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A counterexample concerning contractive projections of real JB*-triples

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Abstract. We describe the complete real polynomial vector fields of a Euclidean disc and we construct a contractive linear projection of a real JB*-triple onto a 2-dimensional subspace with Euclidean norm such that the projected triple product violates the Jordan identity.

1. Introduction

In 1982 the author established [9] that the image by a contractive linear projection of the unit ball of a complex Banach space is holomorphically symmetric whenever the unit ball itself has the same property. As a consequence of this fact, in 1984 Kaup proved [7] by the aid of his Riemann mapping theorem [6] on bounded symmetric domains that the image of a complex JB*-triple by a contractive linear projection is a JB*-triple with the projected product and this latter is the unique operation satisfying the JB*-triple axioms on the image space. This theorem answered positively a long standing conjecture stating that contractive linear images of complex C*-algebras are JB*-triples. Also this result gave rise to the possibility of generalizing the Arens product (defined originally for C*-algebras) to biduals of complex JB*-triples [3].

Recall that by a complex JB*-triple we mean a Banach space E equipped with an operation $\{xyz\}$ $(x, y, z \in E)$ of three arguments (called

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the triple product) which is symmetric complex-bilinear in its outer variables x, z, conjugate-linear in the inner variable y, satisfies the C*-axiom $\|\{xxx\}\| = \|x\|^3$ ($x \in E$), the Jordan identity $\{ab\{xyz\}\} = \{\{abx\}yz\} - \{x\{bay\}z\} + \{xy\{abz\}\}$ ($a, b, x, y, z \in E$), and the spectral axiom stating that, for any $a \in E$, the linear operator $D_a x := \{aax\}$ is E-Hermitian with non-negative spectrum (i.e. $\|\exp(\zeta D_a)\| \leq 1$ whenever $\operatorname{Re}\zeta \leq 0$). In particular complex C*-algebras with the triple product $\{xyz\} := \frac{1}{2}xy^*z + \frac{1}{2}zy^*x$ can be regarded as complex JB*-triples. Given a complex Banach space E, there can be defined a JB*-triple product on E if and only if the unit ball is symmetric holomorphically and this product is uniquely determined in the latter case. Conversely, given an operation $\{\ \} : E^3 \to E$ on a Banach space E, there exists at most one equivalent norm $|\ |$ on E (the so-called JB*-norm of $\{\ \}$) which makes $(E, |\ |, \{\ \})$ a JB*-triple. (For details see e.g. [11].)

Recently considerable efforts are paid to develop a theory of real JB*-triples [1], [11], [5], [8] defined as real subspaces of complex JB*-triples being closed under the underlying triple product. Some positive results [2], [4] have already appeared concerning the problem of contractive projections of real JB*-triples, and several experts raise the conjecture that the contractive linear image of a real JB*-triple is a real JB*-triple with the projected product. The simple example of Section 2 in 4 real dimensions disproves this expectation: the projected product is no Jordan triple product on the range of a rank 2 contractive linear projection P of the realification of a 2 complex dimensional Cartan factor $(E, \|\ \|, \{\ \})$ of Type 1. In our example the intersection D of the unit ball of E with the range of the projection P is a (2-dimensional) Euclidean disc. By the real version [10] of the projection principle, the vector fields of the form $P[c - \{xcx\}]\,{}^{\partial}/\partial x \mid D$ are all complete in D.* However, they do not constitute a Lie-triple with respect to the Lie triple product $\bigl[X(x)\,{}^{\partial}/\partial x, Y(x)\,{}^{\partial}/\partial x, Z(x)\,{}^{\partial}/\partial x\bigr] := \bigl[\bigl[X(x)\,{}^{\partial}/\partial x, Y(x)\,{}^{\partial}/\partial x\bigr], Z(x)\,{}^{\partial}/\partial x\bigr]$ where

$$(1.1)\quad \Bigl[X(x)\,{}^{\partial}/\partial x, Y(x)\,{}^{\partial}/\partial x\Bigr] := \lim_{\tau\downarrow 0}\bigl[X\bigl(x+\tau Y(x)\bigr) - Y\bigl(x+\tau X(x)\bigr)\bigr]\,{}^{\partial}/\partial x$$

