\mathcal{C}'_\flat(Y^h,Z^h,U^h)X^v$$
$$-\frac{1}{2E}\,\frac{1}{E}\mathcal{C}'_\flat(Y^h,Z^h,U^h)(X^vE)C=\frac{1}{E}\mathcal{C}'_\flat(Y^h,Z^h,U^h)\kappa(X^h).$$

This result and (51) yield the formula

(52) $$\Big(\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}}\Big)^*(X^h,Y^h,Z^h,U^h)=\frac{1}{n+1}\Big[\Big(\overset{\circ}{D}_{X^v}\overset{\circ}{\tilde{\mathbb{P}}}\Big)\odot\kappa\Big](Y^h,Z^h,U^h)$$
$$-\kappa(X^h)\Big(\frac{1}{2E}\,\frac{1}{n+1}\,\overset{\circ}{\tilde{\mathbb{P}}}\odot d_JE+\frac{1}{E}\mathcal{C}'\Big)(Y^h,Z^h,U^h).$$

Step 2. Since the tensor $\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}}$ is totally symmetric (**1.7.1**) and $(\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}})^*=\tau\circ\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}}$, the projected tensor $(\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}})^*$ is also totally symmetric. Hence the semibasic (1,4) tensor A defined by

$$A(X^h,Y^h,Z^h,U^h):=\Big(\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}}\Big)^*(X^h,Y^h,Z^h,U^h)-\Big(\overset{\circ}{D}_J\overset{\circ}{\mathbb{P}}\Big)^*(U^h,Y^h,Z^h,X^h)$$

vanishes identically. On the other hand, in view of (52), for any vector fields $X,Y,Z,U\in\mathfrak{X}(M)$ we have

$$A(X^h,Y^h,Z^h,U^h)=\frac{1}{n+1}\Big[\Big(\overset{\circ}{D}_{X^v}\overset{\circ}{\tilde{\mathbb{P}}}\Big)\odot\kappa\Big](Y^h,Z^h,U^h)$$
$$-\kappa(X^h)\Big(\frac{1}{E}\mathcal{C}'_\flat+\frac{1}{2E}\,\frac{1}{n+1}\,\overset{\circ}{\tilde{\mathbb{P}}}\odot d_JE\Big)(Y^h,Z^h,U^h)$$

$$-\frac{1}{n+1}\Big(\overset{\circ}{D}_{U^v}\overset{\widetilde{\circ}}{\mathbb{P}}\odot\kappa\Big)(Y^h,Z^h,X^h)$$
$$+\kappa(U^h)\Big(\frac{1}{E}\mathcal{C}'_\flat+\frac{1}{2E}\,\frac{1}{n+1}\overset{\widetilde{\circ}}{\mathbb{P}}\odot d_J E\Big)(Y^h,Z^h,X^h).$$

Introducing the symmetric semibasic (0,3) tensor

$$\alpha := \frac{1}{E}\mathcal{C}'_\flat+\frac{1}{2E}\,\frac{1}{n+1}\overset{\widetilde{\circ}}{\mathbb{P}}\odot d_J E+\frac{1}{n+1}\overset{\circ}{D}_J\overset{\widetilde{\circ}}{\mathbb{P}}, \tag{53}$$

and taking into account that $\overset{\circ}{D}_{X^v}\overset{\widetilde{\circ}}{\mathbb{P}}=i_{X^h}\overset{\circ}{D}_J\overset{\widetilde{\circ}}{\mathbb{P}}$ $(X\in\mathfrak{X}(M))$, where $\overset{\circ}{D}_J\overset{\widetilde{\circ}}{\mathbb{P}}$ is also totally symmetric by Corollary 4.5 of [15], we finally obtain that

$$A=\alpha\otimes\kappa-\kappa\otimes\alpha.$$

Since the semibasic trace of A clearly vanishes, it follows that

$$\widetilde{\alpha\otimes\kappa}-\widetilde{\kappa\otimes\alpha}=0.$$

Here

$$\widetilde{\alpha\otimes\kappa}\overset{(11)}{=}\widetilde{\kappa}\alpha\overset{(43)}{=}(n-1)\alpha,$$
$$i_X\widetilde{\kappa\otimes\alpha}\overset{(12)}{=}i_{F\kappa(X)}\alpha\quad (X\in\mathfrak{X}(\mathcal{T}M)),$$

and therefore for any vector fields $X,Y,Z\in\mathfrak{X}(M)$,

$$(n-1)\alpha(X^h,Y^h,Z^h)-\alpha(F\kappa(X^h),Y^h,Z^h)=0. \tag{54}$$

To complete the proof, we show that

$$\alpha(F\kappa(X^h),Y^h,Z^h)=\alpha(X^h,Y^h,Z^h). \tag{55}$$

We evaluate the right side of (53) on the triplet $(F\kappa(X^h),Y^h,Z^h)$ term by term.