*In our context, given a function $f : E \to E$, we may identify $f(x)\,{}^{\partial}/\partial x$ simply with f. The vector field $f(x)\,{}^{\partial}/\partial x$ is said to be complete in D if for every $x_0 \in D$ there is a differentiable function $x : \mathbb{R} \to D$ such that $x(0) = x_0$ and $\frac{d}{dt}x(t) = f(x(t))$ ($t \in \mathbb{R}$).

is the usual Lie-commutator of vector fields. Our example based heuristically upon a complete parameterized list of the complete real polynomial vector fields on a (2-dimensional real) Euclidean disc, a result of independent interest which we descuss in Section 3. Among the underlying domains of real Cartan triple factors Hilbert balls play a distinguished role: their gauge functions can be the JB*-norm for several different real JB*-triple factors [8]. This latter fact seems to be one of the main obstacles on the way to a pure real geometric theory of JB*-triples, and it is commonly agreed that a deep understanding of the structure of the complete real polynomial vector fieds of Hilbert balls can be crutial in this direction.

2. Counterexample

Proposition 2.1. *On the* 2*-dimensional complex space* $\mathbb{C}^2$ *let*

$$\{xyz\} := \frac{1}{2}\langle x \mid y\rangle\, z + \frac{1}{2}\langle z \mid y\rangle\, y \qquad (x, y, z \in \mathbb{C}^2) \tag{2.2}$$

be the Jordan triple product of the complex type 1 *Cartan factor structure of* $\mathbb{C}^2$ *with respect to the canonical scalar product* $\langle x \mid y\rangle := x_1\overline{y_1} + x_2\overline{y_2}$ *and conjugation* $\overline{x} := (\overline{x_1}, \overline{x_2})$*, and let* P *denote the real-linear projection*

$$Px := \sum_{k=1}^{2} \operatorname{Re}\langle x \mid e_k\rangle\, e_k \qquad \big(x = (x_1, x_2) \in \mathbb{C}^2\big)$$

onto the real-linear subspace $\mathbb{R}e_1 + \mathbb{R}e_2$ *with the unit vectors* $e_1 := (1, 0)$, $e_2 := (i/\sqrt{2}, 1/\sqrt{2})$*. Then the projection* P *is contractive with respect to the JB*-triple norm* $\|\cdot\|$ *associated with* (2.2) *but the operation*

$$\{xyz\} := P\{xyz\} \qquad (x, y, z \in \mathbb{R}e_1 + \mathbb{R}e_2)$$

violates the Jordan identity.

Proof. It is well-known [8] that the JB*-triple norm of the triple product (2.2) on $\mathbb{C}^2$ coincides with the Hilbert norm associated with the scalar product, i.e.

$$\|x\| = \langle x \mid x\rangle^{1/2} = \Big[\sum_{k=1}^{4} \big(\operatorname{Re}\langle x \mid e_k\rangle\big)^2\Big]^{1/2} \qquad (x \in \mathbb{C}^2)$$

where $e_3 := (-i/\sqrt{2}, 1/\sqrt{2})$ and $e_4 := (0, i)$. Since the system $\{e_1, e_2, e_3, e_4\}$ is orthonormed with respect to the real scalar product $\mathrm{Re}\,\langle x \mid y\rangle$ on $\mathbb{C}^2$, the operator P is an orthogonal projection with respect to $\mathrm{Re}\,\langle x \mid y\rangle$ and in particular contractive with respect to the norm $\|\ \|$. We have to show that

$$\text{(2.3)} \qquad \{ab\}\{xyz\}_P\}_P \neq \{\{abx\}_P yz\}_P - \{x\{bay\}_P z\}_P + \{xy\{abz\}_P\}_P$$

for some $a, b, x, y, z \in \mathbb{R}e_1 + \mathbb{R}e_2$. For

$$\text{(2.4)} \qquad a := e_2, \quad b := e_2, \quad x := e_2, \quad y := e_1, \quad z := e_2$$

we have inequality. Indeed

$$\begin{aligned}
\{e_k e_k e_k\}_P &= P\,\langle e_k \mid e_k\rangle\, e_k = e_k \qquad (k = 1, 2),\\
\{e_2 e_2 e_1\}_P &= \{e_1 e_2 e_2\}_P = \frac{1}{2} P\big[\langle e_2 \mid e_2\rangle\, e_1 + \langle e_1 \mid e_2\rangle\, e_2\big]\\
&= \frac{1}{2} P\left(e_1 - \frac{i}{\sqrt{2}} e_2\right) = P(\frac{3}{4}, -\frac{i}{4}) = P\left(\frac{3}{4} e_1 - \frac{1}{4} e_4\right) = \frac{3}{4} e_1,\\
\{e_2 e_1 e_2\}_P &= P\big[\langle e_2 \mid e_1\rangle\, e_2\big] = P\left(\frac{i}{\sqrt{2}} e_2\right) = P\left(-\frac{1}{2}, \frac{i}{2}\right)\\
&= P\left(-\frac{1}{2} e_1 + \frac{1}{2} e_4\right) = -\frac{1}{2} e_1,\\
\{e_1 e_2 e_1\}_P &= P\big[\langle e_1 \mid e_2\rangle\, e_1\big] = P\left(-\frac{i}{\sqrt{2}} e_1\right) = P\left(-\frac{i}{\sqrt{2}}, 0\right)\\
&= \frac{1}{2} P(e_3 - e_2) = -\frac{1}{2} e_2,
\end{aligned}$$

It follows

$$\begin{aligned}
\{ab\{xyz\}_P\}_P &= \{e_2 e_2 \{e_2 e_1 e_2\}_P\}_P = -\frac{1}{2}\{e_2 e_2 e_1\}_P = -\frac{3}{8} e_1,\\
\{\{abx\}_P yz\}_P &= \{\{e_2 e_2 e_2\}_P e_1 e_2\}_P = \{e_2 e_1 e_2\}_P = -\frac{1}{2} e_1,\\
\{x\{bay\}_P z\}_P &= \{e_2 \{e_2 e_2 e_1\}_P e_2\}_P = \frac{3}{4}\{e_2 e_1 e_2\}_P = -\frac{3}{8} e_1,\\
\{xy\{abz\}_P\}_P &= \{e_2 e_1 \{e_2 e_2 e_2\}_P\}_P = \{e_2 e_1 e_2\}_P = -\frac{1}{2} e_1.
\end{aligned}$$

Therefore the left hand side in (2.3) equals $-3/8e_1$ while the right hand side takes the value $-5/8e_1$ for the choice (2.4). $\square$

Remark 2.5. It turns out from the above proof that $D := P\{x \in \mathbb{C}^2 : \|x\| < 1\} = \{\alpha_1 e_1 + \alpha_2 e_2 : \alpha_1,\ \alpha_2 \in \mathbb{R},\ \alpha_1^2 + \alpha_2^2 < 1\}$ is a 2-dimensional Euclidean disc. Therefore there are even two different real Jordan triple products, namely

$$\{xyz\}_1 := \frac{1}{2}\operatorname{Re}\langle x \mid y\rangle\, z + \frac{1}{2}\operatorname{Re}\langle z \mid y\rangle\, x,$$

$$\{xyz\}_2 := \operatorname{Re}\langle x \mid y\rangle\, z + \operatorname{Re}\langle z \mid y\rangle\, x - \operatorname{Re}\langle x \mid \overline{z}\rangle\, \overline{y}$$