(i) $\frac{1}{E}\mathcal{C}'_\flat(F\kappa(X^h),Y^h,Z^h)=\frac{1}{E}\mathcal{C}'_\flat(X^h,Y^h,Z^h)$
$-\frac{1}{E}\mathcal{C}'_\flat\Big(\frac{1}{2E}(X^vE)S,Y^h,Z^h\Big)\overset{(27)}{=}\frac{1}{E}\mathcal{C}'_\flat(X^h,Y^h,Z^h);$

(ii) $\frac{1}{n+1}\frac{1}{2E}\Big(\tilde{\mathring{\mathbb{P}}}\odot d_J E\Big)(F\kappa(X^h),Y^h,Z^h)=\frac{1}{n+1}\frac{1}{2E}\Big(\tilde{\mathring{\mathbb{P}}}\odot d_J E\Big)$
$\times(X^h,Y^h,Z^h)-\frac{1}{n+1}\frac{1}{4E^2}(X^vE)\Big(\tilde{\mathring{\mathbb{P}}}\odot d_J E\Big)(S,Y^h,Z^h)$
$=\frac{1}{n+1}\frac{1}{2E}\Big(\tilde{\mathring{\mathbb{P}}}\odot d_J E\Big)(X^h,Y^h,Z^h)-\frac{1}{n+1}\frac{1}{4E^2}(X^vE)\tilde{\mathring{\mathbb{P}}}(Y^h,Z^h)C(E)$
$=\frac{1}{n+1}\frac{1}{2E}\Big(\tilde{\mathring{\mathbb{P}}}\odot d_J E\Big)(X^h,Y^h,Z^h)-\frac{1}{n+1}\frac{1}{2E}(X^vE)\tilde{\mathring{\mathbb{P}}}(Y^h,Z^h),$
using the fact that $i_S\tilde{\mathring{\mathbb{P}}}:=\widetilde{i_S\mathring{\mathbb{P}}}\overset{(33)}{=}0$;

(iii) $\frac{1}{n+1}\mathring{D}_J\tilde{\mathring{\mathbb{P}}}(F\kappa(X^h),Y^h,Z^h)=\frac{1}{n+1}\mathring{D}_J\tilde{\mathring{\mathbb{P}}}(X^h,Y^h,Z^h)$
$-\frac{1}{n+1}\frac{1}{2E}(X^vE)\mathring{D}_J\tilde{\mathring{\mathbb{P}}}(S,Y^h,Z^h)=\frac{1}{n+1}\mathring{D}_J\tilde{\mathring{\mathbb{P}}}(X^h,Y^h,Z^h)$
$-\frac{1}{n+1}\frac{1}{2E}(X^vE)\mathring{D}_C\tilde{\mathring{\mathbb{P}}}(Y^h,Z^h)=\frac{1}{n+1}\mathring{D}_J\tilde{\mathring{\mathbb{P}}}(X^h,Y^h,Z^h)$
$+\frac{1}{n+1}\frac{1}{2E}(X^vE)\tilde{\mathring{\mathbb{P}}}(Y^h,Z^h)$, since $\mathring{D}_C\tilde{\mathring{\mathbb{P}}}=-\tilde{\mathring{\mathbb{P}}}$ by (4.5a) of [15].