which make $\operatorname{ran}(P)$ with the norm $\|\ \|$ a 2-dimensional real JB*-triple. That is the vector fields $[c - \{xcx\}_1]\,{}^{\partial}\!/\partial x$ $(c \in \operatorname{ran}(P))$ resp. $[c - \{xcx\}_2] \times {}^{\partial}\!/\partial x$ $(c \in \operatorname{ran}(P))$ are complete in D. Also all the polynomial vector fields $X_c := [c - \{xcx\}_P]\,{}^{\partial}\!/\partial x$ $(c \in \operatorname{ran}(P))$ of degree 2 are complete in D. However, with the commutator of vector fields (1.1),

$$\big\{[X_a, [X_b, X_c]] : a, b, c \in \operatorname{ran}(P)\big\} \not\subset \big\{X_u : u \in \operatorname{ran}(P)\big\}.$$

3. Complete real polynomial vector fields on the disc

Throughout this section let x, y, z denote the coordinate functions

$$x : (\xi, \eta) \mapsto \xi, \quad y : (\xi, \eta) \mapsto \eta, \quad z := x + iy$$

on $\mathbb{R}^2$. Recall that by a polynomial P of the type $\mathbb{R}^2 \to \mathbb{R}$ of degree $\le N$ we mean a function of the form $P = \sum_{\substack{k+\ell \le N \\ k,\ell \ge 0}} \alpha_{k,\ell} x^k y^\ell$ with suitable real coefficients $\alpha_{k,\ell}$. Since $x = (z + \overline{z})/2$ and $y = i(\overline{z} - z)/2$, by induction on N it follows that $\mathbb{R}^2 \to \mathbb{R}$ polynomials of degree N can be written in the complex forms

$$P = \sum_{\substack{k+2\ell \le N \\ k,\ell \ge 0}} |z|^{2\ell} \left[\mu_{k,\ell} z^k + \overline{\mu_{k,\ell}}\, \overline{z}^k\right] = \sum_{m=0}^{N} \Big[p_m(|z|^2) z^m + \overline{p_m(|z|^2)}\overline{z}^m\Big]$$

with suitable complex coefficients $\mu_{k,\ell}$ and some polynomials $p_0, \ldots, p_N : \mathbb{R} \to \mathbb{C}$ (where each p_m is of degree $\le (N - m)/2$). In particular P vanishes at the points of the unit circle $\mathbb{T} := \{(\cos t, \sin t) : t \in \mathbb{R}\}$ if and only if $0 = P(\cos t, \sin t) = \sum_{m=0}^{N} \left[p_m(1)e^{imt} + \overline{p_m(1)}e^{-imt}\right]$ $(t \in \mathbb{R})$ which is equivalent to $p_m(1) = 0$ $(m = 0, \dots, N)$. Since for a polynomial $p : \mathbb{R} \to \mathbb{C}$ we have $p(1) = 0$ iff $p(\rho) = (1-\rho)q(\rho)$ for some polynomial q, we conclude that

$$(3.1)\quad \{P \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{R}^2) : p(\mathbb{T}) = 0\} = \{(1 - |z|^2)Q : Q \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{R}^2)\}.$$

In the sequel we identify $\mathbb{R}^2$ with $\mathbb{C}$ via the complex coordinate z. Thus we regard the point $(\xi, \eta) \in \mathbb{R}^2$ as the complex number $\xi + i\eta$ and the mapping $(\rho\cos\theta, \rho\sin\theta) \mapsto (\rho^m \cos m\theta, \rho^m \sin m\theta)$ is identified with the complex function z^m for $m = 0, 1, 2, \dots$. In terms of this identification we have the following description of the complete real polynomial vector fields of the unit disc $\mathbb{D} := \{(\xi, \eta) \in \mathbb{R}^2 : \xi^2 + \eta^2 < 1\} (\equiv \{\zeta \in \mathbb{C} : |\zeta| < 1\})$.

Theorem 3.2. *Let $P \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{R}^2)$. Then the vector field $P(v)\,{}^{\partial}\!/\partial v$ is complete in $\mathbb{D}$ if and only if P is a finite real linear combination of the functions*

$$iz, \quad \mu\overline{z}^m - \overline{\mu}z^{m+2} \qquad (\mu \in \mathbb{C},\ m = 0, 1, \dots),$$

$$(1 - |z|^2)Q \qquad (Q \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{R}^2) \equiv \mathrm{Pol}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})).$$

Proof. Let $\mathcal{P}$ denote the set of all polynomials $P \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{R}^2)$ such that the vector field $P(v)\,{}^{\partial}\!/\partial v$ is complete in $\mathbb{D}$. Since $\mathbb{D}$ is a (real-analytic) submanifold of $\mathbb{R}^2$ with the analytic boundary $\mathbb{T}$, for a polynomial $P \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{R}^2)$ we have $P \in \mathcal{P}$ if and only if P is tangent to the circle $\mathbb{T}$. That is,

$$(3.3)\quad \begin{aligned} \mathcal{P} &= \{P \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{R}^2) : P(\xi, \eta) \perp (\xi, \eta) \text{ for } \xi, \eta \in \mathbb{R},\ \xi^2 + \eta^2 = 1\} \\ &= \{P \in \mathrm{Pol}(\mathbb{R}^2, \mathbb{R}^2) : \mathrm{Re}(P(e^{i\tau})e^{-i\tau}) = 0\ (\tau \in \mathbb{R})\}. \end{aligned}$$

Let us write t for the natural coordinate function $t : \tau \mapsto \tau$ of the real line $\mathbb{R}$. Notice that, according to the identification $z : \mathbb{R}^2 \leftrightarrow \mathbb{C}$, $P(e^{it})$ is a complex valued trigonometric polynomial of degree N whenever P is a real polynomial $\mathbb{R}^2 \to \mathbb{R}^2$ of degree N. Define

$$\mathcal{T} := \{\text{trigonometric polynomials } \mathbb{R} \to \mathbb{C}\} = \oplus_{k=-\infty}^{\infty} \mathbb{C}e^{ikt},$$

$$\mathcal{S} := \{P(e^{it}) : P \in \mathcal{P}\}$$

where $\oplus$ denotes algebraic direct sum. By (3.3) we have

$$\mathcal{S} = \{T \in \mathcal{T} : \mathrm{Re}(T \cdot e^{-it}) = 0\}. \tag{3.4}$$

Thus $\mathcal{S}$ is a real-linear subspace of $\mathcal{T}$. Given any $T \in \mathcal{S}$, by differentiating the relation $\mathrm{Re}(T \cdot e^{it}) = 0$ we see that also $0 = \mathrm{Re}(T' \cdot e^{it} - iT \cdot e^{it}) = \mathrm{Re}[(T' - iT)e^{it}]$. That is

$$A\mathcal{S} \subset \mathcal{S} \quad \text{where} \quad A(T) := T' - iT \qquad (T \in \mathcal{T}).$$

Observe that the complex-linear operator A acts diagonally with imaginary eigenvalues over the canonical basis of $\mathcal{T}$:

$$Ae^{ikt} = i(k-1)e^{ikt} \qquad (k = 0, \pm 1, \pm 2, \dots).$$

Since $\mathcal{S}$ is an A-invariant real-linear subspace of $\mathcal{T}$ and the eigenvalues of A^2 are real, namely $A^2 e^{ikt} = -(k-1)^2 e^{ikt}$, it follows

$$\begin{aligned} \mathcal{T} &= \oplus_{m=0}^{\infty} \mathcal{T}_m \quad \text{where} \quad \mathcal{T}_m := \{T \in \mathcal{T} : A^2 T = -m^2 T\} \\ &\qquad\qquad\qquad\qquad\qquad\quad = \mathbb{C}e^{i(1+m)t} + \mathbb{C}e^{i(1-m)t}, \\ \mathcal{S} &= \oplus_{m=0}^{\infty} \mathcal{S}_m \quad \text{where} \quad \mathcal{S}_m := \mathcal{S} \cap \mathcal{T}_m. \end{aligned} \tag{3.5}$$