Adding the corresponding sides of (i)–(iii) we obtain the desired relation (55). (54) and (55) imply that

$$(n-2)\alpha=0,$$

from which we see, by the assumption $n>2$, that $\alpha=0$. Owing to (49) and (53) this means that $\mathbb{D}=0$, concluding the proof. □

4. Landsberg manifolds with vanishing Douglas tensor

4.1 Lemma. *Suppose that (M,E) is a Landsberg manifold whose Douglas tensor vanishes. Then the mixed curvature of the Berwald connection can be represented in the form*

$$\mathring{\mathbb{P}}=\frac{1}{n+1}\tilde{\mathring{\mathbb{P}}}\odot\kappa. \tag{56}$$

Proof. This is an immediate consequence of (48), the trivial implication $\mathbb{D}=0\Rightarrow\mathbb{D}^*=0$, and the Proposition 2.1. □

4.2 Proposition. *Let (M, E) be a Landsberg manifold with vanishing Douglas tensor. Then the semibasic trace of the mixed curvature of the Berwald connection can be expressed in the form*

$$\tilde{\mathring{\mathbb{P}}} = \frac{1}{n-1} \widetilde{\tilde{\mathring{\mathbb{P}}}^{\#}} \kappa_\flat, \tag{57}$$

where $\tilde{\mathring{\mathbb{P}}}^{\#}$ is the (1,1) tensor metrically equivalent to the (0,2) tensor $\tilde{\mathring{\mathbb{P}}}$: for any vector fields $X, Y \in \mathfrak{X}(TM)$,

$$g\Big(\tilde{\mathring{\mathbb{P}}}^{\#}(X), JY\Big) = \tilde{\mathring{\mathbb{P}}}(X, Y).$$

Proof. Let $X, Y, Z, U \in \mathfrak{X}(M)$ be arbitrary vector fields. We set out from the fact that the tensor $\mathring{\mathbb{P}}_\flat$ defined by (38) is totally symmetric, whence

$$\begin{aligned}
0 &= \mathring{\mathbb{P}}_\flat(X^h, Y^h, Z^h, U^h) - \mathring{\mathbb{P}}_\flat(X^h, Y^h, U^h, Z^h) \overset{(38)}{=} g\Big(\mathring{\mathbb{P}}(X^h, Y^h)Z^h, JU^h\Big) \\
&- g\Big(\mathring{\mathbb{P}}(X^h, Y^h)U^h, JZ^h\Big) \overset{(56)}{=} \frac{1}{n+1}\Big[g\Big(\tilde{\mathring{\mathbb{P}}}(X^h, Y^h)\kappa(Z^h), U^v\Big) \\
&+ g\Big(\tilde{\mathring{\mathbb{P}}}(Y^h, Z^h)\kappa(X^h), U^v\Big) + g\Big(\tilde{\mathring{\mathbb{P}}}(Z^h, X^h)\kappa(Y^h), U^v\Big) \\
&- g\Big(\tilde{\mathring{\mathbb{P}}}(X^h, Y^h)\kappa(U^h), Z^v\Big) - g\Big(\tilde{\mathring{\mathbb{P}}}(Y^h, U^h)\kappa(X^h), Z^v\Big) \\
&- g\Big(\tilde{\mathring{\mathbb{P}}}(U^h, X^h)\kappa(Y^h), Z^v\Big)\Big] \overset{(44)}{=} \frac{1}{n+1}\Big[\tilde{\mathring{\mathbb{P}}}(X^h, Y^h)\kappa_\flat(Z^h, U^h) \\
&+ \tilde{\mathring{\mathbb{P}}}(Y^h, Z^h)\kappa_\flat(X^h, U^h) + \tilde{\mathring{\mathbb{P}}}(Z^h, X^h)\kappa_\flat(Y^h, U^h) - \tilde{\mathring{\mathbb{P}}}(X^h, Y^h)\kappa_\flat(U^h, Z^h) \\
&- \tilde{\mathring{\mathbb{P}}}(Y^h, U^h)\kappa_\flat(X^h, Z^h) - \tilde{\mathring{\mathbb{P}}}(U^h, X^h)\kappa_\flat(Y^h, Z^h)\Big] \\
&= \frac{1}{n+1}\Big[\tilde{\mathring{\mathbb{P}}}(Y^h, Z^h)\kappa_\flat(X^h, U^h) + \tilde{\mathring{\mathbb{P}}}(Z^h, X^h)\kappa_\flat(Y^h, U^h) \\
&- \tilde{\mathring{\mathbb{P}}}(Y^h, U^h)\kappa_\flat(X^h, Z^h) - \tilde{\mathring{\mathbb{P}}}(U^h, X^h)\kappa_\flat(Y^h, Z^h)\Big],
\end{aligned}$$