Indeed, the decomposition $\mathcal{T} = \oplus_m \mathcal{T}_m$ is trivial; if $T \in \mathcal{S}$ then we can write $T = \sum_{m=0}^{N} T_m$ with suitable N and $T_m \in \mathcal{T}_m$ $(m = 0, \dots, N)$ and here necessarily $T_m = \ell_m(A^2)T \in A^2\mathcal{S} \subset \mathcal{S}$ where ℓ_m is the Lagrange interpolation polynomial of degree N with the property $\ell_m(-k^2) = \delta_{mk}$ $(k = 0, \dots, N)$. By (3.4) and (3.5),

$$\begin{aligned} \mathcal{S}_m &= \{T \in \mathcal{T}_m : \mathrm{Re}(T \cdot e^{-it}) = 0\} \\ &= \left\{ \sum_{\varepsilon = \pm 1} \mu_\varepsilon e^{i(\varepsilon m + 1)t} : \mathrm{Re} \sum_{\varepsilon = \pm 1} \mu_\varepsilon e^{i\varepsilon mt} = 0 \right\} \\ &= \left\{ \sum_{\varepsilon = \pm 1} \mu_\varepsilon e^{i(1 + \varepsilon m)t} : \mathrm{Re}[(\mu_1 + \overline{\mu_{-1}})e^{imt}] = 0 \right\} \\ &= \left\{ \sum_{\varepsilon = \pm 1} \mu_\varepsilon e^{i(1 + \varepsilon m)t} : \mu_1 + \overline{\mu_{-1}} = 0 \right\} \\ &= \{\mu e^{i(1-m)t} - \overline{\mu} e^{i(1+m)t} : \mu \in \mathbb{C}\}. \end{aligned}$$

By setting $Z_{0,\mu} := (\mu - \overline{\mu})z$, $Z_{m,\mu} := \mu\overline{z}^{m-1} - \overline{\mu}z^{m+1}$ $(m > 0,\ \mu \in \mathbb{C})$, we have $\mu e^{i(1-m)t} - \overline{\mu}e^{i(1+m)t} = Z_{m,\mu}(e^{it})$. Thus for each real polynomial $P \in \mathcal{P}$ there exists some real linear combination of the real polynomials $Z_{m,\mu}$ which coincides with P on the boundary $\mathbb{T}$ of $\mathbb{D}$. That is, each element of $\mathcal{P}$ is the sum of some real polynomial vanishing on $\mathbb{T}$ with a real-linear combination of functions of the form $Z_{m,\mu}$. Taking (3.1) into account, this completes the proof. □

References

[1] T. Dang and B. Russo, Real Banach–Jordan triples, *Proc. Amer. Math. Soc* **122** (1994), 135–145.

[2] L. J. Bunce and C.-H. Chu, Real contractive projections on commutative C*-algebras, *Math. Z.* **226** (1997), 85–101.

[3] S. Dineen, Complete holomorphic vector fields on the second dual of a Banach space, *Math. Scandinavica* **59** (1986), 131–142.

[4] Y. Friedman and B. Russo, Contractive projections on $C(K)$, *Trans. Amer. Math. Soc.* **273** (1982), 57–93.

[5] J.-M. Isidro, W. Kaup and A. Rodriguez, On real forms of JB*-triples, *Manuscripta Math.* **86** (1995), 311–335.

[6] W. Kaup, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.* **183** (1983), 503–529.

[7] W. Kaup, Contractive projections on Jordan C*-algebras and generalizations, *Math. Scandinavica* **54** (1984), 95–100.

[8] W. Kaup, On real Cartan factors, *Manuscripta Math.* **92** (1–2) (1997), 191–222.

[9] L. L. Stachó, A projection principle concerning biholomorphic automorphisms, *Acta Sci. Math.* **44** (1982), 99–124.

[10] L. L. Stachó, On nonlinear projections of vector fields, *Josai Mathematical Monographs* **JMM1** (1999), 47–55.

[11] H. Upmeier, Symmetric Banach manifolds a JC*-algebras, Mathematics Studies 104, *North Holland*, 1985.

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