using the symmetry of $\kappa_\flat$ in the last step. In view of the definition of the tensors $\tilde{\mathring{\mathbb{P}}}^{\#}$ and $\kappa_\flat$, we finally obtain the relation

$$
\begin{aligned}
&g\Big(\tilde{\mathring{\mathbb{P}}}^{\#}(Z^h), Y^v\Big)\kappa_\flat(X^h, U^h) + \tilde{\mathring{\mathbb{P}}}(Z^h, X^h)g(\kappa(U^h), Y^v)\\
-&g\Big(\tilde{\mathring{\mathbb{P}}}^{\#}(U^h), Y^v\Big)\kappa_\flat(X^h, Z^h) - \tilde{\mathring{\mathbb{P}}}(U^h, X^h)g(\kappa(Z^h), Y^v) = 0.
\end{aligned}
$$

Since the vector field $Y \in \mathfrak{X}(M)$ is arbitrary and the tensor g is nondegenerate, this implies

$$
\begin{aligned}
(58)\quad &\tilde{\mathring{\mathbb{P}}}^{\#}(Z^h)\kappa_\flat(X^h, U^h) + \tilde{\mathring{\mathbb{P}}}(Z^h, X^h)\kappa(U^h) - \tilde{\mathring{\mathbb{P}}}^{\#}(U^h)\kappa_\flat(X^h, Z^h)\\
&\qquad -\tilde{\mathring{\mathbb{P}}}(U^h, X^h)\kappa(Z^h) = 0.
\end{aligned}
$$

Now let us define the semibasic (1,3) tensors $\overline{B}$ and B as follows: for any vector fields $X, Y, Z \in \mathfrak{X}(M)$,

$$
\begin{aligned}
\overline{B}(X^h, Y^h, Z^h) &:= \tilde{\mathring{\mathbb{P}}}^{\#}(Y^h)\kappa_\flat(X^h, Z^h) + \tilde{\mathring{\mathbb{P}}}(Y^h, X^h)\kappa(Z^h)\\
&\quad - \tilde{\mathring{\mathbb{P}}}^{\#}(Z^h)\kappa_\flat(X^h, Y^h) - \tilde{\mathring{\mathbb{P}}}(Z^h, X^h)\kappa(Y^h);\\
B(X^h, Y^h, Z^h) &:= \overline{B}(Y^h, X^h, Z^h).
\end{aligned}
$$

From (58) it is trivial that $\overline{B} = B = 0$, whence $\widetilde{\overline{B}} = \widetilde{B} = 0$. Let us observe that

$$
B = \tilde{\mathring{\mathbb{P}}}^{\#} \otimes \kappa_\flat + \tilde{\mathring{\mathbb{P}}} \otimes \kappa - \kappa_\flat \otimes \tilde{\mathring{\mathbb{P}}}^{\#} - \kappa \otimes \tilde{\mathring{\mathbb{P}}}.
$$

Thus, applying (11), (12) and (43), it follows that for any vector fields $X, Y \in \mathfrak{X}(M)$,

$$
\begin{aligned}
(59)\qquad \widetilde{B}(X^h, Y^h) &= \kappa_\flat\Big(F\tilde{\mathring{\mathbb{P}}}^{\#}(X^h), Y^h\Big) + (n-1)\tilde{\mathring{\mathbb{P}}}(X^h, Y^h)\\
&\quad - \widetilde{\tilde{\mathring{\mathbb{P}}}^{\#}}\kappa_\flat(X^h, Y^h) - \tilde{\mathring{\mathbb{P}}}(F\kappa(X^h), Y^h) = 0.
\end{aligned}
$$

In this formula

$$
\begin{aligned}
\text{(i)}\qquad \kappa_\flat\Big(F\widetilde{\mathring{\mathbb{P}}}^{\#}(X^h),Y^h\Big) &= g\Big(\kappa F\widetilde{\mathring{\mathbb{P}}}^{\#}(X^h),Y^v\Big)\\
&= g\Big(\widetilde{\mathring{\mathbb{P}}}^{\#}(X^h),Y^v\Big)-\frac{1}{2E}d_JE\Big(F\widetilde{\mathring{\mathbb{P}}}^{\#}(X^h)\Big)g(C,Y^v)\\
&= \widetilde{\mathring{\mathbb{P}}}(X^h,Y^h)-\frac{1}{2E}g\Big(C,\widetilde{\mathring{\mathbb{P}}}^{\#}(X^h)\Big)g(C,Y^v)\\
&= \widetilde{\mathring{\mathbb{P}}}(X^h,Y^h)-\frac{1}{2E}\widetilde{\mathring{\mathbb{P}}}(X^h,S)g(C,Y^v)\overset{(33)}{=}\widetilde{\mathring{\mathbb{P}}}(X^h,Y^h);
\end{aligned}
$$

$$
\text{(ii)}\quad \widetilde{\mathring{\mathbb{P}}}(F\kappa(X^h),Y^h)=\widetilde{\mathring{\mathbb{P}}}(X^h,Y^h)-\frac{1}{2E}(X^vE)\widetilde{\mathring{\mathbb{P}}}(S,Y^h)=\widetilde{\mathring{\mathbb{P}}}(X^h,Y^h),
$$

whence

$$
\widetilde{\mathring{\mathbb{P}}}(X^h,Y^h)+(n-1)\widetilde{\mathring{\mathbb{P}}}(X^h,Y^h)-\widetilde{\widetilde{\mathring{\mathbb{P}}}^{\#}\kappa_\flat}(X^h,Y^h)-\widetilde{\mathring{\mathbb{P}}}(X^h,Y^h)=0,
$$

which proves the relation (57). □

4.3 Corollary. *If (M,E) is a Landsberg manifold with vanishing Douglas tensor, then the mixed curvature of the Berwald connection can be expressed as follows:*

$$
\mathring{\mathbb{P}}=\frac{1}{n^2-1}\widetilde{\widetilde{\mathring{\mathbb{P}}}^{\#}\kappa_\flat}\odot\kappa. \tag{60}
$$

Proof. Immediate from (56) and (57). □

4.4 Corollary. *With the hypothesis as above, for any vector fields $X,Y,Z,U\in\mathfrak{X}(M)$, we have*

$$
\begin{aligned}
\widetilde{\widetilde{\mathring{\mathbb{P}}}^{\#}}\,\big[&\mathcal{C}_\flat(X^h,Z^h,U^h)\kappa(Y^h)+\kappa_\flat(Z^h,U^h)\mathcal{C}(X^h,Y^h)\\
&-\mathcal{C}_\flat(X^h,Y^h,U^h)\kappa(Z^h)-\kappa_\flat(Y^h,U^h)\mathcal{C}(X^h,Z^h)\big]=0.
\end{aligned}
\tag{61}
$$

Proof. Easy calculation, applying (37) and (60). □

4.5 Theorem. *Suppose that (M,E) is an $n>2$ dimensional Landsberg manifold. If the Douglas tensor of (M,E) vanishes, then (M,E) is a Berwald manifold.*

Proof. Let us introduce the semibasic $(1,4)$ tensors $\overline{\beta}$ and β defined by

$$\begin{aligned}&\overline{\beta}(X^h,Y^h,Z^h,U^h) := \textit{left hand side of } (61),\\ &\beta(X^h,Y^h,Z^h,U^h) := \overline{\beta}(Y^h,X^h,U^h,Z^h) \quad (X,Y,Z,U\in\mathfrak{X}(M)).\end{aligned}$$

Then $\overline{\beta}$, β and their semibasic traces vanish identically. It is immediate that for any vector fields $X,Y,Z,U\in\mathfrak{X}(M)$,

$$\begin{aligned}\beta(X^h,Y^h,Z^h,U^h) = {}&\widetilde{\overset{\circ}{\mathbb{P}}}{}^{\#}(\kappa\otimes\mathcal{C}_\flat+\mathcal{C}\otimes\kappa_\flat-\mathcal{C}_\flat\otimes\kappa)(X^h,Y^h,Z^h,U^h)\\ &-\widetilde{\overset{\circ}{\mathbb{P}}}{}^{\#}(\kappa_\flat\otimes\mathcal{C})(X^h,Z^h,Y^h,U^h).\end{aligned}$$

Applying (11)–(14) and (43), the relation $\widetilde{\beta}=0$ yields

$$\begin{aligned}\widetilde{\overset{\circ}{\mathbb{P}}}{}^{\#}\Big[&\mathcal{C}_\flat(F\kappa(X^h),Y^h,Z^h)+\kappa_\flat(F\mathcal{C}(X^h,Y^h),Z^h)\\ &-(n-1)\mathcal{C}_\flat(X^h,Y^h,Z^h)-\kappa_\flat(X^h,Z^h)\widetilde{\mathcal{C}}(Y^h)\Big]=0.\end{aligned}\tag{62}$$

In this formula

(i)
$$\begin{aligned}\mathcal{C}_\flat(F\kappa(X^h),Y^h,Z^h) &= \mathcal{C}_\flat(X^h,Y^h,Z^h)\\ &-\frac{1}{2E}(X^vE)\mathcal{C}_\flat(S,Y^h,Z^h)=\mathcal{C}_\flat(X^h,Y^h,Z^h),\end{aligned}$$

(ii)
$$\kappa_\flat(F\mathcal{C}(X^h,Y^h),Z^h)\overset{(45)}{=}\mathcal{C}_\flat(X^h,Y^h,Z^h),$$

whence

$$\widetilde{\overset{\circ}{\mathbb{P}}}{}^{\#}\Big[(n-3)\mathcal{C}_\flat(X^h,Y^h,Z^h)+\kappa_\flat(X^h,Z^h)\widetilde{\mathcal{C}}(Y^h)\Big]=0.$$

This can also be written in the form

$$\widetilde{\overset{\tilde{\circ}}{\mathbb{P}}{}^{\#}}\left[(n-3)g(\mathcal{C}(X^h,Y^h),Z^v)+g(\widetilde{\mathcal{C}}(Y^h)\kappa(X^h),Z^v)\right]=0.$$

This implies

$$\widetilde{\overset{\tilde{\circ}}{\mathbb{P}}{}^{\#}}\left[(n-3)\mathcal{C}(X^h,Y^h)+\kappa(X^h)\widetilde{\mathcal{C}}(Y^h)\right]=0. \tag{63}$$

Finally let us consider the semibasic $(1,2)$ tensor

$$\gamma:=\widetilde{\overset{\tilde{\circ}}{\mathbb{P}}{}^{\#}}\left[(n-3)\mathcal{C}+\kappa\otimes\widetilde{\mathcal{C}}\right].$$

In view of (63) γ and, at the same time, $\widetilde{\gamma}$ vanish identically. Applying (12) it follows that for any vector field $X\in\mathfrak{X}(M)$,

$$0=\widetilde{\gamma}(X^h)=\widetilde{\overset{\tilde{\circ}}{\mathbb{P}}{}^{\#}}\left[(n-3)\widetilde{\mathcal{C}}(X^h)+\widetilde{\mathcal{C}}(F\kappa(X^h))\right].$$

In this formula

$$\begin{aligned}\widetilde{\mathcal{C}}(F\kappa(X^h))&=\widetilde{\mathcal{C}}(X^h)-\frac{1}{2E}(X^vE)\widetilde{\mathcal{C}}(S)\\&=\widetilde{\mathcal{C}}(X^h)-\frac{1}{2E}(X^vE)\widetilde{i_S\mathcal{C}}\overset{(27)}{=}\widetilde{\mathcal{C}}(X^h),\end{aligned}$$

so we obtain the relation

$$(n-2)\widetilde{\overset{\tilde{\circ}}{\mathbb{P}}{}^{\#}}\,\widetilde{\mathcal{C}}=0.$$

Since $n>2$, from this it follows that $\widetilde{\overset{\tilde{\circ}}{\mathbb{P}}{}^{\#}}=0$ or $\widetilde{\mathcal{C}}=0$. In the first case (60) implies that $\overset{\circ}{\mathbb{P}}=0$. Then, according to Corollary 2.9, (M,E) is a Berwald manifold. In the second case taking the semibasic trace of both sides of the relation $\overset{\circ}{\mathbb{P}}=-D_h\mathcal{C}$ (2.1 (iv)) we obtain that $0=D_h\widetilde{C}=-\overset{\tilde{\circ}}{\mathbb{P}}$, and then (56) yields the desired conclusion $\overset{\circ}{\mathbb{P}}=0$. $\square$

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